What power of two divides a weighted Catalan number?

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January 10, 2006

Key Words: difference operator, divisibility, group actions, Morse links, orbits, power of two, shift operator, weighed Catalan numbers.

AMS subject classification (2000): Primary 05A10; Secondary 11A55, 11B75.

Abstract

Given a sequence of integers $b = (b_0, b_1, b_2, \ldots)$ one gives a Dyck path P of length $2n$ the weight

$$\mathrm{wt}(P) = b_{h_1} b_{h_2} \cdots b_{h_n},$$

where h_i is the height of the ith ascent of P. The corresponding weighted Catalan number is

$$C_n^b = \sum_P \mathrm{wt}(P),$$

where the sum is over all Dyck paths of length $2n$. So, in particular, the ordinary Catalan numbers C_n correspond to $b_i = 1$ for all $i \geq 0$. Let $\xi(n)$ stand for the base two exponent of n, i.e., the largest power of 2 dividing n. We give a condition on b which implies that $\xi(C_n^b) = \xi(C_n)$. In the special case $b_i = (2i+1)^2$, this settles a conjecture of Postnikov about the number of plane Morse links. Our proof generalizes the recent combinatorial proof of Deutsch and Sagan of the classical formula for $\xi(C_n)$.

1 Introduction

The *Catalan numbers*

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

have many interesting arithmetic properties. For example, the following result which essentially dates back to Kummer (see Dickson's book [3] for details) describes their divisibility by powers of 2. Let $s(n)$ be the sum of digits in the binary expansion of n. Also, let $\xi(n)$ denote the base two *exponent* of n, i.e., the largest power of two dividing n.

Theorem 1.1. *We have*

$$\xi(C_n) = s(n+1) - 1. \qquad \square$$

A combinatorial proof of this result was recently given by Deutsch and Sagan [2] using group actions. In this paper, we extend this result and their proof to various weighted Catalan numbers. Note that, unlike the Catalan numbers, weighted Catalan numbers need not have simple multiplicative formulas. So determining their divisibility properties is more subtle than for the usual Catalan numbers.

Let $b = (b_0, b_1, b_2, \dots)$ be a fixed infinite sequence of integers. Define the *weighted Catalan numbers*, C_n^b, as the coefficients of the expansion of the continued fraction:

$$\cfrac{1}{1 - \cfrac{b_0\, x}{1 - \cfrac{b_1\, x}{1 - \cfrac{b_2\, x}{1 - \cfrac{b_3\, x}{1 - \cdots}}}}} = \sum_{n \geq 0} C_n^b\, x^n. \tag{1}$$

If $b = (1, 1, 1, \dots)$, then C_n^b is the usual Catalan number C_n.

Combinatorially, the C_n^b count Dyck paths with certain weights. Recall that a *Dyck path* P of length $2n$ is a sequence of points in the upper half-plane of the integer lattice

$$(x_0, y_0) = (0, 0),\ (x_1, y_1),\ \dots,\ (x_{2n}, y_{2n}) = (2n, 0),$$

such that each step $s_i = [x_i - x_{i-1}, y_i - y_{i-1}]$ has the form $[1, 1]$ or $[1, -1]$. Let us say that step s_i has *height* y_{i-1}. Define the *weight* of a Dyck path P to be the product

$$\mathrm{wt}(P) = b_{h_1} b_{h_2} \cdots b_{h_n},$$

where $h_1, \dots, h_n$ are the heights of its steps of the form $[1, 1]$. Then the following proposition is well known and easy to prove, i.e., see the book of Goulden and Jackson [4, Ch. 5].

Proposition 1.2. *We have*

$$C_n^b = \sum_P \mathrm{wt}(P),$$

where the sum is over all Dyck paths of length $2n$. □

For example, we have

$$C_3^b = b_0\, b_0\, b_0 + b_0\, b_0\, b_1 + b_0\, b_1\, b_0 + b_0\, b_1\, b_1 + b_0\, b_1\, b_2,$$

where the five terms correspond to the five Dyck paths of length 6. As another example, if $b = (1, q, q^2, q^3, \dots)$, then the weighted Catalan number C_n^b is equal to the *q-Catalan number*

$$C_n(q) = \sum_P q^{\mathrm{area}(P)},$$

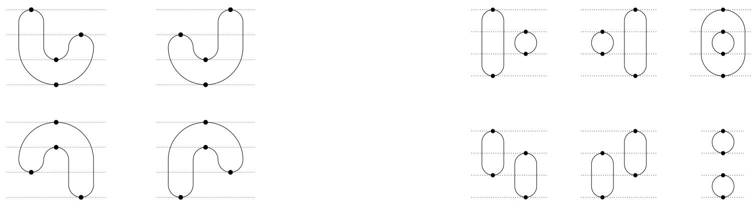
where the sum is over Dyck paths P of length $2n$ and $\mathrm{area}(P)$ denotes the area between P and the lowest possible path. The continued fraction (1) with $b_i = q^i$ is know as the *Ramanujan continued fraction.*

Our main result, Theorem 2.1 below, gives a necessary condition on the sequence b so that

$$\xi(C_n^b) = \xi(C_n) = s(n+1) - 1.$$

As a special case, we obtain a conjecture of Postnikov [5] about plane Morse links. A *plane Morse curve* is a simple curve $f : S^1 \to \mathbb{R}^2$ (i.e., a smooth injective map) such that, for the height function $h : (x, y) \to y$, the map $h \circ f$ has a finite number of isolated nondegenerate critical points with distinct values. All curves are oriented clockwise. See, for example, Figure 1. As one goes around a Morse curve, the sequence formed by the critical values is alternating and returns to where it started. So the number of critical values must be an even integer $2n$, and n is called the *order* of the curve. The *combinatorial type* of a Morse curve is its connected component in the space of all Morse curves. So Figure 1(a) depicts the four plane Morse curves of order 2 up to combinatorial type.

A *plane Morse link* is a disjoint union of plane Morse curves. All our definitions for Morse curves carry over in the natural way to links. In particular, the order of a link is the sum of the orders of its components. Figure 1(b) shows the six disconnected Morse links of order 2, again up to combinatorial type. Let L_n be the number of combinatorial types of plane Morse links of order n. Then we have just seen that $L_2 = 4 + 6 = 10$. The connection with weighted Catalan numbers is made by the following theorem.



(a) Connected links (curves) (b) Disconnected links

Figure 1: All plane Morse links of order 2

Theorem 1.3 ([5]). *The numbers L_n satisfy*

$$\sum_{n\geq 0} L_n\, x^n = \cfrac{1}{1-\cfrac{1^2\,x}{1-\cfrac{3^2\,x}{1-\cfrac{5^2\,x}{1-\cfrac{7^2\,x}{\cdots}}}}}$$

and so

$$L_n = C_n^{(1^2,\ 3^2,\ 5^2,\ 7^2,\ \ldots)}. \qquad \square$$

An easy corollary of our main theorem will be a proof of Conjecture 3.1 from [5]. It can also be found listed as Problem 6.C5(c) in the Catalan Addendum to the second volume of Stanley's *Enumerative Combinatorics* [7].

Conjecture 1.4 ([5, 7]). *We have*

$$\xi(L_n) = \xi(C_n) = s(n+1) - 1. \qquad \square$$

2 The main theorem

Let $\mathbb{Z}_{\geq 0}$ denote the nonnegative integers. The *difference operator*, Δ, acts on functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ by

$$(\Delta f)(x) = f(x+1) - f(x).$$

Note that we can regard our sequence b as such a function where $b(x) = b_x$. We can now state our main result, using $c \mid d$ as usual to mean that c divides evenly into d.

Theorem 2.1. *Assume that the sequence b satisfies*

1. *$b(0)$ is odd, and*

2. *$2^{n+1} \mid (\Delta^n b)(x)$ for all $n \geq 1$ and $x \in \mathbb{Z}_{\geq 0}$.*

Then

$$\xi(C_n^b) = \xi(C_n) = s(n+1) - 1.$$

To prove this, we will also have to consider the *shift operator*, S, acting on functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ by

$$(Sf)(x) = f(x+1).$$

It is well known and easy to verify that we have the product rule

$$\Delta(f \cdot g) = \Delta(f) \cdot g + S(f) \cdot \Delta(g).$$

which generalizes to

$$\Delta^n(f \cdot g) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}(S^k(f)) \cdot \Delta^k(g). \tag{2}$$

Let $\mathcal{F}$ be the set of functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ such that

(a) $f(x)$ is odd for all $x \in \mathbb{Z}_{\geq 0}$, and

(b) $2^{n+1} \mid (\Delta^n f)(x)$ for all $n \geq 1$ and $x \in \mathbb{Z}_{\geq 0}$.

Note that because of (b), we can replace (a) by the seemingly weaker condition that $f(0)$ is odd. We will need the following lemma.

Lemma 2.2. *The set $\mathcal{F}$ is closed under the three operations*

$$f \mapsto Sf, \quad (f,g) \mapsto f \cdot g, \quad \textit{and} \quad (f,g) \mapsto \langle f, g \rangle := \frac{f(x+1)\,g(x) + f(x)\,g(x+1)}{2}.$$

Proof. Closure under S is obvious. Now assume that $f(x), g(x) \in \mathcal{F}$. Then the value $(f \cdot g)(x)$ is clearly odd. And (2) shows that the divisibility criterion is satisfied. Thus $f \cdot g \in \mathcal{F}$.

We can write the second operation as

$$\langle f, g \rangle = f \cdot g + \frac{\Delta(f) \cdot g + f \cdot \Delta(g)}{2}.$$

Since $\Delta(f)$ and $\Delta(g)$ are divisible by 4 and $(f \cdot g)(x)$ is odd, we deduce that $\langle f, g \rangle (x)$ is odd. By (2), $\Delta^n(\Delta(f) \cdot g)$ and $\Delta^n(f \cdot \Delta(g))$ are divisible by 2^{n+2}. Thus $\langle f, g \rangle \in \mathcal{F}$. ∎

Deutsch and Sagan used the interpretation of C_n in terms of binary trees to prove Theorem 1.1. So we will need to review this method and translate our weight function into this setting. A *binary tree*, T, is a rooted tree where every vertex has a right child, a left child, both children, or no children. We also consider the empty tree to be a binary tree. Let $\mathcal{T}_n$ be the set of binary trees with n vertices. Then one of the standard interpretations of the Catalan numbers is that

$$C_n = \#\mathcal{T}_n. \tag{3}$$

Let G_n be the group of symmetries of the binary tree which is complete to depth n (having all of its leaves at distance n from the root). The group G_n is generated by reflections which exchange the left and right subtrees associated with a vertex. Then G_n acts on $\mathcal{T}_n$ with two trees being in the same orbit if they are isomorphic as rooted trees if we forget about the information concerning left and right children. Deutsch and Sagan show in [2, Section 2] that $\#G_n$ is a power of 2, so the cardinality of any G_n-orbit is as well. They also show that the minimal size of a G_n-orbit is 2^s where $s = s(n+1) - 1$. Moreover, orbits of the minimal size can be identified with binary total partitions on the set $\{1, 2, \ldots, s\}$, whose number is $(2s-1)!! = 1 \cdot 3 \cdot 5 \cdots (2s-1)$ as shown by Schröder [6], see also [8, Example 5.2.6]. For ease of reference, we summarize these facts in the following lemma.

Lemma 2.3 ([2]). *Let $\mathcal{O}$ be an orbit of G_n acting on $\mathcal{T}_n$ and let $s = s(n+1) - 1$. Then*

1. *$\#\mathcal{O} = 2^t$ for some $t \geq s$, and*
2. *$\#\mathcal{O} = 2^s$ for exactly $(2s-1)!!$ orbits.* □

It is now easy to prove Theorem 1.1 using this lemma and equation (3). To generalize the proof, consider any fixed function $b \in \mathcal{F}$ and define the corresponding *weight of a binary tree* T to be the function

$$w_b(T) = w_b(T; x) = \prod_{v \in T} b(x + l_v),$$

where the product is over all vertices v of T, and l_v is the number of left edges on the unique path from the root of T to v. If binary tree T corresponds to Dyck path P under the usual depth-first search bijection, then it is easy to see that

$$\mathrm{wt}(P) = \prod_{v \in T} b(l_v) = w_b(T; 0). \tag{4}$$

We need one last lemma for the proof of Theorem 2.1. If $\mathcal{O}$ is an orbit of G_n acting on $\mathcal{T}_n$ then we define it's *weight* to be

$$w_b(\mathcal{O}) = w_b(\mathcal{O}; x) = \sum_{T \in \mathcal{O}} w_b(T). \tag{5}$$

Lemma 2.4. *For any $b \in \mathcal{F}$ and any orbit $\mathcal{O}$ we have*

$$w_b(\mathcal{O}; x) = \#\mathcal{O} \cdot r_b(\mathcal{O}; x),$$

where $r_b(\mathcal{O}; x) \in \mathcal{F}$.

Proof. Write

$$r_b(\mathcal{O}; x) = \frac{w_b(\mathcal{O}; x)}{\#\mathcal{O}}. \tag{6}$$

We induct on n, the number of vertices in a tree of $\mathcal{O}$. If $n = 0$ then $r_b(\mathcal{O}; x) = 1$ for all x which is clearly in $\mathcal{F}$.

Now suppose $n \geq 1$ so that $\mathcal{O}$ contains a nonempty tree T. Let T_1 be the subtree of T consisting of the left child of the root and all its descendents. (So T_1 may be empty.) Similarly, define T_2 for the right child. Suppose T_1 and T_2 are in orbits $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively. If $\mathcal{O}_1 = \mathcal{O}_2$ then $\#\mathcal{O} = \#\mathcal{O}_1 \cdot \#\mathcal{O}_2$ and

$$w_b(T) = b(x) \cdot w_{S(b)}(T_1) \cdot w_b(T_2).$$

If $\mathcal{O}_1 \neq \mathcal{O}_2$, then $\#\mathcal{O} = 2 \cdot \#\mathcal{O}_1 \cdot \#\mathcal{O}_2$ and

$$w_b(T) = b(x)\ [w_{S(b)}(T_1) \cdot w_b(T_2) + w_b(T_1) \cdot w_{S(b)}(T_2)].$$

In the both cases, it follows from equations (5) and (6) that

$$r_b(\mathcal{O}; x) = b(x) \cdot \frac{r_b(\mathcal{O}_1; x+1) \cdot r_b(\mathcal{O}_2, x) + r_b(\mathcal{O}_1; x) \cdot r_b(\mathcal{O}_2; x+1)}{2}.$$

So by Lemma 2.2 and induction we have $r_b(\mathcal{O}; x) \in \mathcal{F}$ as desired. $\square$

Proof of Theorem 2.1. Combining Proposition 1.2, the previous lemma, and equations (4) and (5) gives

$$C_n^b = \sum_P \mathrm{wt}(P) = \sum_{T \in \mathcal{T}_n} w_b(T; 0) = \sum_{\mathcal{O}} \#\mathcal{O} \cdot r_b(\mathcal{O}; 0) \tag{7}$$

where the integers $r_b(\mathcal{O}; 0)$ are all odd since $r_b(\mathcal{O}) \in \mathcal{F}$. So $\xi(\#\mathcal{O} \cdot r_b(\mathcal{O}; 0)) = \xi(\#\mathcal{O})$ for all orbits $\mathcal{O}$. It now follows from Lemma 2.3 that $\xi = s$ for an odd number (namely $(2s-1)!!$) of summands in the last summation in (7) and that $\xi > s$ for the rest. We conclude that $\xi(C_n^b) = s = \xi(C_n)$. $\square$

As corollaries, we can prove Conjecture 1.4 and give information about divisibility of the q-Catalan numbers.

Corollary 2.5. *1. The number of combinatorial types of plane Morse links of order n satisfies*

$$\xi(L_n) = \xi(C_n) = s(n+1) - 1.$$

2. If $q \equiv 1 \pmod 4$ then the q-Catalan numbers satisfy

$$\xi(C_n(q)) = \xi(C_n) = s(n+1) - 1.$$

Proof. For the first assertion, it suffices to show that the function $b(x) = (2x+1)^2$ is in $\mathcal{F}$. Clearly $b(x)$ is odd for all $x \in \mathbb{Z}_{\geq 0}$. Furthermore $\Delta b = 8(x+1)$, $\Delta^2 b = 8$, and $\Delta^n b = 0$ for $n \geq 3$. So the divisibility condition also holds.

For the second statement, we need the function $b(x) = q^x$ to be in $\mathcal{F}$. Since q is odd, so is $b(x)$. Also $\Delta^n b = (q-1)^n q^x$ for $n \geq 1$. So the hypothesis on q implies that $\Delta^n b$ is divisible by $4^n = 2^{2n}$ which is more than needed. $\square$

3 An open problem

Consider the *Catalan sequence*

$$C = (C_0, C_1, C_2, \ldots).$$

Theorem 1.1 implies immediately that C_n is odd if and only if $n = 2^k - 1$ for some $k \geq 0$. It follows that the kth block of zeros in the sequence C taken modulo 2 has length $2^k - 1$ (where we start numbering with the first block). Alter and Kubota [1] have generalized this result to arbitrary primes and prime powers. One of their main theorems is as follows.

Theorem 3.1 ([1]). *Let $p \geq 3$ be a prime and let $q = (p+1)/2$. The length of the kth block of zeros in C modulo p is*

$$\frac{p^{\xi_q(k)+\delta_{3,p}+1} - 3}{2}$$

where $\xi_q(k)$ is the largest power of q dividing k and $\delta_{3,p}$ is the Kronecker delta. $\square$

Deutsch and Sagan [2] have improved on this theorem when $p = 3$ by giving a complete characterization of the residues in C modulo 3. However, the demonstrations of all these results rely heavily on the expression for C_n as a product. It would be interesting to find analogous theorems for C_n^b, but new proof techniques would have to be found.

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