On the X-rays of permutations

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Abstract

The X-ray of a permutation is defined as the sequence of antidiagonal sums in the associated permutation matrix. X-rays of permutation are interesting in the context of Discrete Tomography since many types of integral matrices can be written as linear combinations of permutation matrices. This paper is an invitation to the study of X-rays of permutations from a combinatorial point of view. We present connections between these objects and nondecreasing differences of permutations, zero-sum arrays, decomposable permutations, score sequences of tournaments, queens' problems and rooks' problems.

Keywords: Permutations, X-rays, score sequences of tournaments, zero-sum arrays.

1 Introduction

Let $\mathcal{S}_n$ be the set of all permutations of $[n] = \{1, 2, \dots, n\}$ and let P_π be the permutation matrix corresponding to $\pi \in \mathcal{S}_n$. For $k = 2, \dots, 2n$, the $(k-1)$-th *antidiagonal sum* of P_π is $x_{k-1}(\pi) = \sum_{i+j=k} [P_\pi]_{i,j}$. The sequence of nonnegative integers $x(\pi) = x_1(\pi)x_2(\pi)\dots x_{2n-1}(\pi)$ is called the (*antidiagonal*) *X-ray* of π. The *diagonal X-ray* of π, denoted by $x_d(\pi)$, is similarly defined. Note that $x(\pi) = x(\pi^{-1})$, for every $\pi \in \mathcal{S}_n$. The sequence $x(\pi)$ may be also seen as a word over the alphabet $[n]$. As an example, the following table contains the X-rays of all permutations in $\mathcal{S}_3$:

π	$x(\pi)$	π	$x(\pi)$	π	$x(\pi)$	π	$x(\pi)$	π	$x(\pi)$
123	10101	$231, 312$	01110	132	10020	213	02001	321	00300

Although X-rays of permutations are interesting object on their own, among the reasons why they are of general interest in Discrete Tomography [6] is that many types of integral matrices can be written as linear combinations of permutation matrices (for example, binary matrices with equal row-sums and column-sums, like the adjacency matrices of Cayley graphs). Deciding whether for a given word $w = w_1 \dots w_{2n-1}$ there exists $\pi \in \mathcal{S}_n$ such that $w = x(\pi)$ is an NP-complete problem [3] (see also [5]). The complexity is polynomial if the permutation matrix is promised to be wrapped around a cylinder [4]. It is necessary to keep into account that permutations are not generally specified by their X-rays: just consider the permutation $\pi = 73142865$ and $\sigma = 72413865$; we have $x(\pi) = x(\sigma) = 000110200002100$, $x_d(\pi) = x_d(\sigma) = 00021111100010$ and $\pi \neq \sigma^{-1}$. This hints that an issue concerning X-rays is to quantify how

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much information about π is contained in $x(\pi)$. In this paper we present some connections between X-rays of permutations and a variety of combinatorial objects. From a practical perspective, this may be useful in isolating and approaching special cases of the above problem.

The remainder of the paper is organized as follows. In Section 2 we consider the problem of counting X-rays. We prove a bijection between X-rays and nondecreasing differences of permutations. We define the *degeneracy* of an X-ray $x(\pi)$ as the number of permutations σ such that $x(\pi) = x(\sigma)$, and we characterize the X-rays with the maximum degeneracy. We prove a bijection between X-ray of length $4k + 1$ having maximum degeneracy and zero-sum arrays. In Section 3 we consider the notion of simple permutations. This notion seems to provide a good framework to study the degeneracy of X-rays, but the relation between simple permutations and X-rays with small degeneracy remains unclear. Section 4 is devoted to binary X-rays, that is X-rays whose entries are only zeros and ones. We characterize the X-rays of circulant permutation matrices of odd order. Moreover, we present a relation between binary X-rays, the n-queens problem (see, *e.g.*, [9]), the score sequences of tournaments on n vertices (see [10, Sequence A000571]), and extremal Skolem sequences, see [7, Conjecture 2.2].

A number of conjectures and open problems will be explicitly formulated or will simply stand out from the context. We use the standard notation for integers sequences from the OEIS [10].

2 Counting X-rays

We begin by addressing the following natural question: what is the number of different X-rays of permutations in $\mathcal{S}_n$? Although we are unable to find a generating function for the sequence, we show a bijection between X-rays and nondecreasing differences of permutations. The *difference* of permutations $\pi, \sigma \in \mathcal{S}_n$ is the integers sequence $\pi - \sigma = (w_1, w_2, \ldots, w_n)$, where $w_1 = \pi_1 - \sigma_1, w_2 = \pi_2 - \sigma_2, \ldots, w_n = \pi_n - \sigma_n$. For example, if $\pi = 1234$ and $\sigma = 2413$, we have $e - 2413 = (-1, -2, 2, 1)$. Let x_n be the numbers of different X-rays of permutations in $\mathcal{S}_n$. Let d_n be the number of nondecreasing differences of permutations in $\mathcal{S}_n$. The number d_n equals the number of different differences $e - \sigma$ with entries rearranged in the nondecreasing order. In other words, d_n equals the number of different multisets of the form $M(\sigma) = \{1 - \sigma_1, 2 - \sigma_2, \ldots, n - \sigma_n\}$, with entries rearranged in the nondecreasing order. The entries of $x(\pi)$ are then the entries of the vector $e_{1-\sigma_1} + e_{2-\sigma_2} + \ldots + e_{n-\sigma_n}$, where e_i is the i-th coordinate vector of length $2n - 1$. For example, for $\pi = 3124$ we have $x(3124) = 0101200$ and $e_{1-3} + e_{2-1} + e_{3-2} + e_{4-4} = (0,1,0,0,0,0,0) + (0,0,0,0,1,0,0) + (0,0,0,0,1,0,0) + (0,0,0,1,0,0,0) = (0,1,0,1,2,0,0)$. On the basis of this reasoning we can state the following result.

Proposition 2.1 *The number x_n of different X-rays of permutations in $\mathcal{S}_n$ is equal to the number d_n (see [10,* Sequence A019589]) *of nondecreasing differences of permutations in $\mathcal{S}_n$.*

Let us define and denote the *degeneracy* of an X-ray $x(\pi)$ by

$$\delta(x(\pi)) = |\{\sigma : x(\sigma) = x(\pi)\}|.$$

If $x(\pi)$ is such that $\delta(x(\pi)) \geq \delta(x(\sigma))$ for all $\sigma \in S_n$, we write $x^n_{\max} = x(\pi)$ and we say that $x(\pi)$ has *maximum degeneracy.* The following table contains x_n, $x^n_{\max}$ and $\delta(x^n_{\max})$ for $n = 1, \ldots, 8$.

n	x_n	$x^n_{\max}$	$\delta(x^n_{\max})$	n	x_n	$x^n_{\max}$	$\delta(x^n_{\max})$
1	1	1	1	5	59	001111100	6
2	2	020, 101	1	6	246	00011211000	12
3	5	01110	2	7	1105	0001111111000	28
4	16	0012100	3	8	5270	000011121110000	76

It is not difficult to characterize the X-rays with maximum degeneracy. One can verify by induction that for n even,

$$x^n_{\max} = 00\ldots011\ldots121\ldots110\ldots00,$$

with $n/2$ left-zeros and right-zeros, and $n/2-1$ ones; for n odd,

$$x^n_{\max} = 00..011\ldots110..00,$$

with $(n-1)/2$ left-zeros and right-zeros, and n ones. Notice that if $x(\pi) = x^n_{\max}$ (for n odd) then P_π can be seen as an hexagonal lattice with all sides of length $(n+1)/2$. In each cell of the lattice there is 0 or 1, and 1 is in exactly n cells; the column-sums are 1 and the diagonal and anti-diagonal sums are 0. This observation describes a bijection between permutations of odd order whose X-ray is $x^n_{\max}$ and zero-sum arrays. An $(m, 2n+1)$-*zero-sum* array is an $m \times (2n+1)$ matrix whose m rows are permutations of the $2n+1$ integers

$-n, -n+1, \ldots, n$ and in which the sum of each column is zero [2]. The matrix

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

is an example of $(3,3)$-zero-sum array. Thus we have the next result.

Proposition 2.2 *The number $\delta(x^n_{\max})$ for n odd is equal to the number of $(3, 2n+1)$-zero-sum arrays (see [10,* Sequence A002047]*).*

Before concluding the section, it may be interesting to notice that if we sum entry-wise the X-rays of all permutations in $\mathcal{S}_n$ we obtain the following sequence of $2n-1$ terms:

$$(n-1)!, 2(n-1)!, \ldots, (n-1)(n-1)!, n!, (n-1)(n-1)!, \ldots, 2(n-1)!, (n-1)!.$$

The meaning of the terms of this sequence is clear.

3 Simple permutations and X-rays

In the previous section we have considered the X-rays with maximum degeneracy. What can we say about X-rays with degeneracy 1? If $\delta(x(\pi)) = 1$ then π is an involution (in such a case $P_\pi = P_\pi^{-1}$) but the converse if not necessarily true. In fact consider the involution $\pi = 1267534$. One can verify that $x(\pi) = x(\sigma) = x(\rho) = 1010000212000$, for $\rho = 1275634$ and $\sigma = 1267453$. In a first approach to the problem, it seems useful to study what kind of operations can be done "inside" a permutation matrix P_π in order to obtain another permutation, say P_σ, such that $x(\pi) = x(\sigma)$ and $P_\pi \neq P_\sigma^{-1}$. A intuitively good framework for this task is provided by the notion of *block permutation*. A *segment* and a *range* of a permutation are a set of consecutive positions and a set of consecutive values. For example, in the permutation 34512, the segment formed by the positions $2, 3, 4$ is occupied by the values $4, 5, 1$; the elements $1, 2, 3$ form a range. A *block* is a segment whose values form a range. Every permutation has singleton blocks together with the block $12 \ldots n$. A permutation is called *simple* if these are the only blocks [1]. A permutation is said to be a *block permutation* if it is not simple. Note that if π is simple then it is π^{-1}. Let $S = (\pi_1 \in \mathcal{S}_{n_1}, \ldots, \pi_k \in \mathcal{S}_{n_k})$ be an ordered set and let $\pi \in \mathcal{S}_k$. We assume that in S there exists $1 \leq i \leq k$ such that $n_i > 1$. We denote by $P(\pi, S)$ the $(n_1 + \cdots + n_k)$-dimensional permutation matrix which is partitioned in k^2 blocks, $B_{1,1}, \ldots, B_{k,k}$, such that $B_{i,j} = P_{\pi_i}$ if $\pi(i) = j$ and $B_{i,j} = 0$, otherwise. We denote by $\pi[\pi_1, \ldots, \pi_k]$ (or equivalently by $(\pi)[S]$) the permutation corresponding to $P(\pi, S)$. For example, let $S = (231, 21, 312)$ and $\pi = 231$. Then

$$P(231, S) = \begin{bmatrix} 0 & P_{231} & 0 \\ 0 & 0 & P_{21} \\ P_{312} & 0 & 0 \end{bmatrix}$$

and $231[S] = 231[231, 21, 312] = 56487312$. The matrix $P(231, S)$ can be modified leaving the X-ray of $(\pi)[S]$ invariant:

$$P(231, (312, 21, 312)) = \begin{bmatrix} 0 & P_{312} = P_{231}^T & 0 \\ 0 & 0 & P_{21} \\ P_{312} & 0 & 0 \end{bmatrix}.$$

It is clear that 56487312 is a block permutation. Let π be a simple permutation then possibly $\delta(x(\pi)) > 1$. In fact, the permutation $\pi = 531642$ is simple, but $\delta(x(\pi)) = 6$, since $x(\pi) = 00111011100 = x(526134) = x(461253)$, plus the respective inverses. The permutations 526134 and 461253 are decomposable. This means that there possibly exists a decomposable permutation σ such that $x(\sigma) = x(\pi)$, even if π is simple. There relation between simple permutations and X-rays of small degeneracy is not clear. Intuitively, a simple permutation allows less "freedom of movement" than a block permutation. It is also intuitive that we have low degeneracy when the nonzero entries of the X-ray are "distributed widely" among the $2n - 1$ coordinates. The following result is easily proved.

Proposition 3.1 *Let $\sigma = \pi[S] = \pi[\pi_1, \ldots, \pi_k]$ be a block permutation. Then $\delta(x(\pi)) > 1$ if one of the following two conditions is satisfied:*

(1) If $\pi \neq 12\ldots n$ then there is at least one $\pi_i \in S$ which is not an involution;

(2) If $\pi = 12\ldots n$ then there are at least two $\pi_i, \pi_i \in S$ which are not involution.

Proof. (1) Let $\pi \neq 12\ldots n$ be any permutation. Take π_i^{-1} for some $\pi_i \in S$. Let $\rho = \pi[\pi_1, \ldots, \pi_i^{-1}, \ldots, \pi_k]$. Since σ is a block permutation, $x(\sigma) = x(\rho)$. However, if $\pi_i \neq \pi_i^{-1}$ then $\sigma \neq \rho$ and $\sigma^{-1} \neq \sigma$. It follows that $x(\sigma)$ does not specify σ. (2) Let $\pi = 12\ldots n$. Let all elements of S be involutions except π_i. Take π_i^{-1}. Let $\rho = \pi[\pi_1, \ldots, \pi_i^{-1}, \ldots, \pi_k]$. Again, $x(\sigma) = x(\rho)$, but this time $\rho = \sigma^{-1}$. Then $x(\sigma)$ possibly specifies σ. If, for distinct i, j, there are $\pi_i, \pi_j \in S$ such that $\pi_i \neq \pi_i^{-1}$ and $\pi_j \neq \pi_j^{-1}$ then

$$x(\sigma') = x(\pi[\pi_1, \ldots, \pi_i^{-1}, \ldots, \pi_j^{-1}, \ldots, \pi_k]) = x(\sigma),$$

but $x(\sigma)$ does not specify σ, given that $\rho \neq \sigma^{-1}$. □

This is however not a sufficient condition for having $\delta(x(\pi)) > 1$. Permutations with equal X-rays are said to be in the same *degeneracy class*. The table below contain the number of permutations in $\mathcal{S}_n$ which are in each degeneracy class, and the number of different degeneracy classes with the same cardinality, for $n = 2, \ldots, 8$. These numbers provide a partition on $n!$. We denote by $C(n)$ the total number of degeneracy classes. We write $a(b)$, where a is the number of permutations in the degeneracy class and b the number of degeneracy classes of the same cardinality:

$C(2) = 1$: 1(2)
$C(3) = 2$: 1(4),2(1)
$C(4) = 3$: 1(9),2(6),3(1)
$C(5) = 5$: 1(20),2(26),3(6),4(6),6(1)
$C(6) = 10$: 1(49),2(100),3(19),4(43),5(1),6(19),7(2),8(11),9(1),2(1)
$C(7) = 20$: 1(114),2(345),3(60),4(229),5(18),6(118),7(11),8(98),10(29)
11(2),12(33),14(13),16(14),18(6),20(4),21(1),22(2),26(1),28(1).

.

We conjecture that if $\delta(x(\pi)) = 1$ then $x(\pi)$ does not have more than 2 adjacent nonzero coordinates. However the converse is not true if $\pi \in \mathcal{S}_n$ for $n \geq 8$: for $\pi = 17543628$ and $\sigma = 16547328$, we have $x(\pi) = x(\sigma) = 100000320010001$, but there are no more than 2 adjacent coordinates.

4 Binary X-rays

In general, it does not seem to be an easy task to characterize X-rays. A special case is given by X-rays associated with circulant permutation matrices, for which is available an exact characterization. An X-ray $x(\pi)$ is said to be *binary* if $x_i(\pi) \in \{0, 1\}$ for every $1 \leq i \leq 2n-1$. The set all permutations in $\mathcal{S}_n$ with binary X-ray is denoted by $\mathcal{B}_n$. Counting binary X-rays means solving a modified version of the n-queens problem (see, *e.g.*, [9]) in which two queens do not attack each other if they are in the same NorthWest-SouthEst diagonal. The permutations with binary X-rays associated to circulant matrices are characterized in a straightforward way. Let C_n be the permutation matrix associated with the permutation $c_n = 23\ldots n1$, that is the *basic circulant permutation matrix*. The matrices in the set $\mathcal{C}_n = \{C_n^0, C_n, C_n^2, \ldots, C_n^{n-1}\}$ (C_n^0 is the identity matrix) are called the *circulant permutation matrices*. The matrix C_n^k is associated to c_n^k. Observe that $x(\pi)$ can be seen as a binary number, since $x_i(\pi) \in \{0, 1\}$ for every i. Let

$$d_j(\pi) = 2^{2n-1-j} \cdot x_j(\pi), \quad j = 1, 2, \ldots, 2n-1,$$

and $d(\pi) = \sum_{i=1}^{2n-1} d_i(\pi)$, that is the decimal expansion of $x(\pi)$. The table below lists the X-rays of $\mathcal{C}_3, \mathcal{C}_5$ and $\mathcal{C}_7$, and their decimal expansions:

π	$x(\pi)$	$d(\pi)$	π	$x(\pi)$	$d(\pi)$
123	10101	21	12345	101010101	341
231	01110	14	23451	010111010	186
			34512	001111100	124

.

For $\pi = c_n^k$, one can verify that

$$\begin{aligned} d(\pi) &= \tfrac{1}{6}2^{\frac{3}{2}n+\frac{1}{2}+k} - \tfrac{1}{6}2^{\frac{1}{2}n+\frac{1}{2}+k} + \tfrac{1}{3}2^{\frac{3}{2}n+\frac{1}{2}-k} - \tfrac{1}{3}2^{\frac{1}{2}n+\frac{1}{2}-k} \\ &= a(k)\,(2^n-1)\,(2^n-1)\,2^{\frac{1}{2}n-k+\frac{1}{2}}, \end{aligned}$$

where $a(k) = (2^{2k-1}+1)/3$ (A007583).

In the attempt to count binary X-rays, we are able to establish a bijection between these objects and score sequences of tournaments. A *tournament* is a loopless digraph such that for every two distinct vertices i and j either (i, j) or (j, i) is an arc [8]. The *score sequence* of an tournament on n vertices is the vector of length n whose entries are the out-degrees of the vertices of the tournament rearranged in nondecreasing order.

Proposition 4.1 *Let b_n be the number of binary X-rays of permutations in S_n and let s_n be the number of different score sequences of tournaments on n vertices (see* [10, Sequence A000571]*). Then $b_n \leq s_n$.*

Proof. The number s_n equals the number of integers lattice points $(p_0, \ldots, p_n)$ in the polytope P_n given by the inequalities $p_0 = p_n = 0$, $2p_i - p_{i+1} - p_{i-1} \leq 1$ and $p_i \geq 0$, for $i = 1, \ldots, n-1$, see [8]. Let $x_1, \ldots, x_n$ be the coordinates related to $p_1, \ldots, p_n$ by $p_i = x_1 + \ldots + x_i - i^2$, for $i = 1, \ldots, n$. We can rewrite the inequalities defining the polytope P_n in these coordinates as follows: $x_1 + \ldots + x_i \geq i^2$, $x_{i+1} \geq x_i + 1$ and $x_1 + \ldots + x_n = n^2$. For a permutation $w \in \mathcal{S}_n$ with a binary X-ray, let $l_i = l_i(w)$ be the position of the i-th '1' in its X-ray. In other words, the sequence $(l_1, \ldots, l_n)$ is the increasing rearrangement of the sequence $(w_1, w_2 + 1, w_3 + 2, \ldots, w_n + n - 1)$. Then the numbers $l_1, \ldots, l_n$ satisfy the inequalities defining the polytope P_n (in the x-coordinates). Indeed, $l_1 + \ldots + l_n = w_1 + (w_2 + 1) + \cdots + (w_n + n - 1) = n^2$; $l_{i+1} \geq l_i + 1$; and the minimal possible value of $l_1 + \cdots + l_i$ is $(1+0)+(2+1)+\cdots+(i+(i-1)) = i^2$. This finishes the proof. In order, to prove that $b_n = s_n$ it is enough to show that, for any integer point $(x_1, \ldots, x_n)$ satisfying the above inequalities, we can find a permutation $w \in \mathcal{S}_n$ with $x_i = l_i(w)$. □

Conjecture 4.2 *All binary X-rays of permutations in $\mathcal{S}_n$ are in a bijective correspondence with integer lattice points $(x_1, \ldots, x_n)$ of the polytope given by the inequalities*

$$x_1 + \cdots + x_i \geq i^2, \qquad i = 1, \ldots, n;$$
$$x_1 + \cdots + x_n = n^2,$$
$$x_{i+1} - x_i \geq 1, \qquad i = 1, \ldots, n-1.$$

For a permutation $w \in S_n$, the corresponding sequence $(x_1, \ldots, x_n)$ is defined as the increasing rearrangement of the sequence $(w_1, w_2 + 1, w_3 + 2, \ldots, w_n + n - 1)$.

Again, it is clear that X-rays injectively map into the integer points of the above polytope. One needs to show that there will be no gaps in the image. Also, it can be shown that the above conjecture, restricted to binary X-rays, is equivalent to Conjecture 2.2 from [7] concerning extremal Skolem sequences.

We conjecture also that the number of different X-rays of permutations in $\mathcal{S}_n$ whose possible entries are 0 and 2 is equal to the number of score sequences in tournament with n players, when 3 points are awarded in each game (see [10, Sequence A047729]).

5 Palindromic X-rays

What can we say about X-rays with special symmetries? The *reverse* of $x(\pi)$, denoted by $\overline{x}(\pi)$, is the mirror image of $x(\pi)$. If $x(\pi) = \overline{x}(\pi)$ then π is said to be *palindromic*. The *reverse* of π, denoted by $\overline{\pi}$, is mirror image of π. For example, if $\pi = 25143$ then $\overline{\pi} = 34152$. The permutation matrix $P_{\overline{\pi}}$ is obtained by writing the rows of P_π in reverse order. In general $\overline{x}(\pi) \neq x(\overline{\pi})$. In fact, for $\pi = 25143$, we have $x(\pi) = 0011001200$, $\overline{x}(\pi) = 0021001100$ and $x(\overline{\pi}) = 0020011010$. We denote by $|M$ and $\underline{M}$ the matrices obtained by writing the columns and the rows of a matrix M in reverse order, respectively.

Proposition 5.1 *Let l_n be the number of permutations in $\mathcal{S}_n$ with palindromic X-rays and let i_n be the number of involutions in $\mathcal{S}_n$ (see* [10, Sequence A000085]*). Then, in general, $l_n > i_n$.*

Proof. Recall that a permutation π is an *involution* if $\pi = \pi^{-1}$. Since $P_\pi = P_\pi^T$, it is clear that the diagonal X-ray of an involution π is palindromic. The X-ray of σ such that $P_\sigma = |P_\pi$ is then also palindromic. This shows that $l_n \geq i_n$. Now, consider a permutation matrix of the form

$$P_\sigma = \begin{bmatrix} P_\rho & 0 \\ 0 & P_\rho^T \end{bmatrix},$$

for some permutation ρ which is not an involution. Then $P_\rho \neq P_\rho^T$, $P_\sigma \neq P_\sigma^T$ and σ is not an involution, but the diagonal X-ray of σ is palindromic. The X-ray of π such that $P_\pi = |P_\sigma$ is then also palindromic. This proves the proposition. □

The next contains the values of l_n for small n:

n	l_n	n	l_n	n	l_n	n	l_n
2	2	4	12	6	128	8	2110
3	4	5	32	7	436	9	8814

.

Proposition 5.2 *Let $l_{n,A=D}$ be the number of permutations in $\mathcal{S}_n$ with:*

(1) equal diagonal and antidiagonal X-rays;

(2) palindromic X-rays.

Let r_n be the number of permutations in $\mathcal{S}_n$ invariant under the operation of first reversing and then taking the inverse (see [10, Sequnce A097296]*). Then, in general, $l_{n,A=D} > r_n$.*

Proof. We first construct the permutations which are invariant under the operation of first reversing and then taking the inverse. Let $\pi \in \mathcal{S}_n$ where $n = 2k$. We look at P_π as partitioned in 4 blocks:

$$P_\pi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

If

$$P_\pi = (\underline{P_\pi})^T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} \underline{B} & \underline{A} \\ \underline{D} & \underline{C} \end{bmatrix}^T = \begin{bmatrix} (\underline{B})^T & (\underline{D})^T \\ (\underline{A})^T & (\underline{C})^T \end{bmatrix}$$

then $A = (\underline{B})^T, B = (\underline{D})^T, C = (\underline{A})^T$ and $D = (\underline{C})^T$. This implies the X-ray of P_π being palindromic and, moreover, the diagonal and antidiagonal X-rays being equal. Note that we can construct P_π only if $n \equiv 0 (\bmod 4)$, and in this case $r_n \neq 0$. However, fixed $n \equiv 0 (\bmod 4)$, we have $r_n = r_{n+1}$, since the permutation matrix

$$P_\sigma = \begin{bmatrix} A & & \mathbf{0} \\ & 1 & \\ \mathbf{0} & & D \end{bmatrix} + \begin{bmatrix} \mathbf{0} & & B \\ & 1 & \\ C & & \mathbf{0} \end{bmatrix}$$

can be always constructed from P_π. (Permutation matrices like P_π and P_σ provide the solutions of the "rotationally invariant" n-rooks problem. This points out that A097296 and A037224 are indeed the same sequence.) Now, the proposition is easily proved by observing that, for $\rho = 369274185$, P_ρ is not of the form of P_σ. A direct calculation shows that $r_9 = 12$ and $l_{9,A=D} = 20$.$\square$

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