ENUMERATION OF SPANNING TREES OF GRAPHS

Igor Pak, Alexander Postnikov

Harvard University
Massachusetts Institute of Technology

November 1, 1994

Abstract. In this paper, we define and study polynomials that enumerate spanning trees according to degrees of all vertices. These polynomials have a certain reciprocity. A formula that reduces the polynomials for multipartite graphs to the polynomials for component graphs is also found. There is a generalization of Prüfer's coding that corresponds to the latter formula. All results are extended to the case of oriented nets.

Contents

Introduction

A *spanning tree* T in a graph G is a connected subgraph in G that contains all vertices of G and has no cycles. Enumeration of spanning trees in a graph G is classical combinatorial combinatorics (e.g. see [HP]).

The first approach to this problem is the computation of the number $t(G)$ of spanning trees. For example, the famous A. Cayley's formula [Cay] $t(K_n) = n^{n-2}$ gives the number of spanning trees in the complete graph K_n, in other words, the number of all trees with n labelled vertices. Another general result is the matrix-tree theorem (see section 9) that expresses the number $t(G)$ as the determinant of a matrix.

Key words and phrases. Spanning tree, Reciprocity, Oriented net, Multipartite graph, Prüfer's coding, Matrix-tree theorem.

However the matrix-tree theorem does not give the final answer to all problems concerning enumeration of trees because of two reasons. First, in many cases the computation of determinant is a rather complicated task, thought the number $t(G)$ could be expressed by a simple formula; such as, for example, Cayley's formula, or the formula $t(K_{rs}) = r^{s-1}\, s^{r-1}$ for the number of spanning trees of the complete bipartite graph K_{rs}. Second, matrix-tree theorem does not present any algorithm for enumeration of trees (in other words, for composition of the complete list of all spanning trees).

The first such algorithm was found by H. Prüfer [Pr] who presented a simple and fast method for coding of all labelled trees. More precisely, he constructed a bijection between the set of all trees with n labelled vertices (spanning trees of K_n) and the set of sequences $(a_1, a_2, \ldots, a_{n-2})$ of integers from 1 to n. Later, similar algorithms for enumeration of other classes of trees were found. Among them A. Rényi's coding for spanning trees of the complete bipartite graph [Ré1].

In the present paper, we develop a new approach to the problem of enumeration of trees.

We will see that it is more convenient to consider not just the number $t(G)$ of spanning trees but the polynomial f_G that enumerate trees according to degrees of all vertices.

A. Rényi [Ré2] noticed that one can use these polynomials for simple inductive proofs (see Section 6). (Maybe A. Cayley already know this.) The polynomials f_G also possess the remarkable property of reciprocity (Theorems 2.1 and 4.2).

Another and perhaps the most significant reason is the possibility to express the polynomials for certain complex graphs using the polynomials for more simple graphs. Let $\Gamma = \Gamma(G; G_1, \ldots, G_k)$ be the graph that is obtained by substitution of graphs $G_1, \ldots, G_k$ instead of the vertices of a graph G. (See Section 5 for a more strict definition). We call such graphs *multipartite graphs*. We will show that the polynomial f_Γ can be obtained from the polynomials f_G and f_{G_i} for $i = 1, 2, \ldots, k$ (see Theorem 5.1).

In Section 11 we will show that there is an analogue of the above mentioned expression in terms of Prüfer's coding. The polynomial f_G is defined as a sum of a monomials over all spanning trees of the extended graph $\widetilde{G}$ which is obtained from G by adding a vertex connected with all vertices of G. The analogue of Prüfer's coding for a multipartite graph Γ enables us to reduce the enumeration of spanning trees of the graph $\widetilde{\Gamma}$ to the enumeration of trees of graphs $\widetilde{G}$ and $\widetilde{G_i}$, $i = 1, 2, \ldots, k$.

Note also that we generalize the results concerning the reciprocity and multipartite graphs to oriented nets, i.e oriented graphs with conductivities that are assigned to the edges (Section 4).

As illustrations of these results, the formula for increasing trees (Section 7) and Hurwitz's generalization of binomial theorem (Section 8) are proved.

In the paper we use standard notations from graph theory (e.g. see [W]).

1. Enumerators for spanning trees

In this section we construct the polynomial f_G for a graph G.

Let $G = (V, E)$ be a *graph* on the set of *vertices* V with the set of *edges* $E \subset \binom{V}{2} := \{e \subset V : |e| = 2\}$. We associate the variable x_v with each vertex $v \in V$.

For each tree T on the set of vertices V, $|V| \geq 2$, we define a monomial of variables x_v:

$$m(T) = \prod_{v \in V} {x_v}^{\rho_T(v)-1}, \tag{1-1}$$

where $\rho_T(v)$ denotes *degree* of the vertex v in the tree T, i.e. the number of edges of T incident to the vertex v.

For the graph G we construct the polynomial t_G of variables x_v:

$$t_G := \sum_T m(T),$$

where the sum is over all *spanning trees* T in the graph G, i.e. T is a connected subgraph in G which contains all vertices of G and has no cycles.

The polynomial t_G in the case of the complete graph K_n first was considered by A. Cayley [Cay], who found the exact formula for it (see formula (3-3) below). the polynomial t_G for arbitrary graph was defined by A. Rényi [Ré2].

Let $0 \notin V$ and $\widetilde{V} := V \cup \{0\}$. For a graph G on the set V the *extended graph* $\widetilde{G}$ on the set $\widetilde{V}$ is obtained from G by adding edges $\{0, v\}$ for all vertices $v \in V$.

Let the variable x be associated with added vertex 0.

For nonempty graph G we construct another polynomial f_G of variables x and x_v, $v \in V$:

$$f_G := t_{\widetilde{G}} \tag{1-2}$$

Let $V = \{1, 2, \dots, n\}$ and $f_G = f_G(x; x_1, x_2, \dots, x_n)$.

The monomials of the polynomial f_G, which do not include x, correspond to spanning trees of the graph G such that the degree of the vertex 0 is equal to 1. Hence, the vertex 0 is connected by an edge with certain vertex i of the graph G, $i = 1, 2, \dots, n$. Therefore

$$t_G(x_1, x_2, \dots, x_n) \cdot (x_1 + x_2 + \dots + x_n) = f_G(0; x_1, x_2, \dots, x_n) \tag{1-3}$$

Formulas (1-2), (1-3) show that the polynomials f_G and t_G define each other. But formulas for f_G are usually more simple than the corresponding formulas for t_G. So we will use the polynomial f_G further on.

Remark. It is easy to see that spanning trees in $\widetilde{G}$ correspond to *spanning rooted forests* in G, i.e. subgraphs in G without cycles containing all vertices of G, with a root chosen in each component. Hence f_G is the sum over all spanning rooted forests in G.

2. Reciprocity theorem for polynomials f_G

A graph $\overline{G} = (V, \overline{E})$ is called *complimentary* to a graph $G = (V, E)$ if $\overline{E} = \binom{V}{2} \setminus E$, in other words e is an edge of $\overline{G}$ iff e is not an edge of G.

The following theorem presents a reciprocity property for polynomials f_G.

Theorem 2.1. *Let G be a graph on the set of vertices $V = \{1, 2, \dots, n\}$. Then*

$$f_{\overline{G}}(x; x_1, x_2, \dots, x_n) = (-1)^{n-1} \cdot f_G(-x - x_1 - \dots - x_n; x_1, x_2, \dots, x_n). \tag{2-1}$$

In particular case $x_1 = x_2 = \dots = x_n = 1$ formula (2-1) was found by S. D. Bedrosian [Bed] and A. K. Kelmans [Kel].

3. Computations for certain graphs

Let G_1 and G_2 be two graphs on disjoint sets of vertices. Let $G_1 + G_2$ denote the disjoint union of the graphs. We associate variables $y_1, y_2, \ldots, y_r$ to the vertices of G_1 and variables $z_1, z_2, \ldots, z_s$ to the vertices of G_2. Then the following formula holds:

$$f_{G_1+G_2}(x; y_1, \ldots, y_r, z_1, \ldots, z_t) = x f_{G_1}(x; y_1, \ldots, y_r) \cdot f_{G_2}(x; z_1, \ldots, z_s) \tag{3-1}$$

Indeed, every spanning tree T in the graph $\widetilde{G_1 + G_2}$ splits into two spanning trees T_1 and T_2 in the graphs $\widetilde{G_1}$ and $\widetilde{G_2}$ respectively. And $\rho_T(0) - 1 = (\rho_{T_1}(0) - 1) + (\rho_{T_2}(0) - 1) + 1$, hence the factor x in the right hand part occurs.

Theorem 2.1, accompanied by formula (3-1), enables us to compute f_G for certain graphs.

Example 3.1. Let O_n denote the empty graph on the set $\{1, 2, \ldots, n\}$. It is clear, that $f_{O_1} = 1$. Consequently applying formula (3-1), we get $f_{O_n} = x^{n-1}$. For the complete graph K_n on the set $\{1, 2, \ldots, n\}$ by theorem 2.1 we get

$$f_{K_n} = f_{\overline{O_n}} = (-1)^{n-1}(-x - x_1 - \cdots - x_n)^{n-1} = (x + x_1 + \cdots + x_n)^{n-1}. \tag{3-2}$$

Example 3.2. Let K_{rs} be the full bipartite graph, variables $y_1, \ldots, y_r$ correspond to r vertices of the first part and variables $z_1, \ldots, z_s$ correspond to s vertices of the second part. Then by formula (3-1) we get

$$f_{K_r+K_s} = x \cdot (x + y_1 + \cdots + y_r)^{r-1} \cdot (x + z_1 + \cdots + z_s)^{s-1}.$$

By Theorem 2.1:

$$\begin{aligned} f_{K_{rs}} &= f_{\overline{K_r+K_S}} = (-1)^{r+s-1}(-x - y_1 - \cdots - y_r - z_1 - \cdots - z_s)\cdot \\ &\cdot (-x - z_1 - \cdots - z_s)^{r-1} \cdot (-x - y_1 - \cdots - y_r)^{s-1} = \\ &= (x + y_1 + \cdots + y_r + z_1 + \cdots + z_s)(x + z_1 + \cdots + z_s)^{r-1} \cdot (x + y_1 + \cdots + y_r)^{s-1}. \end{aligned}$$

We can find the corresponding formulas for t_G by (1-3):

$$t_{K_n} = (x_1 + \cdots + x_n)^{n-2}, \tag{3-3}$$

$$t_{K_{rs}} = (y_1 + \cdots + y_r)^{s-1} \cdot (z_1 + \cdots + z_s)^{r-1}. \tag{3-4}$$

Let $t(G)$ denote the number of spanning trees of a graph G. Substituting $x_1 = \cdots = x_n = y_1 = \cdots = y_r = z_1 = \cdots = z_s = 1$ into (3-3) and (3-4) we get

$$t(K_n) = n^{n-2}, \tag{3-5}$$

$$t(K_{rs}) = r^{s-1} \cdot s^{r-1}. \tag{3-6}$$

4. Generalization to oriented nets

An *oriented net* (or simply *net*) on the set of vertices V is defined by the set of *conductivities* $g_{vw} \in \mathbb{R}$ assigned to every ordered pair of vertices $v, w \in V$.

We associate a net with each graph as follows

$$g_{vw} = g_{wv} = \begin{cases} 1 & \text{if } \{v, w\} \text{ is an edge of the graph;} \\ 0 & \text{else} \end{cases}$$

If it does not lead to a confusion we denote the graph and the corresponding net by the same letter G.

When displaying a net graphically we draw an *oriented graph* with assigned edge conductivities (see fig. 4.1).

We will consider only nets $G = (g_{vw})$, $v, w \in V$, without *loops*, i.e. $g_{vv} = 0$ for all $v \in V$.

Let T be a tree on the set of vertices $\widetilde{V} = V \cup \{0\}$. Let us orient the tree T from the root in the vertex 0. The *multiplicity* of T in the net G is the number

$$k_G(T) = \prod_{(v,w)} g_{vw},$$

where the product is over all ordered pairs $(v, w) \in V \times V$ such that (v, w) is an edge of T (exactly in this orientation). If T consists only of edges $(0, v)$ and does not contain any edge $(v, w) \in V \times V$, we assume that $k_G(T) = 1$.

Note that if G is the net associated with a graph then

$$k_G(T) = \begin{cases} 1 & \text{if } T \text{ is a spanning tree of the graph } \widetilde{G}; \\ 0 & \text{else} \end{cases}$$

Now one can define the polynomial f_G for a net G:

$$f_G(T) := \sum_T k_G(T) \cdot m(T),$$

where the sum is over all trees on the set of vertices $\widetilde{V}$; and $m(T)$ is defined by (1-1).

Example 4.1. Let $V = \{1, 2\}$, $g_{12} = \alpha, g_{21} = \beta$. Then $f_G = x + \alpha x_1 + \beta x_2$ (see fig. 4.1).

Fig. 4.1

A net $\overline{G} = (\overline{g_{vw}})$ on the set V is called *complimentary* to the net $G = (g_{vw}$ on the same set V if $\overline{g}_{vw} = 1 - g_{vw}$ for all $v \neq w$ (for $v = w$ $\overline{g_{vv}} = g_{vv} = 0$).

It is clear that in the case when the net G corresponds to a graph the concepts *complementary net* and *complementary graph* coincide.

The formula (2-1) remains true for nets.

Theorem 4.2. *Let G be an oriented net on the set of vertices $\{1, 2, \ldots, n\}$. Then*

$$f_{\overline{G}}(x; x_1, \ldots, x_n) = (-1)^{n-1} \cdot f_G(-x - x_1 - \cdots - x_n; x_1, \ldots, x_n).$$

Example 4.3. For the net from Example 4.1 we get:

$$f_{\overline{G}}(x; x_1, x_2) = x + (1-\alpha)x_1 + (1-\beta)x_2 = (-1) \cdot ((-x - x_1 - x_2) + \alpha x_1 + \beta x_2 = \\ = (-1)^{2-1} \cdot f_G(-x - x_1 - x_2; x_1, x_2).$$

We will prove Theorem 4.2 in Section 6.

5. Multipartite graphs and nets

Let $G_i = (V_i, E_i), i = 1, 2, \ldots, k$, be a collection of graphs on a disjoint sets of vertices $V_1, V_2, \ldots, V_k$; G be a graph on the set of vertices $\{\bar{1}, \bar{2}, \ldots, \bar{k}\}$.

We define $\Gamma = \Gamma(G; G_1, G_2, \ldots, G_k)$ be the graph on the set of vertices $V = \bigcup_i V_i$ such that two vertices $v \in V_i$ and $w \in V_j$ are connected by an edge in Γ iff either of two following conditions hold:

(1) $i = j$ and (v, w) is en edge of G_i or
(2) $i \neq j$ and $(\bar{i}, \bar{j})$ is an edge of G.

See an example on fig. 5.1.

Fig. 5.1

We will call graphs of the type $\Gamma = \Gamma(G; G_1, G_2, \ldots, G_k)$ *multipartite graphs.*

This definition can be extended onto oriented nets as follows.

Let G_i be a collection of nets on a disjoint sets of vertices V_i with conductivities $g^{(i)}_{vw}$, $v, w \in V_i$, where $i = 1, 2, \ldots, k$; G be a net on the set of vertices $\{\bar{1}, \bar{2}, \ldots, \bar{k}\}$ with conductivities $g_{\bar{i}\bar{j}}$, $i, j \in \{1, 2, \ldots, k\}$.

Let $\Gamma = \Gamma(G; D_1, G_2, \dots, G_k)$ be the net on the set of vertices $V = \bigcup_i V_i$ defined by collection of conductivities γ_{vw}, $v \in V_i, w \in V_j$, where

$$\gamma_{vw} = \begin{cases} g^{(i)}_{vw} & \text{if } \; i = j; \\ g_{\bar{i}\bar{j}} & \text{if } \; i \neq j. \end{cases}$$

The following theorem reduces computation of the polynomial f_Γ to calculation of polynomials $f_G, f_{G_1}, \dots, f_{G_k}$.

Let for $i = 1, 2, \dots, k$ the vertices of G_i be pairs $(i, r), 1 \le r \le n_i$, where $n_i := |V_i|$; and variables $x_{i1}, x_{i2}, \dots, x_{in_i}$ correspond to these vertices. Let $X_i := x_{i1} + x_{i2} + \dots + x_{in_i}$ and $Y_i := \sum_{j=1}^k X_j \, g_{\bar{j}\bar{i}}$ (recall that $g_{\bar{i}\bar{i}} = 0$).

Theorem 5.1. *Let $\Gamma = \Gamma(G; G_1, \dots, G_k)$. Then*

$$f_\Gamma(x; x_{ir}) = f_G(x; X_1, \dots, X_k) \cdot \prod_{i=1}^{k} f_{G_i}(x + Y_i; x_{i1}, \dots, x_{in_i}). \tag{5-1}$$

6. Proofs of Theorems 4.2 and 5.1

In this section we apply essentially the same method as A. Rényi used in [Ré2] for the proof of formula (3-3).

In the beginning we prove several simple lemmas.

Let G be a net on the set V, $v \in V$. Let $G \setminus v$ denote the *restriction* of G onto the set $V \setminus \{v\}$, i.e. $V \setminus v$ is the net on the set $V \setminus \{v\}$ with the same conductivities as the net G.

Lemma 6.1.

$$f_G|_{x_v=0} = (x + \sum_{w \in V} x_w \, g_{wv}) \cdot f_{G \setminus v}. \tag{6-1}$$

Proof. The polynomial $f_G|_{x_v=0}$ consists of monomials corresponding to trees T such that the vertex v is an endpoint of T. This vertex is connected by the edge in T with certain vertex $w \in \widetilde{V} \setminus \{v\}$. Let T' be the tree on the set $\widetilde{V} \setminus \{v\}$ obtained from T by deletion the vertex v $(T' = T \setminus v)$. Then

$$k_G(T)\, m(T) = \begin{cases} x \cdot k_{G \setminus v}(T')\, m(T') & \text{if } w = 0, \\ x_w \cdot g_{wv} \, k_{G \setminus v}(T')\, m(T') & \text{if } w \in V. \end{cases}$$

Hence the formula (6-1) holds.

Note that (6-1) is analogous to (1-3).

Lemma 6.2. *Let G be a graph (or net) with n vertices. Then f_G is a homogeneous polynomial of degree $n-1$. (We assume that* $\deg x = \deg x_v = 1, v \in V$*.)*

Proof. By (1-1), for a tree T on the set of vertices $\widetilde{V}$, $|\widetilde{V}| = n + 1$ we have

$$\deg m(T) = \sum_{v \in \widetilde{V}} (\rho_T(v) - 1) = 2n - (n+1) = n - 1.$$

The second equality holds because the sum of degrees of all vertices of a tree is twice the number of edges of the tree.

Lemma 6.3. *Let h be a polynomial of variables $x_1, x_2, \ldots, x_n$ with degree strictly less then n and $h|_{x_i=0} = 0$ for all $i = 1, 2, \ldots, n$. Then $h \equiv 0$.*

Proof. Suppose that $h \neq 0$. Then we can find a monomial in h with nonzero coefficient. Since $\deg h < n$, we can find an integer $i \in \{1, 2, \ldots, n\}$ such that the variable x_i does not appear in the monomial. Hence $h|_{x_i=0} \neq 0$, that contradicts to the conditions.

Proof of Theorem 4.2.
We prove be by induction on $n = |V|$ that the polynomial

$$\begin{aligned} h_G(x; x_1, \ldots, x_n) :=& f_{\overline{G}}(x; x_1, \ldots, x_n) - \\ -(-1)^{n-1}\, & f_G(-x - x_1 - \cdots - x_n; x_1, \ldots, x_n) \end{aligned}$$

is equal to zero.
1°. Let $n = 1$. There is a unique graph (net) with one vertex $G = K_1$ for which $f_{K_1} = 1$. Hence $h_{K_1} = 1 - 1 = 0$.
2°. Let $n \geq 2$. We obtain by Lemma 6.1 that

$$\begin{aligned} & h_G(x; x_1, \ldots, x_n)|_{x_i=0} = (x + \sum_{j:j\neq i} (1 - g_{ji}) x_j) f_{\overline{G\setminus i}}(x; x_1, \ldots, \widehat{x_i}, \ldots, x_n) - \\ & - (-1)^{n-1}((-x - x_1 - \cdots - \widehat{x_i} - \cdots - x_n) + \sum_{j:j\neq i} g_{ji} x_i) \cdot \\ & \cdot f_{G\setminus i}(-x - x_1 - \cdots - \widehat{x_i} - \cdots - x_n; x_1, \ldots, \widehat{x_i}, \ldots, x_n) = \\ & = (x + \sum_{j:j\neq i} (1 - g_{ji} x_j)) \cdot h_{G\setminus i}(x; x_1, \ldots, \widehat{x_i}, \ldots, x_n). \end{aligned}$$

Here the symbol $\widehat{\ }$ denotes that corresponding element is omitted.
By inductive hypothesis $h_{G\setminus i} = 0$. Therefore

$$h_G|_{x_i=0} = 0 \text{ for } i = 1, 2, \ldots, n.$$

By Lemma 6.2 $\deg h_G < n$. Hence by Lemma 6.3 $h_G = 0$. Q.E.D.

Proof of Theorem 5.1.
Let $\Gamma = \Gamma(G; G_1, \ldots, G_k)$, γ_{vw} be conductivities of Γ;

$$H_\Gamma := f_\Gamma(x; x_{ir}) - f_G(x; X_1, \ldots, X_k) \cdot \prod_{i=1}^{k} f_{G_i}(x + Y_i; x_{i1}, \ldots, x_{in_i}),$$

where $n_i = |V_i|$ is the number of vertices in ith part of Γ.
We prove by induction that $H_\Gamma = 0$. The induction will be by the number n of vertices of the graph Γ $n = n_1 + \cdots + n_k$.
1°. It is clear that for $n = 1$ $H_\Gamma = 0$.
2°. Let $n \geq 2$. Let the vertex $v = (j, r) \in V_j$ Now we calculate $H_\Gamma|_{x_v=0}$. We consider two cases:
A. Let $n_j = |V_j| \geq 2$. In this case

$$\Gamma \setminus v = \Gamma(G; G_1, \ldots, G_{j-1}, G_j \setminus v, G_{j+1}, \ldots, G_k).$$

By Lemma 6.1

$$H_\Gamma|_{x_v=0} = (x + \sum_{w\in V} x_w\gamma_{wv})\, f_{\Gamma\setminus v} - f_G(x; X_1, \dots, X_k)|_{x_v=0} \cdot$$
$$(\prod_{\substack{i=1,\dots,k\\ i\neq j}} f_{G_i}(x+Y_i; x_{i1},\dots,x_{in_i}))\cdot((x+Y_j)+$$
$$+\sum_{w\in V_j} x_w\, g^{(j)}_{wv})\cdot f_{G_j\setminus v}(x+Y_j; x_{j1},\dots,x_{jn_j}) =$$
$$= (x+\sum_{w\in V} x_w\,\gamma_{wv})\cdot H_{\Gamma\setminus v}.$$

Now we apply the equality

$$\sum_{w\in V} x_w\,\gamma_{wv} = Y_j + \sum_{w\in V_j} x_w\, g^{(j)}_{wv},$$

which is true by definition of Y_j.

B. Let $n_j = |V_j| = 1$. In this case

$$\Gamma\setminus v = \Gamma(G\setminus\bar{j}; G_1,\dots,\widehat{G_j},\dots,G_k).$$

By Lemma 6.1

$$H_\Gamma|_{x_v=0} = (x+\sum_{w\in V} x_w\,\gamma_{wv}) f_{\Gamma\setminus v} - (x+\sum_{i=1}^{k} X_i\, g_{\bar{i}\bar{j}})\cdot$$
$$\cdot f_{G\setminus\bar{j}}(x; X_1,\dots,\widehat{X_j},\dots,X_k)\cdot \prod_{\substack{i=1,\dots,k\\ i\neq j}} f_{G_i}(x+Y_i; x_{i1},\dots,x_{in_i})|_{x_v=0} =$$
$$= (x+\sum_{w\in V} x_w\,\gamma_{wv})\cdot H_{\Gamma\setminus v}.$$

In the both cases assuming by induction that $H_{\Gamma\setminus v}=0$, we obtain that

$$H_\Gamma|_{x_v} = 0 \text{ for all } v\in V.$$

By Lemma 6.2 $\deg H_\Gamma < n$, hence, by Lemma 6.3, $H_\Gamma = 0$. Q.E.D.

7. Increasing trees

This and the subsequent chapters present two examples of using Theorem 5.1.

Let T be a tree on the set of vertices $\{0,1,2,\dots,n\}$ oriented from the root in the vertex 0. The tree T is said to be *increasing* if for each oriented edge (i,j) of T $i<j$ (see fig. 7.1).

Let I_n be the oriented net on the set of vertices $\{1,2,\dots,\}$ with conductivities:

$$g_{ij} = \begin{cases} 1 & \text{if } i<j \\ 0 & \text{if } i\geq j \end{cases}$$

FIG. 7.1

Then

$$k_{I_n}(T) = \begin{cases} 1 & \text{if } T \text{ is an increasing tree} \\ 0 & \text{else} \end{cases}$$

Thus f_{I_n} is the enumerator for increasing trees.

Now we demonstrate how Theorem 5.1 helps us to calculate the polynomial f_{I_n}.

Split the set of vertices of I_n onto two classes: $\{1, 2, \dots, n-1\}$ and $\{n\}$. Let G be the net with two vertices $\bar{1}, \bar{2}$ and conductivities $g_{\bar{1}\bar{2}} = 1, g_{\bar{2}\bar{1}} = 0$. It is clear that $I_n = \Gamma(G; I_{n-1}, K_1)$, where I_{n-1} is the net on the vertices $1, 2, \dots, n-1$ and K_1 in the unique graph with one vertex n.

Let the variables $x_1, x_2, \dots, x_n$ be associated with vertices $1, 2, \dots, n$. In the designations of Section 5 $X_1 = x_1 + x_2 + \dots + x_{n-1}, X_2 = x_n, Y_1 = 0, Y_2 = x_1 + \dots + x_{n-1}$.

We have already found f_G (see Example 4.1 with $\alpha = 1$ and $\beta = 0$) $f_G(x; X_1, X_2) = x + X_1$; $f_{K_1} = 1$.

Hence by Theorem 5.1 we obtain:

$$\begin{aligned} f_{I_n}(x; x_1, \dots, x_n) &= f_G(x; X_1, X_2) \cdot f_{I_{n-1}}(x + Y_1; x_1, \dots, x_{n-1}) \cdot f_{K_1}(x + Y_2; x_n) = \\ &= (x + x_1 + \dots + x_{n-1})\, f_{I_{n-1}}(x; x_1, \dots, x_{n-1}). \end{aligned}$$

By induction we obtain

$$f_{I_n}(x; x_1, \dots, x_n) = \prod_{i=1}^{n-1} (x + x_1 + \dots + x_i) \tag{7-1}$$

Substituting into (7-1) $x = x_1 = x_2 = \dots = x_n$, we find that the number of increasing trees on the set of vertices $\{0, 1 \dots, n\}$ is equal to $f_{I_n}(1; 1, \dots, 1) = n!$.

Of course it is not difficult to obtain formula (7-1) without using Theorem 5.1. The following example is less trivial.

8. HURWITZ'S IDENTITY

A. Hurwitz found the following generalization of the binomial theorem:

(8-1)
$$(x+y)(x+y+z_1+\cdots+z_n)^{n-1}=$$
$$=\sum_{(I,J)} x(x+z_{i_1}+z_{i_2}+\cdots+z_{i_k})^{k-1}\cdot y(y+z_{j_1}+z_{j_2}+\cdots+z_{j_l})^{l-1},$$

where the sum is over all 2^n pairs of disjoint subsets $I=\{i_1,i_2,\ldots,i_k\}, J=\{j_1,j_2,\ldots,j_l\}\in\{1,2,\ldots,n\}$ such that $k+l=n$.

In this section we demonstrate how to interpret and prove the identity (8-1) in terms of polynomials f_G.

Consider a net Γ on the set of vertices $\{v\}\cup\{1,2,\ldots,n\}$, such that $\Gamma=\Gamma(G;K_1,K_n)$, where K_1 is the graph with one vertex v, K_n is the complete graph on the set of vertices $1,2,\ldots,n$, and G is the net with two vertices $\overline{1},\overline{2}$ and conductivities $g_{\overline{12}}=1, g_{\overline{21}}=0$ (see fig. 8.1).

Fig. 8.1

Now we apply Theorem 5.1 for the calculation of f_G Let the variable y corresponds to the vertex v; the variables $z_1,z_2,\ldots,z_n$ to the vertices $1,2,\ldots,n$, correspondingly; the variable x, as usually, to the added vertex 0.

Let X_1,X_2,Y_1,Y_2 denote the same as in Section 5. Then $X_1=y, X_2=z_1+z_2+\cdots+z_n, Y_1=0, Y_2=y$; $f_G(x;X_1,X_2)=x+X_1, f_{K_1}=1$, and, by (3-2) $f_{K_n}(x;z_1,\ldots,z_n)=(x+z_1+\cdots+z_n)^{n-1}$. Therefore, by Theorem 5.1 we get

(8-2)
$$f_\Gamma=f_G(x;X_1,X_2)\cdot f_{K_1}(x+Y_1;y)\cdot f_{K_n}(x+Y_2;z_1,\ldots,z_n)=$$
$$=(x+y)(x+y+z_1+\cdots+z_n)^{n-1}.$$

There is another method for calculation of the polynomial f_Γ. Note that there no such edge of Γ that enter to the vertex v (i.e. $g_{wv}=0$ for any vertex w of Γ). Hence any tree T on the set of vertices $\{0\}\cup\{v\}\cup\{1,2,\ldots,n\}$ necessarily contains the edge $(0,v)$.

When we delete the edge $(0,v)$ from T, the tree T falls into two trees T' and T'' with roots 0 and v correspondingly (see an example on fig. 8.2).

FIG. 8.2

Let T' has vertices $0, i_1, i_2, \ldots, i_k$ and T'' has vertices $v, j_1, j_2, \ldots, j_l$. Denote $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_l\}$. Then $I \cap J = \emptyset$ and $I \cup J = \{1, 2, \ldots, n\}$.

By the formula (3-2)

$$\sum_{T'} m(T') = (x + z_{i_1} + \cdots + z_{i_k})^{k-1},$$

where the sum is over all trees with the set of vertices $\{0\} \cup I$.

Analogously,

$$\sum_{T''} m(T'') = (y + z_{j_1} + \cdots + z_{j_l})^{l-1},$$

where the sum is over all trees with vertices $\{v\} \cup J$.

Eventually, we get

$$f_\Gamma = \sum_{(I,J)} (xy)(x + z_{i_1} + \cdots + z_{i_k})^{k-1} \cdot (y + z_{j_1} + \cdots + z_{j_l})^{l_1}, \tag{8-3}$$

where the sum is over all pairs of sets (I, J) such that $I \cap J = \emptyset$ and $I \cup J = \{1, 2, \ldots, n\}$. The factor xy correspond to the edge $(0, v)$.

Comparing two expressions (8-2) and (8-3) for f_Γ, we obtain Hurwitz's identity (8-1).

9. MATRIX-TREE THEOREM

One can express the polynomial f_G as determinant of a matrix.

In the beginning we formulate Tutte's generalization of matrix-tree theorem [Tut].

Let z_{vw}, $v, w \in \widetilde{V} = V \cup \{0\}$, be the collection of commutative variables, we assume that $z_{vv} = 0$ for $v \in \widetilde{V}$.

With any tree T on the set of vertices $\widetilde{V}$ we we associate a monomial $M(T)$ of variables z_{vw}. Let us orient the tree T from the root in vertex 0 and assume that

$$M(T) := \prod_{(v,w)} z_{vw},$$

where the product is over all pairs $(v,w) \in \widetilde{V} \times \widetilde{V}$ which are oriented edges of T (in this orientation). Now we denote

$$F_V = \sum_T M(T),$$

where the sum is over all trees on the set of vertices $\widetilde{V}$.

Without loss of generality we can assume that $V = \{1, 2, \ldots, n\}$. In this case $F_n := F_{\{1,\ldots,n\}}$ is a polynomial of z_{ij}, $0 \le i, j \le n$.

Let *Kirchoff's matrix* be $n \times n$ matrix $A = (a_{ij})$, $i, j \in \{1, 2, \ldots, n\}$, where

$$a_{ij} = \begin{cases} \sum\limits_{l=0}^{n} z_{li} \text{ if } i = j, \\ -z_{ij} \text{ if } i \ne j. \end{cases} \tag{9-1}$$

Theorem 9.1. (Matrix-tree theorem).

$$F_n = \det A.$$

Let now G be a net on the set of vertices V with conductivities g_{vw}. Assume that

$$\begin{aligned} z_{vw} &= x_v g_{vw}, \qquad v, w \in V, \\ z_{0v} &= x, \qquad v \in V. \end{aligned} \tag{9-2}$$

It is clear (see fig. 9.1) that

$$x\, k_G(T)\, m(T) = M(T),$$

where $m(T)$ and $k_G(T)$ are such as in Sections 1 and 4.

FIG. 9.1

Substituting (9-2) into (9-1), we get the following

Corollary 9.2.

$$x f_G(x; x_1, \ldots, x_n) = \det B,$$

where $B = (b_{ij})$, $1 \le i, j \le n$, is $n \times n$-matrix

$$b_{ij} = \begin{cases} x + \sum\limits_{l=1}^{n} x_l g_{li}, \qquad i = j, \\ -x_i g_{ij}, \qquad i \ne j. \end{cases}$$

10. Coding of trees

As we already knew (see formula (3-5)), the number of spanning trees of the complete graph K_n is equal to n^{n-2}. But in many cases it is not sufficient to know only the number of trees, a simple and quick method for enumeration of trees is also required.

A method that enables us to enumerate all trees with n labelled vertices is *Prüfer's coding* [Prü]. Since the Prüfer's construction is of importance in the subsequent part of the paper, we recall it in this section.

Prüfer's coding establishes a bijection π between on the one hand, the set $\mathrm{Tr}(V)$ of all trees on the set of vertices V, $|V| \geq 2$, and on the over hand, the set of sequences $(a_1, a_2, \dots, a_{n-2}) \in V^{n-2}$:

$$\pi : \mathrm{Tr}(V) \to V^{n-2}.$$

Note that if one construct π and the inverse bijection π^{-1}, he thereby get a combinatorial proof of Cayley's identity (3-5).

Let us suppose that V be a linear ordered set. Let $T \in \mathrm{Tr}(V)$. Construct by induction a sequence of trees $T^{(i)}$ and codes $C^{(i)} \in V^i$, $i = 0, 1, \dots, n-2$:

1°. For $i = 0$ $T^{(0)} = T \in \mathrm{Tr}(V), C^{(0)} = \emptyset$ (the empty sequence).

2°. For $i = 1, 2, \dots, n-2$. Let b_i be the maximal endpoint of the tree $T^{(i-1)}$ (in the linear order on the set of vertices). Then there is unique edge $\{a_i, b_i\}$ in the tree $T^{(i-1)}$ that incident to the vertex b_i. Then put $T^{(i)}$ to be the tree obtained by deleting the vertex b_i and the edge $\{a_i, b_i\}$ from $T^{(i-1)}$; and $C^{(i)} = (a_1, a_2, \dots, a_i)$, i.d. the sequence $C^{(i)}$ is obtained from $C^{(i-1)}$ by adding a_i to the right.

Finally put $\pi(T) = C = C^{(n-2)} = (a_1, a_2, \dots, a_{n-2})$.

Example 10.1. See fig. 10.1.

Note that the following property holds for the sequence $C = (a_1, a_2, \dots, a_{n-2}) = \pi(T)$

Proposition 10.2. *Let $v \in V$ be a vertex of the tree $T \in \mathrm{Tr}(V)$. Then v occurs in the sequence $C = \pi(T)$ $\rho_T(v) - 1$ times exactly, where $\rho_T(v)$ is the degree of the vertex v in the tree T.*

Proof. In the first place, note that for $i = 1, 2, \dots, n-2$ the vertex a_i which is added into the sequence C cannot be an endpoint of the tree $T^{(i-1)}$.

Let a vertex v is connected in the tree T with $\rho = \rho_T(v)$ vertices $v_1, v_2, \dots, v_\rho$. The vertex v is added into the sequence C every time when we delete one of the vertices $v_1, v_2, \dots, v_\rho$ from the tree. The vertex v becomes an endpoint of a tree $T^{(i)}$ when only one of the vertices $v_1, v_2, \dots, v_\rho$ remains undeleted. Hence before v becomes an endpoint we should add v into the sequence C $\rho - 1$ times. Thereafter, by the previous notice, we will not add v into the sequence C.

Remark 10.3. In particular, if v is an endpoint of the tree T then v does not occur in the sequence $C = \pi(T)$.

Now one can find the method for decoding, i.e the construction of the inverse bijection

$$\tau : V^{n-2} \to \mathrm{Tr}(V).$$

Let $C = (a_1, a_2, \dots, a_{n-2}) \in V^{n-2}$. Construct by induction a sequence of forests $F_{(i)}$ and a sequence of sets $V_{(i)}$, $i = 0, 1, 2, \dots, n-2$:

FIG. 10.1

1°. Let $V_{(0)} = V$ and $F_{(0)} = \emptyset$ (the empty forest).

2°. For $i = 1, 2, \ldots, n-2$ let b_i be the maximal vertex from the set $V_{(i-1)}$ that is not contained in the sequence $C_{(i)} := (a_i, a_{i+1}, \ldots, a_{n-2})$. Then we put $V_{(i)} = V_{(i-1)} \setminus \{b_i\}$, and $F_{(i)}$ is the forest obtained from F_{i-1} by adding the edge $\{a_i, b_i\}$.

Finally, the set $V_{(n-2)}$ consists of two elements $V_{(n-2)} = \{c, d\}$. We put T to be the tree that is obtained from $F_{(n-2)}$ by adding the edge $\{c, d\}$ and define $\tau(V) = T$.

The connection between the sequence $T^{(i)}$ that was constructed for coding and the sequences $V_{(i)}$, $F_{(i)}$, and $C_{(i)}$ that were constructed for decoding is the following: the set $V_{(i)}$ is the set of vertices of the tree $T^{(i)}$; the forest $F^{(i)}$ consists of all edges of the tree T which do not belong to the tree $T^{(i)}$; and $C_{(i)}$ is Prüfer's code for the tree $T^{(i)}$.

Example 10.4. See fig. 10.2.

It easily follows by induction from the constructions of π and τ and from Remark 10.3 that $\tau \circ \pi = \mathrm{id}_{\mathrm{Tr}(V)}$ and $\pi \circ \tau = \mathrm{id}_{V^{n-2}}$. Therefore one can obtain the following

FIG. 10.2

Proposition 10.5. *The map π is a bijection from* $\mathrm{Tr}(V)$ *to* V^{n-2} *and τ is the inverse bijection.*

Note that Propositions 10.2, 10.5 give a combinatorial proof not only to the formula (3-5) but also to the formula (3-3).

11. CODING OF MULTIPARTITE GRAPHS

In this section we construct a coding for multipartite graphs. This construction presents an independent combinatorial proof of Theorem 5.1.

Let, as in Section 5, G_i be a graph on the set of vertices $V_i = \{(i,1),(i,2),\dots,(i,n_i)\}$, G be a graph with vertices $\bar{1},\bar{2},\dots,\bar{k}$, and $\Gamma = \Gamma(G;G_1,\dots,G_k)$ be the multipartite graph on the set of vertices $V = \cup_i V_i = \{(i,r) : 1 \le i \le k, 1 \le r \le n_i\}$.

Let "$\le$ be the lexicographical order on the set V, that is for $(i',r'),(i'',r'') \in V$ the expression $(i',r') \le (i'',r'')$ denotes that either $i' < i''$ or both $i' = i''$ and $r' \le r''$. And let $\widetilde{V}$ be the set that is obtained from V by adding one minimal element 0. Then $\widetilde{V}$ is a linear ordered set.

In the subsequent part of the section we describe a method for coding of spanning trees T of the extended graph $\widetilde{\Gamma}$. We associate with a tree T a collection $R, P_1, P_2, \dots, P_k$ of sequences of elements of $\widetilde{V}$ of lengths $k-1, n_1-1, n_2-1, \dots, n_k-1$ correspondingly.

Let T be a spanning tree of $\widetilde{\Gamma}$. Orient the tree T to the root in the vertex 0. Let $T|_{V_i}$ be the restriction of T on the set of vertices V_i, $i = 1, 2, \dots, k$. Then $T|_{V_i}$ is an oriented forest (i.e. collection of mutually disjoint oriented trees) every component of which is a tree oriented to its own root. Let T_i be the tree on the set $\widetilde{V_i} = V_i \cup \{0\}$ which is obtained from $T|_{V_i}$ by adding the vertex 0 connected with all roots of components of $|T|_{V_i}$. Then T_i is a spanning tree of the graph $\widetilde{G_i}$.

Let P_i' be the Prüfer's code (see Section 10) for the tree T_i which we write into the sequence P_i.

Let T' be the tree obtained from T by contraction of all forests $T|_{V_i}$ to its roots. Let $(b_1, a_1), (b_2, a_2), \dots$ be Prüfer's sequence of edges for the tree T' (see Section 10). Note that $b_1, b_2, \dots$ cannot be 0, i.e $b_i \in V, \quad i = 1, 2, dots$.

If $b_1 \in V_i$ then we change the first occurrence of 0 in the sequence P_i to such vertex a_1' that (b_1, a_1') is an oriented edge of T (the vertex a_1' is determined uniquely by this condition).

We proceed this operation with vertices b_2, b_3 etc. in the similar manner.

If on certain rth step $b_r \in V_i$ but there are no 0's left in P_i then we write a_r' on the first unoccupied place in the sequence R.

We will repeat one of these operations until we eventually get the code R, P_1, $P_2, \dots, P_k$.

Example 11.1. See fig. 11.1

Lemma 11.2. *The sequences $R, P_1, P_2, \dots, P_k$ have lengths $k-1, n_1-1, n_2-1, \dots$, and n_k-1, correspondingly, and satisfy the following conditions:*

(1) *If $v \in V_i$ is an element of the sequence P_j the i=j or $(\bar{i}, \bar{j})$ is an edge of the graph G;*
(2) *If the sequence P_i' is obtained from P_i by changing all elements v of P_i, $v \notin V_i$ to 0 then P_i' is the Prüfer's code for certain spanning tree of $\widetilde{G_i}$;*
(3) *If the sequence R' is obtained from R by changing all elements $v \in V_i$ to $\bar{i}$, $i = 1, 2, \dots, k$, then R' is Prüfer's code for certain spanning tree of $\widetilde{G}$.*

References

[Bed] S. D. Bedrosian, *Generating formulas for the number of trees in a graph*, J. Franklin Inst. **227** (1964), no. 4, 313–326.

[Cay] A. Cayley, *A theorem on trees*, Quart. J. Pure. Appl. Math. **23** (1889), 376–378.

[CDS] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.

[HP] F. Harary, E. M. Palmer, *Graphical Enumeration*, Academic Press, New York, 1973.

[Kn] D. E. Knuth, *The Art of Computer Programming,* Vol. 1, Fundamental Algorithms, Addison-Wesley Publishing Company, 1968.

[Mo] J. W. Moon, *Counting Labelled Trees,* Canadian Math. Monographs, No. 1, 1970.

[Prü] H. Prüfer, *Neuer Beweis eines Satzes über Permutationen*, Arch. Math. Phys. **27** (1918), 742–744.

[Ré1] A. Rényi, J. Mayar Tud. Akad. Mat. Fiz. Oszt. Közl **12** (1966), 77–105.

[St] R. P. Stanley, *Enumerative Combinatorics,* Vol. 1, Wadsworth & Brooks/Cole, California, 1986.

Fig. 11.1

[Tut] W. T. Tutte, *Graph Theory,* Encyclopedia of Math. and its Appl., Vol. 21 (G.-C. Rota, ed.), Addison-Wesley Publishing Company, 1984, pp. 182–188.

Harvard, Dept of Mathematics, Cambridge MA 021??
E-mail address: pak@math.harvard.edu

MIT, Dept of Mathematics, Room 2–324, Cambridge MA 02139
E-mail address: apost@math.mit.edu