# Reflections on cyclicity & positivity
## (permutohedra & Grassmannian)

Alex Postnikov

Jan. 10, 2022

On the occasion of Ron Adin's and Yuval Roichman's birthdays

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Keywords from works of Ron & Yuval:

cyclic, reduced decompositions, Schur positivity, polytopes, spanning trees, ...

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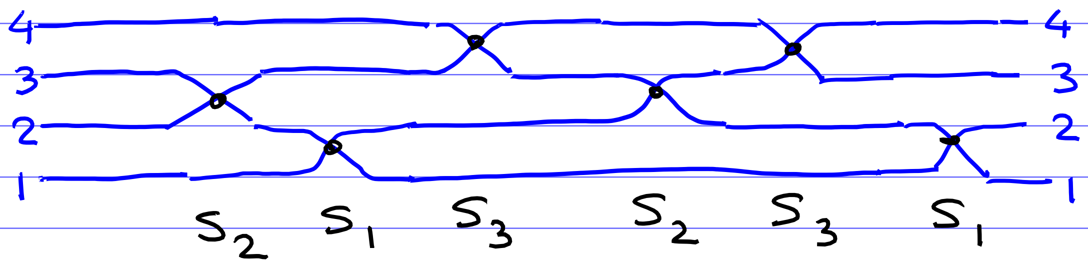
positive Grassmennian $\overset{?}{\longleftrightarrow}$ generalized permutohedra

# Reduced decompositions in $S_n$

<u>Example</u> A wiring diagram:




$(2, 1, 3, 2, 3, 1)$ a <u>reduced word</u>
for $w_0 \in S_4$

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Christian Gaetz, Yibo Gao,
Pakawut (Pro) Jiradilok, Gleb Nenashev, P.:
The maximum multiplicity of
a generator in a reduced word.
FPSAC, tomorrow @ 19:05 ...

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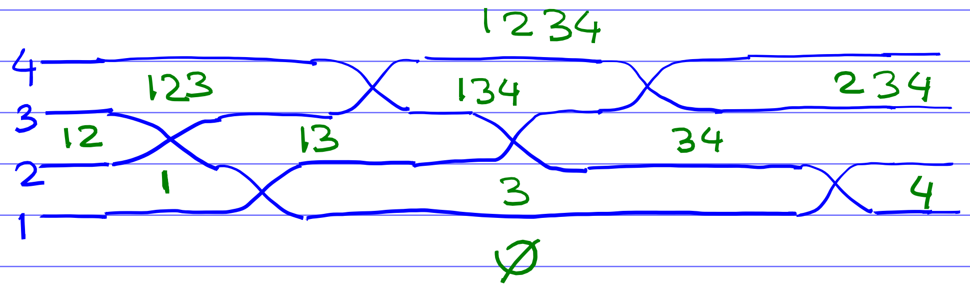
reduced words

$\updownarrow$ ?

cyclic symmetry

What is the
"cyclic analog" of
reduced words ?

The labelling of regions in a wiring diagram by subsets of $[n] := \{1,2,\dots,n\}$ :




$$\mathcal{C} = \{\varnothing, 1, 3, 4, 12, 13, 34,$$
$$123, 124, 234, 1234\}$$
$$\mathcal{C} \subset 2^{[n]}$$

**Definition** [Leclerc - Zelevinsky, 1998]
A collection $\mathcal{C}$ of subsets of $[n]$ is strongly separated if
$\forall\, I, J \in \mathcal{C}$

$I \setminus J \prec J \setminus I$ or
$J \setminus I \prec I \setminus J$

any element of $I \setminus J$ is less than any element of $J \setminus I$

**Theorem** [LZ] The collections $\mathcal{C}$ coming from reduced decompositions of $w_0 \in S_n$ are exactly the maximal (by containment) strongly separated collections in $2^{[n]}$.

**Corollary** The maximal cardinality of a strongly separated collection in $2^{[n]}$ is $\frac{n(n+1)}{2}+1$ ← # regions in a wiring diagram for $w_0$

**Question.** What is the maximal cardinality of a strongly separated collection in $\binom{[n]}{k}$? ← the set of $k$-element subsets of $[n]$

Equiv. to the question about max. multiplicity of a generator in a reduced word.

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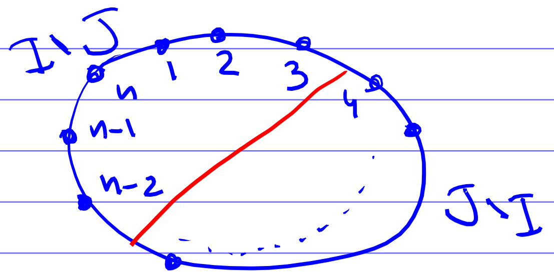
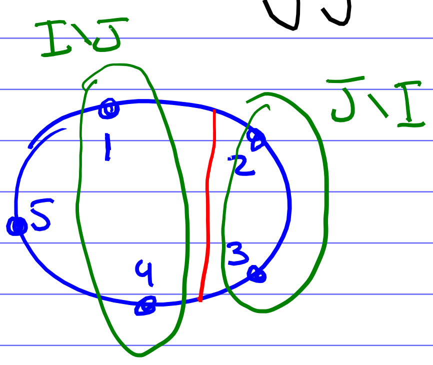
[Leclerc - Zelevinsky] also defined the notion of weakly separated collections.

**Definition** [LZ] A collection $\mathcal{C} \subset \binom{[n]}{k}$ is **weakly separated** if $\forall\ I, J \in \mathcal{C}$ the subsets $I \setminus J$ and $J \setminus I$ are separated by a chord on the circle (with indices $1, 2, \dots, n$ arranged clockwise):



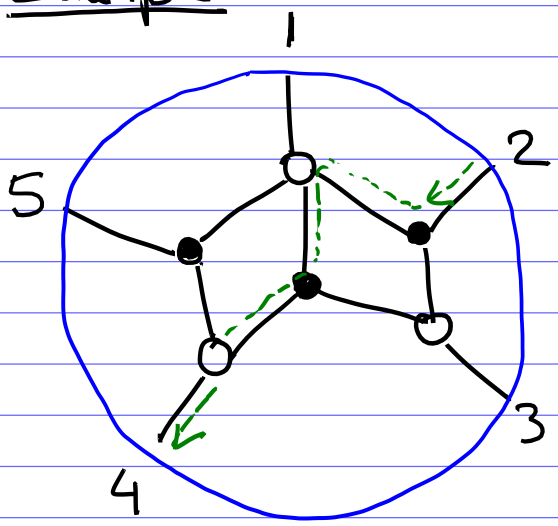

**Example** $\mathcal{C} = \{\{1,4,5\}, \{2,3,5\}\}$ is weakly separated but not strongly separated.



# Reduced plabic graphs

(the "cyclic analog" of reduced words)

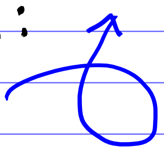
**Example**



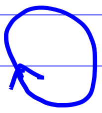

- drawn inside a circle with boundary vertices labelled $1, 2, \ldots, n$ clockwise

- internal vertices are colored in 2 colors

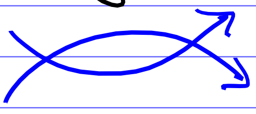
- no self-intersect. strands:



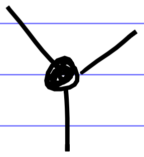
- no closed strands.

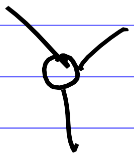


- no "bad double crossing" strands.



Strands are the walks that follow the rules of the road:

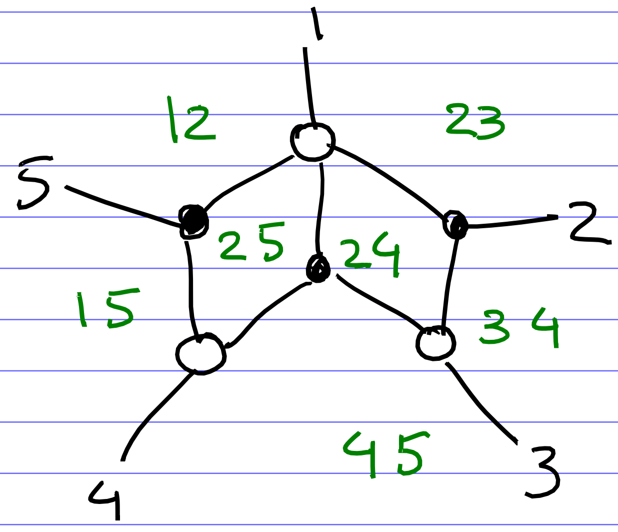
 turn right

 turn left

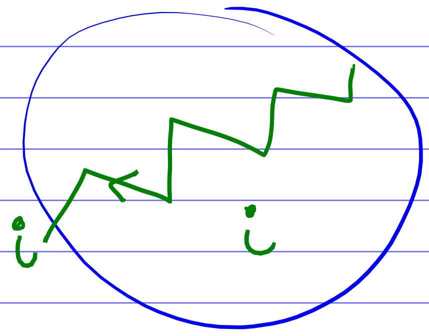
A plabic graph has the strand permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ given by: the strand that starts at $i^{th}$ boundary vertex ends at $\pi_i^{th}$ —— " —— for $i = 1, 2, \ldots, n$.

The labelling of regions of a reduced plabic graph by $k$-element subsets of $[n]$:

Example




Rule: Place the label $i$ in all regions to the left of the strand that ends at $i^{th}$ boundary vertex:




$n=5$, $k=2$

$\mathcal{C} = \{12, 23, 34, 45, 51, 25, 24\} \in \binom{[5]}{2}$

Theorem [Oh–P.–Speyer '15]

Maximal (by containment) weakly separated collections $\mathcal{C} \subset \binom{[n]}{k}$ are exactly the collections of labels of reduced plabic graphs with the strand permutation

$\pi : [n] \to [n]$ s.t.

$\pi : i \longmapsto i+k \bmod n$.

Any such maximal weakly separated collection has the cardinality $k(n-k)+1$.

All these combin. structures are closely related to algebra & geometry:

- quazi-commuting collections of quantum minors [Lecl.-Zel.]
- cluster algebras
- the totally nonnegative Grassmannian.

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## The Grassmannian

Fix $1 \le k \le n$.

$Gr_{kn} = Gr_{kn}(\mathbb{R}) :=$

$:= \left\{ \begin{matrix} k\times n \text{ matrices } / \mathbb{R} \\ \text{of rank } k \end{matrix} \right\} \Big/ \text{row operations}$

$$A = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kn} \end{bmatrix}$$

$I$

For $I \in \binom{[n]}{k}$,

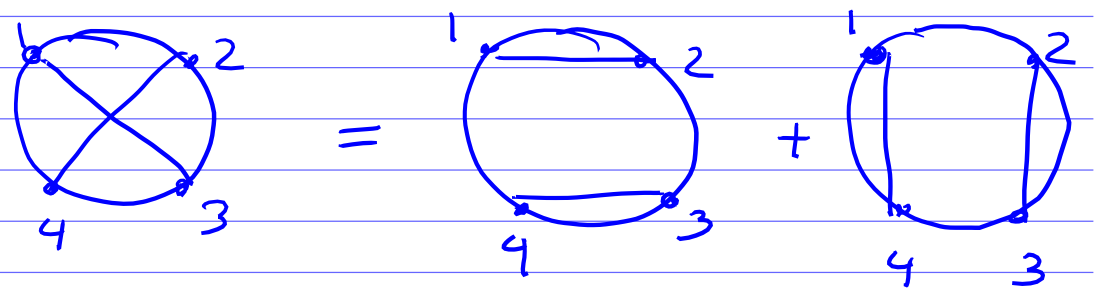
$\Delta_I = \Delta_I(A)$ the $k\times k$ minor of $A$ in the column set $I$ (& row set $\{1,2,\ldots,k\}$)

The $\Delta_I$ form projective coordinates on $Gr_{kn}$, called the Plücker coordinates.

The $\Delta_I$ are not algebraically independent. They satisfy certain quadratic relations called the Plücker relations.

Example $n=4$, $k=2$.

$$\boxed{\Delta_{13}\cdot\Delta_{24} = \Delta_{12}\cdot\Delta_{34} + \Delta_{14}\cdot\Delta_{23}}$$




## The totally nonnegative Grassmannian

$Gr_{kn}^{\ge 0} := \{ A \in Gr_{kn}(\mathbb{R}) \mid \text{all } \Delta_I(A) \ge 0 \}$

**Theorem** A maximal WS-collection $\mathcal{C} \subset \binom{[n]}{k}$ (i.e. the collection of labels of a reduced plabic graph) gives an algebraically independent set $\{\Delta_I \mid I \in \mathcal{C}\}$ of Plücker coordinates on $Gr_{kn}$.

Any other Plücker coordinate $\Delta_J$ can be expressed by a subtraction-free Laurent polynomial in terms of the $\Delta_I$, $I \in \mathcal{C}$.

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**Remark**. These are not all algebraically indep. coordinate systems on $Gr_{kn}$ given by subsets of Plücker coord.

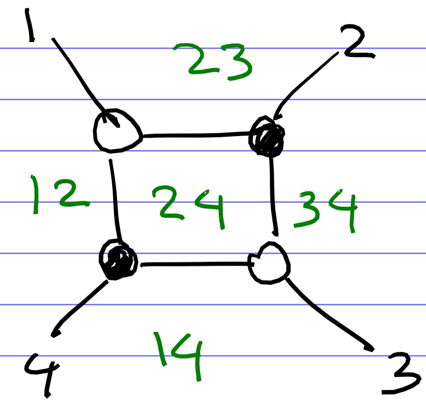
**Example** $n=4$, $k=2$

$$\Delta_{13} \cdot \Delta_{24} = \Delta_{12} \cdot \Delta_{34} + \Delta_{14} \cdot \Delta_{23}$$

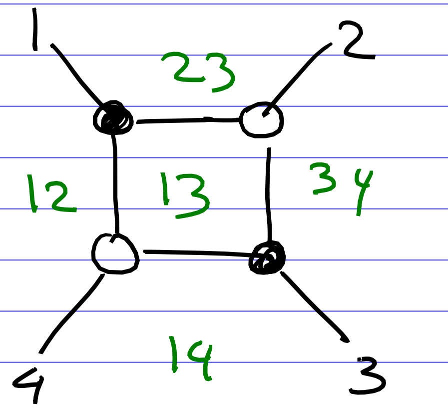
Any 5-element subset $\mathcal{C}$ in $\binom{[4]}{2}$ gives an alg. indep. coord. system on $Gr_{24}$.

However only $\binom{[4]}{2} \setminus \{\{1,3\}\}$ and $\binom{[4]}{2} \setminus \{\{2,4\}\}$ have the subtraction-free property.

They correspond to the 2 plebic graphs:




and



Question: How to describe all algebraically independent subsets of the Plücker coordinates on $Gr_{kn}$ ?

In general, this is a hard question.

It is related to the ongoing joint project with Jiyang Gao, Yibo Gao, Zixuan Xu "Low rank matrix completion"

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For $k=2$, Daniel Bernstein gave the following criterion using tropical geometry.

A subset $\mathcal{E} \subset \binom{[n]}{2}$ $\leftrightsquigarrow$ the edge set of some subgraph $G$ of $K_n$

Theorem [Bernstein, 2017] Alg. indep. subsets of Plücker coordinates in $Gr_{2n}$ correspond to graphs $G$ that have an acyclic orient. without alternating closed trails.

closed walks on $G$ with alternating edge directions w/o repeating edges.

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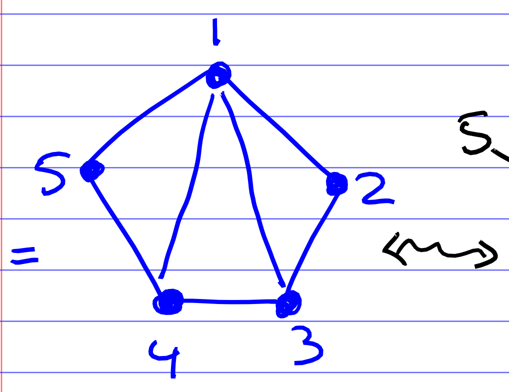
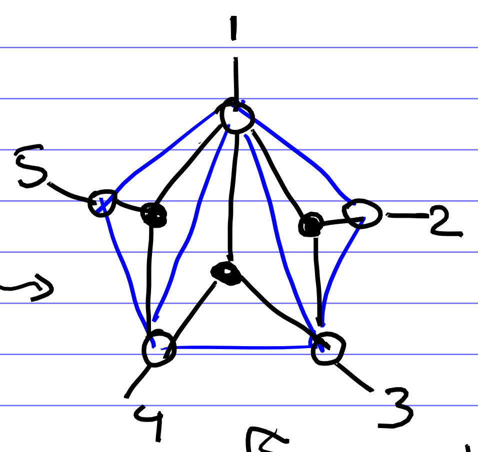
For $k=2$, reduced plabic graphs correspond to triangulations of an n-gon

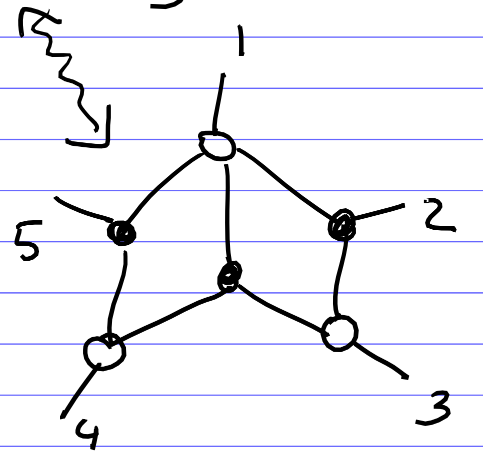
Example

$G=$ 


$\leftrightsquigarrow$



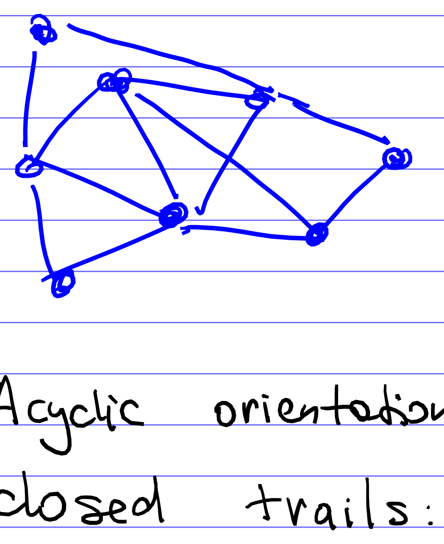

a triangulation of 5-gon

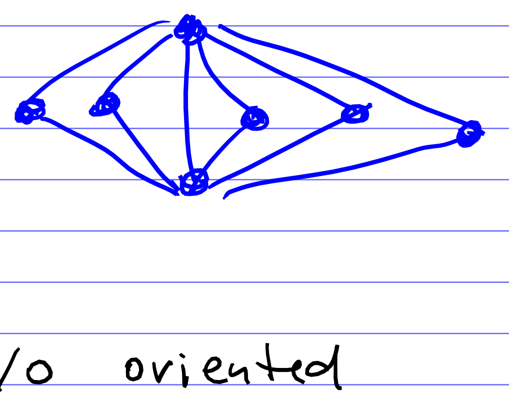



a reduced plabic graph.

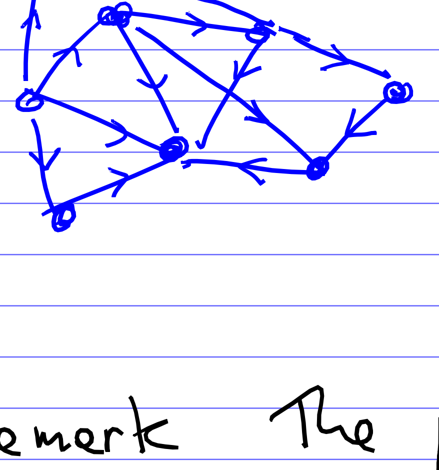
Any triangulation of an n-gon satisfies Bernstein's criterion.

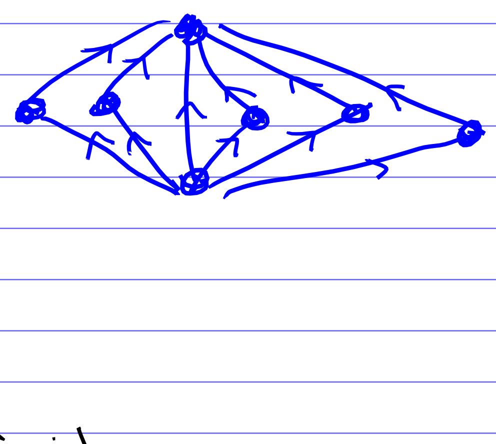
But here are some graphs that satisfy Bernstein's criterion, which are not triangulations:





Acyclic orientations w/o oriented closed trails:





Remark The positivity (i.e. subtraction-free Laurentness) implies the graph $G$ is a triangulation of an n-gon, i.e. it has a "circular structure".

**Definition.** A positroid is a subset $\mathcal{M} \subseteq \binom{[n]}{k}$ s.t. $\exists$ $A \in Gr^{\geq 0}_{kn}$ s.t.

$$\Delta_I(A) \gneq 0 \quad \text{iff} \quad I \in \mathcal{M}$$

(and $\Delta_I(A) = 0$ iff $I \notin \mathcal{M}$).

Positroids form a special class of realizable matroids. ($I \in \mathcal{M}$ are called <u>bases</u>)

The <u>cyclic symmetry</u> of $Gr^{\geq 0}_{kn}$

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in Gr^{\geq 0}_{kn}$$

$$c(A) = \begin{bmatrix} | & | & & | & \\ v_2 & v_3 & \cdots & v_n & (-1)^{k-1} v_1 \\ | & | & & | & \end{bmatrix}$$

This operation preserves the signs of all Plücker coord.

So we get a well-defined map

$$c : Gr^{\geq 0}_{kn} \longrightarrow Gr^{\geq 0}_{kn}$$

**Remark.** The notion of positroids is invariant under <u>cyclic shifts</u> of indices $i \in [n]$ : $i \mapsto i+1 \mod n$,

but not under any permutation of $[n]$.

The hierarchy of combinatorial structures:

$$\begin{array}{ccccc} \text{graphs} & \subset & \text{hyperplane arrangements} & \subset & \text{matroids} \\ \cap & & & & \cap \\ \text{hypergraphs} & & \subset & & \text{polymatroids} \\ & & & & \cap \\ & & & & \text{Coxeter polymatroids} \end{array}$$

All these objects are associated with some special kinds of convex polytopes.

How do positroids & plabic graphs fit into this picture?

In the joint work with Thomas Lam titled "Polypositroids" 2020 we discussed extensions of positroids to polymatroids & Coxeter polymatroids.

# Matroid polytopes

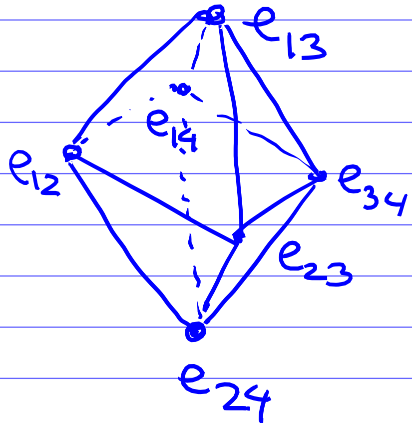
$e_1, e_2, \ldots, e_n$ the standard basis of $\mathbb{R}^n$

For $I \subseteq [n]$, $e_I := \sum_{i \in I} e_i$.

For $\mathcal{M} \subseteq \binom{[n]}{k}$, let

$$P_{\mathcal{M}} := \mathrm{Conv}\left( e_I \mid I \in \mathcal{M} \right)$$

**Example** $P_{\binom{[4]}{2}} =$




octahedron

**Theorem** (Gelfand, Goresky, Macpherson, Serganova, 1987)

$\mathcal{M} \subseteq \binom{[n]}{k}$ is the set of bases of a matroid iff every edge of the polytope $P_{\mathcal{M}}$ is $\parallel$ some vector $e_i - e_j$.

# Positroid polytopes

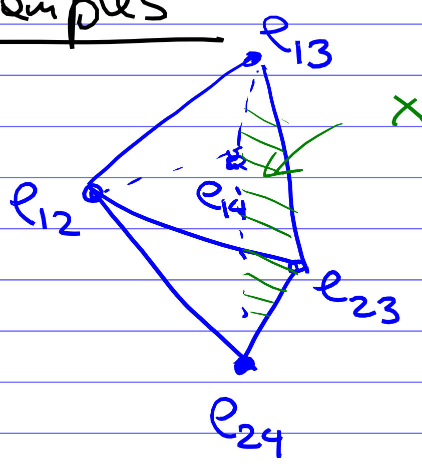
**Theorem** [Lam–P.]

$\mathcal{M} \subseteq \binom{[n]}{k}$ is a positroid iff $P_\mathcal{M}$ satisfies:

- every edge of $P_\mathcal{M}$ is $\parallel e_i - e_j$ for some $i \neq j \in [n]$
- every facet of $P_\mathcal{M}$ is given by equation $x_i + x_{i+1} + \dots + x_j = C$ for some cyclic interval

$$[i,j] := \begin{cases} \{i, i+1, \dots, j\}, & i \le j \\ \{i, i+1, \dots, n, 1, \dots, j\}, & i > j \end{cases}$$

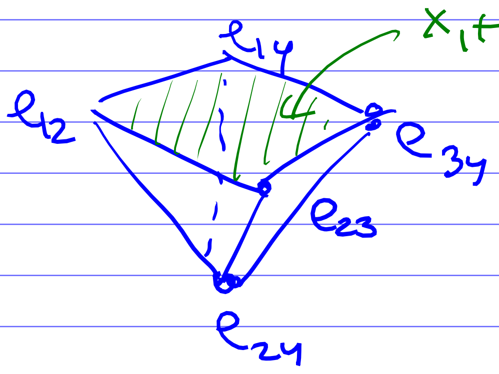
**Examples**




$x_1 + x_2 = 2$

$\{1,2\}$ is a cyclic interval

$\binom{[4]}{2} \setminus \{\{3,4\}\}$ is a positroid




$x_1 + x_3 = 2$

$\{1,3\}$ is not a cyclic interval

$\binom{[4]}{2} \setminus \{\{1,3\}\}$ is a matroid but not a positroid.

We can easily see this from the Plücker relation

$$\Delta_{13} \cdot \Delta_{24} = \Delta_{12} \cdot \Delta_{34} + \Delta_{14} \cdot \Delta_{23}$$

If we require that all $\Delta_I \geq 0$, we can have $\Delta_{34} = 0$ and all other $\Delta_I \gneq 0$,

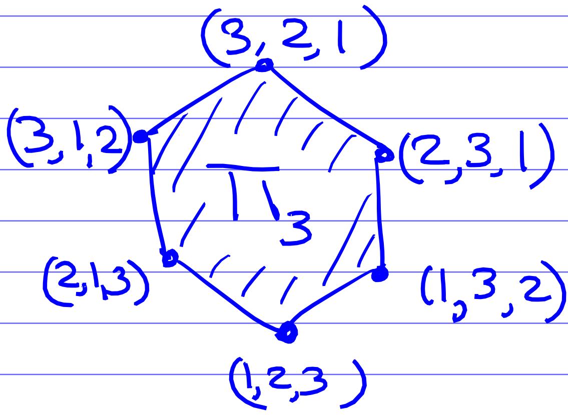
but we cannot have $\Delta_{13} = 0$ and all other $\Delta_I \gneq 0$.

# Generalized permutohedra

The standard permutohedron is

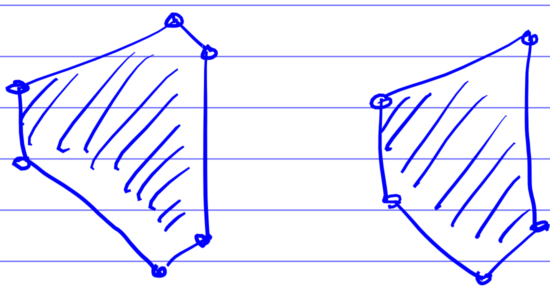
$$\Pi_n := \mathrm{Conv}\left( (w(1), \ldots, w(n)) \mid w \in S_n \right)$$

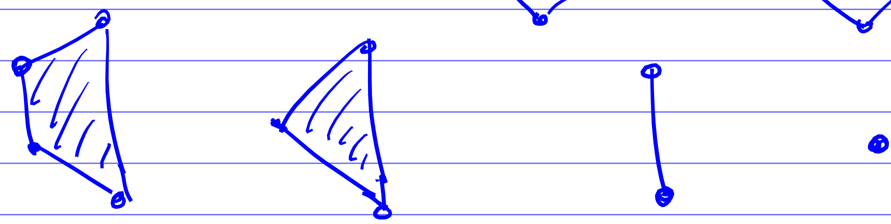
Example




A generalized permutohedron is a polytope obtained from $\Pi_n$ by parallel translation of its facets. Some edges can collapse, but after a collapse no further translations are allowed.

Examples





More precisely, gen. permutohedra $P \subset \{x_1 + \dots + x_n = \text{Const.}\} \subset \mathbb{R}^n$ can be defined in several equivalent ways [P.-Reiner-Williams '08]:

- polytopes $P$ s.t. the normal fan of $P$ is a coarsening of the normal fan of $\Pi_n$
- polytopes $P$ s.t. every edge is $\parallel e_i - e_j$
- polytopes $P_f$ given by

$$P_f := \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i \in I} x_i \le f(I), \ \forall I \subseteq [n] \quad x_1 + \dots + x_n = f([n]) \},$$

where $f : 2^{[n]} \to \mathbb{R}$ is a <u>submodular function</u>

$$f(I) + f(J) \geq f(I \cup J) + f(I \cap J) \quad \forall I, J \subseteq [n]$$

$$f(\emptyset) = 0$$

Gen. permutohedra are essentially the same thing as <u>polymatroids.</u>

**Definition** (Lam–P.)

A polypositroid is a polytope $P \subset \mathbb{R}^n$ s.t.

- $P$ is a gen. permutohedron, and
- any facet of $P$ is given by equation $x_i + x_{i+1} + \dots + x_j = \mathrm{Const.}$ for some cyclic interval $[i,j]$.

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All the nice things that we know about positroids can be extended to polypositroids...

There are several explicit constructions of polypositroids.

generalized permutohedra $\xleftrightarrow{\text{bij.}}$ submodular functions $f: 2^{[n]} \to \mathbb{R}$

polypositroids $\xleftrightarrow{\text{bij.}}$ "submodular functions on cyclic intervals"

Let $\mathrm{Cyc}_n \subset 2^{[n]}$ be the set of all cyclic intervals $[i,j] \subseteq [n]$, (together with $\emptyset$ & $[n]$)

<u>Theorem.</u> [LP] Any polypositroid $P \subset \mathbb{R}^n$ has the form

$$P = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{r \in I} x_r \leq f(I), \text{ for any } I \in \mathrm{Cyc}_n, \text{ and } x_1 + \dots + x_n = f([n])\},$$

where $f : \mathrm{Cyc}_n \to \mathbb{R}$ is a function such that

$f(I) + f(J) \geq f(I \cup J) + f(I \cap J)$ whenever $I, J, I \cup J, I \cap J \in \mathrm{Cyc}_n$
and $f(\emptyset) = 0$.

These conditions define the "cone of polypositroids".

Some differences between gen. permutohedra & polypositroids:

- The Minkowski sum of two gen. permutohedra is a gen. perm. But this may not be true for polypositroids

- Any generic gen. permutohedron is combinatorially equiv. to $\Pi_n$. But, for $n \geq 4$, there are several different combinatorial types of polypositroids.

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One combinatorial type of generic polypositroids is the cyclohedron

Minkowsi sum

$$\sum_{\substack{\text{cyclic intervals} \\ [i,j] \subseteq [n]}} \mathrm{Conv}(e_r \mid r \in [i,j])$$

a coordinate simplex

But all these different comb. types have the same $f$-vector

Theorem [LP] Any generic polypositroid in $\mathbb{R}^n$ has the same $f$-vector as the cyclohedron.

In particular, it has $\binom{2n-2}{n-1}$ vertices.

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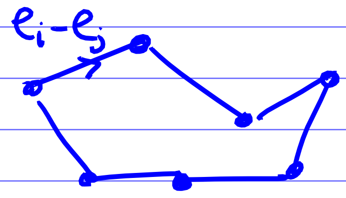
Remark. A generic gen. permutohedron has $n!$ vertices.

How do reduced plabic graphs fit into this "polyhedral framework"?

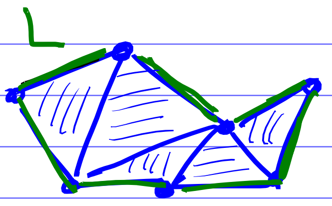
There several related polyhedral descriptions of plabic graphs (& their generalizations): [Galashin '16], [Farber-P. '16], [P. '18] [Galashin-P.-Williams '19], [Lam-P. '20].

<u>Definition.</u> [LP] A <u>loop</u> $L$ is a closed piecewise-linear curve in $\mathbb{R}^n$

$L =$ 

s.t. every vertex $\in \mathbb{Z}^n$ and every edge is a translation of $e_i - e_j$

A <u>membrane</u> $M$ with boundary loop $L$ is a piecewise linear 2-dim surface in $\mathbb{R}^n$ homeomorphic to a disk which is triangulated into triangles with vertices $\in \mathbb{Z}^n$ and edges of the form $e_i - e_j$

$M =$ 

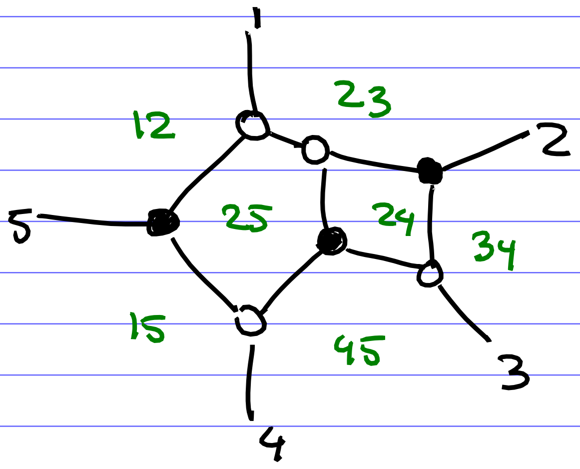

a membrane

A membrane is <u>minimal</u> if it has the minimal possible surface area among all membranes with the same boundary loop.

Basically, we have

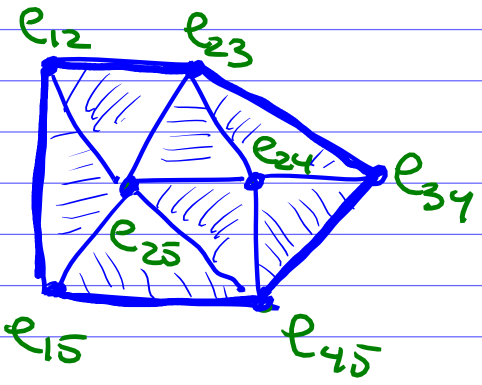
<u>Theorem</u> [P '18], [Lam-P. '20]
Reduced 3-valent plabic graphs with a given strand permutation $\pi$ correspond to minimal membranes with given boundary loop $L = L(\pi)$.

## Exemple




a plabic graph

dualize

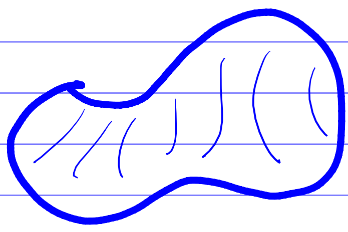



the corresponding membrane

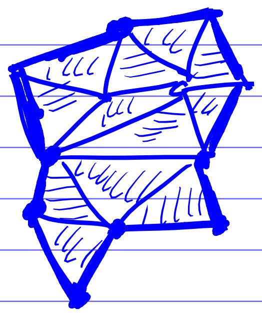
## Discrete Plateau's Problem

For a given loop $L$, describe all minimal membranes $M$ with boundary loop $L$. Find the surface area of $M$.

**Remark** This is a discrete version of Plateau's problem from geometric measure theory about existence of minimal surfaces (aka soap bubbles)



vs.



In [LP] we solve this (discrete) problem, give a combinatorial formula for the surface area, etc.

יום הולדת שמח,

רון ויובל!