Affine Weyl Groups in K-theory Representation Theory, and Combinatorics

Alex Postnikov
Department of Mathematics
Massachusetts Institute of Technology
`apost@math.mit.edu`

(joint with Cristian Lenart)

June 14, 2004

available as preprint `arXiv:math.RT/0309207`

Notations:

G – complex semisimple Lie group
T – maximal torus
B – Borel subgroup, $G \supset B \supset T$
G/B – generalized flag variety (e.g., SL_n/B)
Λ – weight lattice and Φ – root system
W – Weyl group (generated by s_α, $\alpha \in \Phi$)
$X_w^o = BwB/B$, $w \in W$ – Schubert cells

$$G/B = \bigcup_{w \in W} X_w^o \quad \text{(Schubert decomposition)}$$

$X_w = \overline{X_w^o}$ – Schubert varieties
$\mathcal{O}_w = \mathcal{O}_{X_w}$ – structure sheaf of X_w
$\mathcal{L}_\lambda$, $\lambda \in \Lambda$ – line bundle on G/B
$R(T) \simeq \mathbb{Z}[\Lambda]$ – representation ring of torus T
(ring of linear combinations of e^λ, $\lambda \in \Lambda$)

$K_T(G/B)$ – Grothendieck ring of T-equivariant sheaves on G/B; $[\mathcal{O}_w], [\mathcal{L}_\lambda] \in K_T(G/B)$

Claim. [Kostant,Kumar] *$K_T(G/B)$ is a free $R(T)$-module with basis given by $[\mathcal{O}_w]$, $w \in W$.*

Problem: *Give a combinatorial formula for coefficients* $c_{u,w}^{\lambda,\mu} \in \mathbb{Z}$ *in*

$$[\mathcal{L}_\lambda] \cdot [\mathcal{O}_u] = \sum_{w \in W,\ \mu \in \Lambda} c_{u,w}^{\lambda,\mu}\, e^{\mu}\, [\mathcal{O}_w]$$

(K_T-Chevalley formula)

This would generalize *Chevalley formula* for $H^*(G/B)$: $[\lambda] \cdot [X_u] = \sum_{u s_\alpha \lessdot u} (\lambda, \alpha^\vee)\, [X_{u\, s_\alpha}]$ (called Monk's or Pieri's formula in type A)

Pittie and Ram gave a formula for $c_{u,w}^{\lambda,\mu}$, for dominant λ, in terms of Littelmann paths. This formula involves several recursive procedures. Hard to use for explicit computations.

We present a simpler and more explicit K_T-Chevalley formula for arbitrary weights λ. It is closer to the original Chevalley formula.

Application: model for characters

Assume that λ is dominant.

V_λ – irreducible representation of G
$V_{\lambda,u}$, $u \in W$ – Demazure B-module
In particular, $V_\lambda \simeq V_{\lambda,w_\circ}$, where $w_\circ$ is the longest element in W.

Lemma.

$$ch(V_\lambda) = \sum_{w,\mu} c^{\lambda,\mu}_{w_\circ,w}\, e^\mu \quad \textit{(characters of irreps } V_\lambda\textit{)}$$

$$ch(V_{\lambda,u}) = \sum_{w,\mu} c^{\lambda,\mu}_{u,w}\, e^\mu \quad \textit{(Demazure characters)}$$

Our formula implies a simple subtraction-free combinatorial formula for $ch(V_\lambda)$ and $ch(V_{\lambda,u})$. Simpler than Littelmann path model.

λ-chains

$\mathcal{A} = \{H_{\alpha,k} \mid \alpha \in \Phi,\, k \in \mathbb{Z}\}$ – affine Coxeter arrangement for $G^\vee$. Its regions, called *alcoves*, correspond to elements of W_{aff}.

Fix a weight $\lambda \in \Lambda$. Let $\pi(t)$ be a continuous path in $\mathfrak{h}^*_{\mathbb{R}}$ such that $\pi(0) \in$ (fund. alcove) and $\pi(1) = \pi(0) + \lambda$. It crosses affine hyperplanes $H_1, \ldots, H_l \in \mathcal{A}$. Let β_i be the root perpendicular to H_i. Call such a collection of roots $(\beta_1, \ldots, \beta_l)$ a λ-*chain*. λ-chains are in 1-1 correspondence with decompositions $v_\lambda = s_{i_1} \cdots s_{i_l}$ of a certain element in W_{aff}.

Example: (type A_2)

Bruhat operators

For positive root α, define operator B_α by

$$B_\alpha : [\mathcal{O}_w] \longmapsto \begin{cases} [\mathcal{O}_{ws_\alpha}] & \text{if } \ell(ws_\alpha) = \ell(w) - 1 \\ 0 & \text{otherwise} \end{cases}$$

$B_{-\alpha} = -B_\alpha$

$$R_\alpha = 1 + B_\alpha \qquad (R\text{-matrix})$$

This R-matrix satisfies the *Yang-Baxter equation* (in Cherednik sense). In particular, for a root subsystem in Φ of type A_2 generated by (α, β), we have

$$R_\alpha \, R_{\alpha+\beta} \, R_\beta = R_\beta \, R_{\alpha+\beta} \, R_\alpha$$

similar relations for type B_2 and G_2 subsystems

For a λ-chain $(\beta_1, \ldots, \beta_l)$, define

$$R^{[\lambda]} = R_{\beta_1} \cdots R_{\beta_l}$$

Yang-Baxter equation implies that $R^{[\lambda]}$ does not depend on a choice of λ-chain.

Main result

Theorem. (K-Chevalley formula) *The operator $R^{[\lambda]}$ acts on $K(G/B)$ as the operator of multiplication by $[\mathcal{L}_\lambda]$.*

Let X^λ be the $R(T)$-linear operator given by

$$X^\lambda : [\mathcal{O}_w] \mapsto e^{w(\lambda/h^\vee)}\,[\mathcal{O}_w],$$

where $h^\vee$ is the dual Coxeter number

$\tilde{R}_\alpha = X^\rho\,(X^\alpha + B_\alpha)\,X^{-\rho}$, where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$

$\tilde{R}^{[\lambda]} = \tilde{R}_{\beta_1}\cdots\tilde{R}_{\beta_l} = X^\rho(X^{\beta_1}+B_{\beta_1})\cdots(X^{\beta_l}+B_{\beta_l})X^{-\rho}$

Theorem (K_T-Chevalley formula) *The operator $\tilde{R}^{[\lambda]}$ acts on $K_T(G/B)$ as the operator of multiplication by $[\mathcal{L}_\lambda]$.*

This implies that the basis expansion of the product $[\mathcal{L}_\lambda]\cdot[\mathcal{O}_u]$ is given as a certain sum over saturated chains in the Bruhat order.

Formula for character of V_λ

Assume that λ is dominant. Let $(\beta_1, \dots, \beta_l)$ be a λ-chain, let $H_1, \dots, H_l$ be the corresponding collection of affine hyperplanes, and let

$$r_j = s_{\beta_j, k_j} = \text{affine reflection w.r.t. } H_j$$

$r_1, \dots, r_l \in W_{\text{aff}}$

Corollary.

$$ch(V_\lambda) = \sum_J e^{-r_{j_1} \cdots r_{j_l}(-\lambda)},$$

where the sum is over $J = \{j_1 < \cdots < j_s\} \subset \{1, \dots, l\}$ *such that*

$$1 \lessdot s_{\beta_{j_1}} \lessdot s_{\beta_{j_1}} s_{\beta_{j_2}} \lessdot \cdots \lessdot s_{\beta_{j_1}} \cdots s_{\beta_{j_s}}$$

is a saturated increasing chain in the Bruhat order on W.

Products with special Schubert classes

$[\mathcal{O}_{w_\circ s_i}]$ – *special* classes for codimension 1 Schubert varieties. They generate $K_T(G/B)$ as an algebra over $R(T)$.

Lemma. cf. [Brion] $[\mathcal{O}_{w_\circ s_i}] = 1 - e^{w_\circ(\omega_i)}[\mathcal{L}_{-\omega_i}]$

Here $\omega_i \in \Lambda$ are the fundamental weights.

Our formula implies a rule for coefficients in

$$[\mathcal{O}_{w_\circ s_i}] \cdot [\mathcal{O}_u] = \sum_{w,\, \mu} \dots\, e^{\mu}\, [\mathcal{O}_w]$$

It is hard to directly apply Pittie-Ram's formula because this expression involves *negative* fundamental weights $-\omega_i$.

Two duality formulas

Two involutions $u \mapsto u\, w_\circ$ and $u \mapsto w_\circ\, u$ on W map saturated increasing chains in the Bruhat order to saturated decreasing chains. Our K_T-Chevalley formula easily implies the following two symmetries.

Corollary. $c_{u,w}^{\lambda,\mu} = (-1)^{\ell(u)-\ell(w)} c_{w\, w_\circ,\, u\, w_\circ}^{w_\circ(\lambda),\, \mu}$

[Brion] proved this for $K(G/B)$ using an involved geometric argument.

New duality:

Corollary. $c_{u,w}^{\lambda,\mu} = (-1)^{\ell(u)-\ell(w)} c_{w_\circ\, w,\, w_\circ\, u}^{-\lambda,\, -w_\circ(\mu)}$

Dual K_T-Chevalley formula

$\mathcal{I}_w$ – sheaf given by the exact sequence

$$0 \to \mathcal{I}_{X_w} \to \mathcal{O}_{X_w} \to \mathcal{O}_{\partial X_w} \to 0,$$

where $\partial X_w = \bigcup_{u<w} X_u$ – boundary of X_w

The classes $[\mathcal{I}_w]$, $w \in W$, form an $R(T)$-basis of $K_T(G/B)$ (studied by [Kostant-Kumar]).

$$[\mathcal{I}_w] = \sum_{u \le w} (-1)^{\ell(u)} [\mathcal{O}_u]$$

$$[\mathcal{O}_w] = \sum_{u \le w} (-1)^{\ell(u)} [\mathcal{I}_u]$$

(Möbius inversion on the Bruhat order)

Lemma.

$$[\mathcal{L}_\lambda] \cdot [\mathcal{I}_u] = \sum_{w \in W,\ \mu \in \Lambda} c_{u,w}^{-\lambda,-\mu}\, e^{\mu}\, [\mathcal{I}_w]$$

Our K_T-Chevalley formula immediately gives a rule for the expansion of $[\mathcal{L}_\lambda] \cdot [\mathcal{I}_u]$.

Idea of proof of K_T-Chevalley formula:

Let T_i be the *Demazure operators.* They act $R(T)$-linearly on $K_T(G/B)$. (In type A, these are the *isobaric divided differences operators.*) They satisfy Hecke algebra relations for $q=0$. ($T_i^2 = T_i$ and the Coxeter relations)

According to [Kostant, Kumar], for a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$,

$$[\mathcal{O}_w] = T_{i_l} \cdots T_{i_1}([\mathcal{O}_1]).$$

In order to show that $\tilde{R}^{[\lambda]} = [\mathcal{L}_\lambda]\times$ (operator of multiplication by $[\mathcal{L}_\lambda]$), it is enough to show that the operators $\tilde{R}^{[\lambda]}$ satisfy the same commutation relations with T_i's as the operators $[\mathcal{L}_\lambda]\times$ do. (Affine Hecke algebra relations.)

First, we give commutation relations for the operators R_α and T_i, then deduce the affine Hecke algebra relations for $\tilde{R}^{[\lambda]}$ and T_i. Q.E.D.

Quantum K-theory

Quantum Bruhat operators, for a root α,

$$Q_\alpha : [\mathcal{O}_w] \mapsto \begin{cases} [\mathcal{O}_{ws_\alpha}] & \ell(ws_\alpha) = \ell(w) - 1 \\ q^{\alpha^\vee} [\mathcal{O}_{ws_\alpha}] & \ell(ws_\alpha) = \ell(w) + 2\,|\alpha^\vee| - 1 \\ 0 & \text{otherwise} \end{cases}$$

where $|\alpha^\vee| = (\rho, \alpha^\vee)$ (height of coroot $\alpha^\vee$), and $q^{\alpha^\vee} = q_1^{d_1} \cdots q_r^{d_r}$, for $\alpha^\vee = d_1\alpha_1^\vee + \cdots + d_r\alpha_r^\vee$.

In [Brenti,Fomin,Postnikov], we proved that Q_α's satisfy the Yang-Baxter equation. One can write the *quantum Chevalley formula* for the quantum cohomology $QH^*(G/B)$, using these operators.

[Lee] and [Givental,Lee] defined and study *quantum K-theory $QK(G/B)$*. It involves certain K-invariants of Gromov-Witten type.

Conjecture. *We obtain a Chevalley-type product formula for $QK(G/B)$ if we replace operators B_α in our K-Chevalley formula with Q_α.*