TOTAL POSITIVITY, GRASSMANNIANS, AND NETWORKS

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ABSTRACT. The aim of this paper is to discuss a relationship between total positivity and planar directed networks. We show that the inverse boundary problem for these networks is naturally linked with the study of the totally nonnegative Grassmannian. We investigate its cell decomposition, where the cells are the totally nonnegative parts of the matroid strata. The boundary measurements of networks give parametrizations of the cells. We present several different combinatorial descriptions of the cells, study the partial order on the cells, and describe how they are glued to each other.

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1. Introduction

A *totally positive* matrix is a matrix with all positive minors. We extend this classical notion, as follows. Define the *totally nonnegative Grassmannian* Gr_{kn}^{tnn} as the set of elements in the Grassmannian $Gr_{kn}(\mathbb{R})$ with all nonnegative Plücker coordinates. The classical set of totally positive matrices is embedded into Gr_{kn}^{tnn} as an open subset. The intersections $S_{\mathcal{M}}^{\mathrm{tnn}}$ of the *matroid strata* with Gr_{kn}^{tnn}, which we call the *nonnegative Grassmann cells,* give an interesting subdivision of the totally nonnegative Grassmannian. These intersections are actually cells (unlike the matroid strata that might have a nontrivial geometric structure) and they form a CW-complex. Conjecturally, this is a regular CW-complex and the closures of the cells are homeomorphic to balls. Fomin-Zelevinsky's [FZ1] *double Bruhat cells* (for type A) are included into Gr_{kn}^{tnn} as certain special nonnegative Grassmann cells $S_{\mathcal{M}}^{\mathrm{tnn}}$. Note that the subdivision of Gr_{kn}^{tnn} into the cells $S_{\mathcal{M}}^{\mathrm{tnn}}$ is a finer subdivision than the Schubert decomposition.

Our "elementary" approach agrees with Lusztig's general theory total positivity [Lus1, Lus2, Lus3] and with the cellular decomposition of the nonnegative part of G/P conjectured by Lusztig and proved by Rietsch [Rie1, Rie2].

Another main object of the paper is a planar directed *network* drawn inside a disk with the boundary vertices $b_1, \dots, b_n$ (and some number of internal vertices) and with positive weights x_e on the edges. We assume that k boundary vertices b_i are sources and the remaining $(n-k)$ boundary vertices b_j are sinks. We allow the sources and sinks to be interlaced with each other in any fashion. For an acyclic network, we define the *boundary measurements* as $M_{ij} = \sum_{P\,:\,b_i \to b_j} \prod_{e\in P} x_e$, where the sum is over all directed paths P in the network from a source b_i to a sink b_j, and the product is over all edges e of P. If a network has directed cycles, we introduce the sign $(-1)^{wind(P)}$ into the weight of P, where $wind(P)$ is the winding index that counts the number of full $360°$ turns the path P makes. We show that the power series for M_{ij} (which might be infinite if G has directed cycles) always sums to a *subtraction-free* rational expression.

We discuss the *inverse boundary problem* for such planar directed networks. In other words, we are interested in the information about networks that can be recovered from all boundary measurements M_{ij}. We characterize all possible collections of the measurements, describe all transformations of networks that preserve the measurements, and show how to reconstruct a network from the measurements (up to these transformations). Our work on this problem is parallel to results of Curtis-Ingerman-Morrow [CIM, Ing, CM] on the inverse problem for (undirected) transistor networks.

The inverse boundary problem for directed networks has deep connections with total positivity. The collection of all boundary measurements M_{ij} of a network can be encoded as a certain element of the Grassmannian Gr_{kn}. This gives the boundary measurement map $Meas : \{networks\} \to Gr_{kn}$. We show that the image of the map *Meas* is exactly the totally nonnegative Grassmannian Gr_{kn}^{tnn}. The proof of the fact that the image of *Meas* belongs to Gr_{kn}^{tnn} is based on a modification of Fomin's work [Fom] on *loop-erased walks.* On the other hand, for each point in Gr_{kn}^{tnn}, we construct a network of special kind that maps into this point.

The image of the set of networks with fixed combinatorial structure given by a graph G (with arbitrary positive weights on the edges) is a certain nonnegative Grassmann cell $S_{\mathcal{M}}^{\mathrm{tnn}}$. This gives the map from graphs G to the set of Grassmann cells. If the graph G is reduced (that is minimal in a certain sense) then the map *Meas* induces a rational subtraction-free parametrization of the associated cell $S_{\mathcal{M}}^{\mathrm{tnn}}$.

For each cell $S_{\mathcal{M}}^{\mathrm{tnn}}$ we describe one particular graph G associated with this cell. This graph is given by a Ⅎ-diagram, which is a filling of a Young diagram λ with 0's and 1's that satisfies certain Ⅎ-property. The shape λ corresponds to the Schubert cell Ω_λ containing $S_{\mathcal{M}}^{\mathrm{tnn}}$. The Ⅎ-diagrams have interesting combinatorial properties.

There are several types of transformations of networks that preserve the boundary measurements. First of all, there are quite obvious rescaling of the edge weights x_e at each internal vertex, which we call the *gauge transformations.* Then there are transformations that allow up to switch directions of edges in the network. We can easily transform any network into a special form (called a perfect network) and then color the vertices into two colors according to some rule. We prove that the boundary measurement map *Meas* is invariant under switching directions of edges that preserve colors of vertices. Thus the boundary measurement map can now be defined for *undirected* planar networks with vertices colored in two colors. We call them *plabic networks* (abbreviation for "planar bicolored"). Finally, there are several *moves* (that is local structure transformations) of plabic networks that preserve the boundary measurement map. We prove that any two networks with the same boundary measurements can be obtained from each other by a sequence of these transformations.

Essentially, the only nontrivial transformation of networks is a certain *square move.* This move can be related to cluster transformation from Fomin-Zelevinsky theory of *cluster algebras* [FZ2, FZ3, FZ4, BFZ2]. It is a variant of the *octahedron recurrence* in a disguised form. In terms of dual graphs, the square move corresponds *graph mutations* from Fomin-Zelevinsky's theory. However, we allow to perform mutations of dual graphs only at vertices of degree 4. By this restriction we preserve planarity of graphs.

We show how to transform each plabic graph into a *reduced* graph. We define *trips* in such graph as directed paths in these (undirected) graphs that connect boundary vertices b_i and obey certain "rules of the road." The trips give the *decorated trip permutation* of the boundary vertices. We show that any two reduced plabic graphs are related by the moves and correspond to the same Grassmann cell $S_{\mathcal{M}}^{\mathrm{tnn}}$ if and only if they have the same decorated trip permutation. Thus the cells $S_{\mathcal{M}}^{\mathrm{tnn}}$ are in one-to-one correspondence with decorated permutations.

Plabic graphs can be thought of as generalized wiring diagrams, which are graphical representations of reduced decompositions in the symmetric group. The moves of plabic graphs are analogues of the *Coxeter moves* of wiring diagrams. Plabic graphs also generalize Fomin-Zelevinsky's *double wiring diagrams* [FZ1].

We define *alternating strand diagrams,* which are in bijection with plabic graphs. These diagrams consist of n directed strands that connect n points on a circle and intersect with each other inside the circle in an alternating fashion. Scott [Sco1, Sco2] used our alternating strand diagrams to study Leclerc-Zelevinsky's [LZ] quasi-commuting families of quantum minors and cluster algebra on the Grassmannian.

We discuss the partial order on the cells $S_{\mathcal{M}}^{\mathrm{tnn}}$ by containment of their closures and describe it in terms of decorated permutations. We call this order the *circular Bruhat order* because it reminds the usual (strong) Bruhat order on the symmetric group. Actually, the usual Bruhat order is a certain interval in the circular Bruhat order.

We use our network parametrizations of the cells to describe how they are glued to each other. The gluing of a cell $S_{\mathcal{M}}^{\mathrm{tnn}}$ to the lower dimensional cells inside of its closure $\overline{S_{\mathcal{M}}^{\mathrm{tnn}}}$ is described by sending some of the edge weights x_e to 0. Thus, for the cell $S_{\mathcal{M}}^{\mathrm{tnn}}$ associated with a graph G, the lower dimensional cells in its closure are associated with subgraphs $H \subseteq G$ obtained from G by removing some edges. In a sense, this is an analogue of the statement that, for a Weyl group element with a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, all elements below w in the Bruhat order are obtained by taking subwords in the reduced decomposition.

For each plabic graph G associated with a cell $S_{\mathcal{M}}^{\mathrm{tnn}}$, we describe a different parametrization of the cell by a certain subset of the Plücker coordinates. This parametrization is related to the boundary measurement parametrization by the *chamber anzatz* and a certain *twist map* $S_{\mathcal{M}}^{\mathrm{tnn}}/T \to S_{\mathcal{M}}^{\mathrm{tnn}}/T$, where T is the "positive torus" $T = \mathbb{R}_{>0}^n$ acting on Gr_{kn}^{tnn}. This construction is analogous to a similar construction of Berenstein-Fomin-Zelevinsky [BFZ1, FZ1] for double Bruhat cells. In our setup, instead of chambers in (double) wiring diagrams, we work with regions of plabic graphs.

As an application, we obtain a description of Berenstein-Zelevinsky's *string cones* and polytopes [BZ1, BZ2] (of type A) as *tropicalizations* of the boundary measurements M_{ij}. Integer lattice points in these polytopes count the *Littlewood-Richardson coefficients*. This explains the combinatorial description of the string cones from our earlier work [GP] and the rule for the Littlewood-Richardson coefficients.

Our construction produces several different combinatorial objects associated with the cells $S_{\mathcal{M}}^{\mathrm{tnn}}$. We give explicit bijections between all these objects. Here is the (incomplete) list of various objects: totally nonnegative Grassmann cells, totally nonnegative matroids, ⅃-diagrams, decorated permutations, circular chains, move-equivalence classes of (reduced) plabic graph, move-equivalence classes of alternating strand diagrams.

We also construct bijections between ⅃-diagrams and other combinatorial objects such as permutations of a certain kind, rook placements on skew Young diagrams, etc. Williams [W1], Steingrimsson-Williams [SW], and Corteel-Williams [CW] obtained several enumerative results on ⅃-diagrams and related objects, and studied their combinatorial properties.

Throughout the paper we use the following notation $[n] := \{1, \dots, n\}$ and $[k, l] := \{k, k+1, \dots, l\}$. The word "network" means a graph (directed or undirected) together with some weights assigned to edges or faces of the graph.

Many results of this paper were obtained in 2001. Some results were announced by Williams in [W1, Sect. 2–3] and [W2, Appendix].

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2. Grassmannian

In this section we review some classical facts about Grassmannians, their stratifications, and matroids. For more details, see [Fult].

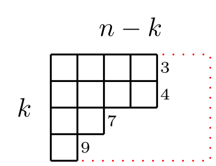
2.1. **Schubert cells.**

For $n \geq k \geq 0$, let the *Grassmannian* Gr_{kn} be the manifold of k-dimensional subspaces $V \subset \mathbb{R}^n$. It can be presented as the quotient $Gr_{kn} = GL_k \backslash \mathrm{Mat}^*_{kn}$, where Mat^*_{kn} is the space of real $k \times n$-matrices of rank k. Here we assume that the subspace V associated with a $k \times n$-matrix A is spanned by the row vectors of A.

Recall that a *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ is a weakly decreasing sequence of nonnegative integers. It is graphically represented by its Young diagram which is the collection of boxes with indexes (i, j) such that $1 \leq i \leq k$, $1 \leq j \leq \lambda_i$ arranged on the plane in the same fashion as one would arrange matrix elements.

There is a cellular decomposition of the Grassmannian Gr_{kn} into a disjoint union of Schubert cells Ω_λ indexed by partitions $\lambda \subseteq (n-k)^k$ whose Young diagrams fit inside the $k \times (n-k)$-rectangle $(n-k)^k$, that is $n - k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$.

The partitions $\lambda \subseteq (n-k)^k$ are in one-to-one correspondence with k-element subsets $I \subset [n]$. The boundary of the Young diagram of such partition λ forms a lattice path from the the upper-right corner to the lower-left corner of the rectangle $(n-k)^k$. Let us label the n steps in this path by the numbers $1, \dots, n$ consecutively, and define $I = I(\lambda)$ as set of labels of k vertical steps in the path. The inverse map $I = \{i_1 < \cdots < i_k\} \mapsto \lambda$ is given by $\lambda_j = |[i_j, n] \setminus I|$, for $j = 1, \dots, k$. As an example, Figure 2.1 shows a Young diagram of shape $\lambda = (4, 4, 2, 1) \subseteq 6^4$ that corresponds to the subset $I(\lambda) = \{3, 4, 7, 9\} \subseteq [10]$.



$n = 10$, $k = 4$
$\lambda = (4, 4, 2, 1)$
$I(\lambda) = \{3, 4, 7, 9\}$

FIGURE 2.1. A Young diagram λ and the corresponding subset $I(\lambda)$

For $\lambda \subseteq (n-k)^k$, the *Schubert cell* Ω_λ in Gr_{kn} is defined as the set of k-dimensional subspaces $V \subset \mathbb{R}^n$ with prescribed dimensions of intersections with the elements of the opposite coordinate flag:

$$\Omega_\lambda := \{V \in Gr_{kn} \mid \dim(V \cap \langle e_i, \dots, e_n \rangle) = |I(\lambda) \cap [i, n]|, \text{ for } i = 1, \dots, n\},$$

where $\langle e_i, \dots, e_n \rangle$ is the linear span of coordinate vectors.

The decomposition of Gr_{rk} into Schubert cells can be described using *Gaussian elimination.* We can think of points in Gr_{kn} as nondegenerate $k \times n$-matrices modulo row operations. Recall that according to Gaussian elimination any nondegenerate $k \times n$-matrix can be transformed by row operations to the canonical matrix in *echelon form,* that is a matrix A such that $a_{1,i_1} = \cdots = a_{k,i_k} = 1$ for some $I = \{i_1 < \cdots < i_k\} \subset [n]$, and all entries of A to the left of these 1's and in same columns as the 1's are zero. In other words, matrices in echelon form are representatives of the left cosets in $GL_k \backslash \mathrm{Mat}^*_{kn} = Gr_{kn}$. Let us also say that such echelon matrix A is in *I-echelon form* if we want to specify that the 1's are located in the column set I. For example, a matrix in $\{3,4,7,9\}$-echelon form, for $n = 10$ and $k = 4$, looks like

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}$$

where "$*$" stands for any element of $\mathbb{R}$.

The Schubert cell Ω_λ is exactly the set of elements in the Grassmannian Gr_{kn} that are represented by matrices A in I-echelon form, where $I = I(\lambda)$. If we remove the columns with indices $i \in I$ from A, i.e., the columns with the 1's, and reflect the result with respect the vertical axis, the pattern formed by the $*$'s is exactly the Young diagram of shape λ. So an I-echelon matrix has exactly $|\lambda| := \lambda_1 + \cdots + \lambda_k$ such $*$'s, which can be any elements in $\mathbb{R}$. This shows that the Schubert cell Ω_λ homeomorphic to $\mathbb{R}^{|\lambda|}$. Thus the Grassmannian Gr_{kn} has the disjoint decomposition

$$Gr_{kn} = \bigcup_{\lambda \subseteq (n-k)^k} \Omega_\lambda \simeq \bigcup_{\lambda \subseteq (n-k)^k} \mathbb{R}^{|\lambda|}.$$

For example, the $*$'s in the $\{3,4,7,9\}$-echelon form above correspond to boxes of the Young diagram of shape $\lambda = (4,4,2,1)$. Thus the Schubert cell $\Omega_{(4,4,2,1)}$, whose elements are represented by matrices in $\{3,4,7,9\}$-echelon form, is isomorphic to $\mathbb{R}^{|\lambda|} = \mathbb{R}^{11}$.

2.2. **Plücker coordinates.** For a $k \times n$-matrix A and a k-element subset $I \subset [n]$, let A_I denote the $k \times k$-submatrix of A in the column set I, and let $\Delta_I(A) := \det(A_I)$ denote the corresponding *maximal minor* of A. If we multiply A by $B \in GL_k$ on the left, all minors $\Delta_I(A)$ are rescaled by the same factor $\det(B)$. If $A = (a_{ij})$ is in I-echelon form then $A_I = Id_k$ and $a_{ij} = \pm\Delta_{(I \setminus \{i\}) \cup \{j\}}(A)$. Thus the Δ_I form projective coordinates on the Grassmannian Gr_{kn}, called the *Plücker coordinates,* and the map $A \mapsto (\Delta_I)$ induces the *Plücker embedding* $Gr_{kn} \hookrightarrow \mathbb{RP}^{\binom{n}{k}-1}$ of the Grassmannian into the projective space. The image of the Grassmannian Gr_{kn} under the Plücker embedding is the algebraic subvariety in $\mathbb{RP}^{\binom{n}{k}-1}$ given by the *Grassmann-Plücker relations:*

$$\Delta_{(i_1,\dots,i_k)} \cdot \Delta_{(j_1,\dots,j_k)} = \sum_{s=1}^{k} \Delta_{(j_s,i_2,\dots,i_k)} \cdot \Delta_{(j_1,\dots,j_{s-1},i_1,j_{s+1},\dots,j_k)},$$

for any $i_1,\dots,i_k,j_1,\dots,j_k \in [n]$. Here we assume that $\Delta_{(i_1,\dots,i_k)}$ (labelled by an ordered sequence rather than a subset) equals to $\Delta_{\{i_1,\dots,i_k\}}$ if $i_1 < \cdots < i_k$ and $\Delta_{(i_1,\dots,i_k)} = (-1)^{\mathrm{sign}(w)} \Delta_{(i_{w(1)},\dots,i_{w(k)})}$ for any permutation $w \in S_k$.

2.3. **Matroid strata.** An element in the Grassmannian Gr_{kn} can also be understood as a collection of n vectors $v_1,\dots,v_n \in \mathbb{R}^k$ spanning the space $\mathbb{R}^k$, modulo the simultaneous action of GL_k on the vectors. The vectors v_i are the columns of a $k \times n$-matrix A that represents the element of the Grassmannian.

Recall that a *matroid* of *rank* k on the set $[n]$ is a nonempty collection $\mathcal{M} \subseteq \binom{[n]}{k}$ of k-element subsets in $[n]$, called *bases* of $\mathcal{M}$, that satisfies the *exchange axiom:*

For any $I, J \in \mathcal{M}$ and $i \in I$ there exists $j \in J$ such that $(I \setminus \{i\}) \cup \{j\} \in M$.

An element $V \in Gr_{kn}$ of the Grassmannian represented by a $k \times n$-matrix A gives the matroid $\mathcal{M}_V$ whose bases are the k-subsets $I \subset [n]$ such that $\Delta_I(A) \neq 0$, or equivalently, I is a base of $\mathcal{M}_V$ whenever $\{v_i \mid i \in I\}$ is a basis of $\mathbb{R}^k$. This collection of bases satisfies the exchange axiom because, if the left-hand side in a Grassmann-Plücker relation is nonzero, then at least one term in the right-hand side is nonzero.

The Grassmannian Gr_{kn} has a subdivision into *matroid strata,* also known as *Gelfand-Serganova strata,* $S_\mathcal{M}$ labelled by some matroids $\mathcal{M}$:

$$S_\mathcal{M} := \{V \in Gr_{kn} \mid \mathcal{M}_V = \mathcal{M}\}$$

In other words, the elements of the stratum $S_\mathcal{M}$ are represented by matrices A such that $\Delta_I(A) \neq 0$ if and only if $I \in \mathcal{M}$. The matroids $\mathcal{M}$ with nonempty strata $S_\mathcal{M}$ are called *realizable* over $\mathbb{R}$. The geometrical structure of the matroid strata $S_\mathcal{M}$ can be highly nontrivial. Mnëv [Mnë] showed that they can be as complicated as essentially any algebraic variety.

Note that, for an element V of the Schubert cell Ω_λ, the subset $I(\lambda)$ is exactly the lexicographically minimal base of the matroid $\mathcal{M}_V$. This fact it transparent when V is represented by a matrix in I-echelon form. In other words, the Schubert cells can also be described as

$$\Omega_\lambda = \{V \in Gr_{kn} \mid I(\lambda) \text{ is the lexicographically minimal base of } \mathcal{M}_V\}.$$

This implies that the decomposition of Gr_{kn} into matroid strata $S_\mathcal{M}$ is a finer subdivision than the decomposition into Schubert cells Ω_λ.

The Schubert decomposition depends on a choice of ordering of the coordinates in $\mathbb{R}^n$. The symmetric group S_n acts on $\mathbb{R}^n$ by permutations of the coordinates. For a permutation $w \in S_n$, let $\Omega_\lambda^w := w(\Omega_\lambda)$ be the permuted Schubert cell. In other words, the cell Ω_λ^w is the set of elements $V \in Gr_{kn}$ such that $I(\lambda)$ is the lexicographically minimal base of $\mathcal{M}_V$ with respect to the total order $w(1) < w(2) < \cdots < w(n)$ of the set $[n]$.

Remark 2.1. The decomposition of the Grassmannian Gr_{kn} into matroid strata $S_\mathcal{M}$ is the common refinement of the $n!$ permuted Schubert decompositions $Gr_{kn} = \bigcup_{\lambda \subseteq (n-k)^k} \Omega_\lambda^w$, for $w \in S_n$; see [GGMS]. Indeed, if we know the lexicographically minimal base in $\mathcal{M}_V$ with respect any total order on $[n]$ then we can determine all bases of $\mathcal{M}_V$ and thus determine the matroid stratum containing the element $V \in Gr_{kn}$.

It will be convenient for us use local affine coordinates on the Grassmannian. Let us pick a k-subset $I \subset [n]$. Let A be a $k \times n$-matrix that represents an element in Gr_{kn} such that $\Delta_I(A) \neq 0$, that is I is a base of the corresponding matroid. Then $A' = (A_I)^{-1} A$ is the unique representative the left coset of $GL_k \cdot A$ such that A'_I is the identity matrix. Then matrix elements of A' located in columns indexed $j \notin I$ give local affine coordinates on Gr_{kn}. In other words, we have the rational isomorphism

$$Gr_{kn} \setminus \{\Delta_I = 0\} \simeq \mathbb{R}^{k(n-k)}.$$

In the case when I is the lexicographically minimal base, the matrix A' is exactly the representative in echelon form.

3. Totally nonnegative Grassmannian

A matrix is called *totally positive* (resp., *totally nonnegative*) if all its minors of all sizes are strictly positive (resp., nonnegative). In this section we discuss analogues of these classical notions for the Grassmannian.

Definition 3.1. Let us define the *totally nonnegative Grassmannian* $Gr_{kn}^{\mathrm{tnn}} \subset Gr_{kn}$ as the quotient $Gr_{kn}^{\mathrm{tnn}} = \mathrm{GL}_k^+ \backslash \mathrm{Mat}_{kn}^{\mathrm{tnn}}$, where $\mathrm{Mat}_{kn}^{\mathrm{tnn}}$ is the set of real $k \times n$-matrices A of rank k with nonnegative *maximal* minors $\Delta_I(A) \geq 0$ and GL_k^+ is the group of $k \times k$-matrices with positive determinant. The *totally positive Grassmannian* $Gr_{kn}^{\mathrm{tp}} \subset Gr_{kn}^{\mathrm{tnn}}$ is the subset of Gr_{kn} whose elements can be represented by $k \times n$-matrices with strictly positive maximal minors $\Delta_I(A) > 0$.

For example, the totally positive Grassmannian contains all $k \times n$-matrices $A = (x_j^i)$ with $x_1 < \cdots < x_n$, because any maximal minor $\Delta_I(A)$ of such matrix is a positive Vandermonde determinant. Clearly, Gr_{kn}^{tp} is an open subset in Gr_{kn} and Gr_{kn}^{tnn} is a closed subset in Gr_{kn} of dimension $k(n-k) = \dim Gr_{kn}$.

Definition 3.2. Let us define *totally nonnegative Grassmann cells* $S_{\mathcal{M}}^{\mathrm{tnn}}$ in Gr_{kn}^{tnn} as the intersections $S_{\mathcal{M}}^{\mathrm{tnn}} = S_{\mathcal{M}} \cap Gr_{kn}^{\mathrm{tnn}}$ of the matroid strata $S_{\mathcal{M}}$ with the totally nonnegative Grassmannian, i.e.,

$$S_{\mathcal{M}}^{\mathrm{tnn}} = \{GL_k^+ \cdot A \in Gr_{kn}^{\mathrm{tnn}} \mid \Delta_I(A) > 0 \text{ for } I \in \mathcal{M}, \text{ and } \Delta_I(A) = 0 \text{ for } I \notin \mathcal{M}\}.$$

Let us say that a matroid $\mathcal{M}$ is *totally nonnegative* if the cell $S_{\mathcal{M}}^{\mathrm{tnn}}$ is nonempty.

Note that the totally positive Grassmannian Gr_{kn}^{tp} is just the top dimensional cell $S_{\mathcal{M}}^{\mathrm{tnn}} \subset Gr_{kn}^{\mathrm{tnn}}$, that is the cell corresponding to the complete matroid $\mathcal{M} = \binom{[n]}{k}$.

Remark 3.3. Clearly, the notion of total positivity is not invariant under permutations of the coordinates in $\mathbb{R}^n$, and the class of totally nonnegative matroids is not preserved under permutations of the elements. This notion does however have some symmetries. For a $k \times n$-matrix $A = (v_1, \ldots, v_k)$ with the column vectors $v_i \in \mathbb{R}^k$, let $A' = (v_2, \ldots, v_n, (-1)^{k-1} v_1)$ be the matrix obtained from A by the cyclic shift of the columns and then multiplying the last column by $(-1)^{k-1}$. Note that $\Delta_I(A) = \Delta_{I'}(A')$ where I' is the cyclic shift of the subset I. Thus A is totally nonnegative (resp., totally positive) if and only if A' is totally nonnegative (resp., totally positive). This gives an action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on the sets Gr_{kn}^{tnn} and Gr_{kn}^{tp}. This also implies that cyclic shifts of elements in $[n]$ preserve the class of totally nonnegative matroids on $[n]$.

Example 3.4. For $n = 4$ and $k = 2$, there are only three rank 2 matroids on $[4]$ which are not totally nonnegative: $\mathcal{M} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\}, \mathcal{M} \cup \{\{1,3\}\}$, $\mathcal{M} \cup \{\{2,4\}\}$. This set of matroids is closed under cyclic shifts of $[4]$.

Interestingly, the totally nonnegative Grassmann cells $S_{\mathcal{M}}^{\mathrm{tnn}}$ have a much simpler geometric structure than the matroid strata $S_{\mathcal{M}}$.

Theorem 3.5. *Each totally nonnegative Grassmann cell $S_{\mathcal{M}}^{\mathrm{tnn}}$ is homeomorphic to an open ball of appropriate dimension. The decomposition of the totally nonnegative Grassmannian Gr_{kn}^{tnn} into the union of the cells $S_{\mathcal{M}}^{\mathrm{tnn}}$ is a CW-complex.*

In Section 6 we will explicitly construct a rational parametrization for each cell $S_{\mathcal{M}}^{\text{tnn}}$, i.e., an isomorphism between the space $\mathbb{R}_{>0}^d$ and $S_{\mathcal{M}}^{\text{tnn}}$; see Theorem 6.5. In Section 18 we will describe how the cells are glued to each other.

This next conjecture follows a similar conjecture by Fomin-Zelevinsky [FZ1] on double Bruhat cells.

Conjecture 3.6. *The CW-complex formed by the cells $S_{\mathcal{M}}^{\text{tnn}}$ is regular. The closure of each cell is homeomorphic to a closed ball of appropriate dimension.*

According to Remark 2.1, the matroid stratification of the Grassmannian is the common refinement of $n!$ Schubert decompositions. For the totally nonnegative part of the Grassmannian it is enough to take just n Schubert decompositions.

Theorem 3.7. *The decomposition of Gr_{kn}^{tnn} into the cells $S_{\mathcal{M}}^{\text{tnn}}$ is the common refinement of the n Schubert decompositions $Gr_{kn}^{\text{tnn}} = \bigcup_{\lambda\subseteq(n-k)^k}(\Omega_\lambda^w \cap Gr_{kn}^{\text{tnn}})$, where w run over cyclic shifts $w : i \mapsto i+k \pmod n$, for $k \in [n]$.*

This theorem will follow from Theorem 17.2 in Section 17.

Lusztig [Lus1, Lus2, Lus3] developed general theory of total positivity for a reductive group G using canonical bases. He defined the totally nonnegative part $(G/P)_{\geq 0}$ of any generalized partial flag manifold G/P and conjectured that it is made up of cells. This conjecture was proved by Rietsch [Rie1, Rie2]. Marsh-Rietsch [MR] gave a simpler proof and constructed parametrization of the totally nonnegative cells in $(G/B)_{\geq 0}$. This general approach to totally positivity agrees with our "elementary" approach.

Theorem 3.8. *In case of the Grassmannian Gr_{kn}, Rietsch's cell decomposition coincides with the decomposition of Gr_{kn}^{tnn} into the cells $S_{\mathcal{M}}^{\text{tnn}}$.*

I thank Xuhua He and Konni Rietsch for the following explanation. According to [MR, Proposition 12.1], Rietsch cells are given by conditions $\Delta_I > 0$ and $\Delta_J = 0$ for *some* minors. Actually, the paper [MR] concerns with the case of G/B, but the case of G/P (which includes the Grassmannian) can be obtained by applying the projection map $G/B \to G/P$, as it was explained in [Rie1]. It follows that our cell decomposition of Gr_{kn}^{tnn} into the cells $S_{\mathcal{M}}^{\text{tnn}}$ is a *refinement* of Rietsch's cell decomposition. In Section 19 we will construct a combinatorial bijection between objects that label our cells and objects that label Rietsch's cells, which will prove Theorem 3.8.

Let us show how total positivity on the Grassmannian is related to the classical notion of total positivity of matrices. For a $k \times n$-matrix A such that the square submatrix $A_{[k]}$ in the first k columns is the identity matrix $A_{[k]} = Id_k$, define $\phi(A) = B$, where $B = (b_{ij})$ is the $k \times (n-k)$-matrix with entries $b_{ij} = (-1)^{k-j} a_{i+k,j}$:

$$\phi : \begin{pmatrix} 1 & \cdots & 0 & 0 & a_{1,k+1} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & a_{k-1,k+1} & \cdots & a_{k-1,n} \\ 0 & \cdots & 0 & 1 & a_{k,k+1} & \cdots & a_{kn} \end{pmatrix} \mapsto \begin{pmatrix} \pm a_{1,k+1} & \cdots & \pm a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{k-1,k+1} & \cdots & -a_{k-1,n} \\ a_{k,k+1} & \cdots & a_{kn} \end{pmatrix}.$$

Let $\Delta_{I,J}(B)$ denote the minor of matrix B (not necessarily maximal) in the row set I and the column set J. By convention, we assume that $\Delta_{\emptyset,\emptyset}(B) = 1$.

Lemma 3.9. *Suppose that $B = \phi(A)$. There is a correspondence between the maximal minors of A and all minors of B such that each maximal minor of A*

equals to the corresponding minor of B*. Explicitly,* $\Delta_{I,J}(B) = \Delta_{([k]\setminus I)\cup\tilde{J}}(A)$*, where* $\tilde{J}$ *is obtained by increasing all elements in* J *by* k*.*

Proof. Exercise for the reader. □

Note that matrices A with $A_{[k]} = Id_k$ are representatives (in echelon form) of elements of the top Schubert cell $\Omega_{(n-k)^k} \subset Gr_{kn}$, i.e., the set of elements in the Grassmannian with nonzero first Plücker coordinate $\Delta_{[k]} \neq 0$. Thus ϕ gives the isomorphism $\phi : \Omega_{(n-k)^k} \to \mathrm{Mat}_{k,n-k}$.

Proposition 3.10. *The map* ϕ *induces the isomorphism between* $\Omega_{(n-k)^k} \cap Gr_{kn}^{\mathrm{tnn}}$ *and the set of classical totally nonnegative* $k \times (n-k)$*-matrices, i.e., matrices with nonnegative minors of all sizes. This map induces the isomorphism between each totally nonnegative cell* $S_{\mathcal{M}}^{\mathrm{tnn}} \subset \Omega_{(n-k)^k} \cap Gr_{kn}^{\mathrm{tnn}}$ *and a set of* $k \times (n-k)$*-matrices given by prescribing some minors to be positive and the remaining minors to be zero. In particular, it induces the isomorphism between the totally positive part* Gr_{kn}^{tp} *of the Grassmannian and the classical set of all totally positive* $k \times (n-k)$*-matrices.*

Remark 3.11. Fomin-Zelevinsky [FZ1] investigated the decomposition of the totally nonnegative part of GL_k into cells, called the *double Bruhat cells.* These cells are parametrized by pairs of permutations in S_k. The partial order by containment of closures of the cells is isomorphic to the direct product of two copies of the Bruhat order on S_k. The map ϕ induces the isomorphism between the totally nonnegative part of the Grassmannian $Gr_{k,2k}$ such that $\Delta_{[k]} \neq 0$ and $\Delta_{[k+1,2k]} \neq 0$ and the totally nonnegative part of GL_k. Moreover, it gives isomorphisms between the double Bruhat cells in GL_k and *some* totally nonnegative cells $S_{\mathcal{M}}^{\mathrm{tnn}} \subset Gr_{k,2k}^{\mathrm{tnn}}$; namely, the cells such that $[k], [k+1, 2k] \in \mathcal{M}$.

In this paper we will extend Fomin-Zelevinsky's [FZ1] results on (type A) double Bruhat cells to all totally nonnegative cells $S_{\mathcal{M}}^{\mathrm{tnn}}$ in the Grassmannian. We will see that these cells have a rich combinatorial structure and lead to new combinatorial objects.

4. Planar networks

Definition 4.1. A *planar directed graph* G is a directed graph drawn inside a disk (and considered modulo homotopy). We will assume that G has finitely many vertices and edges. We allow G to have loops and multiple edges. We will assume that G has n *boundary vertices* on the boundary of the disk labelled $b_1, \dots, b_n$ clockwise. The remaining vertices, called the *internal vertices,* are located strictly inside the disk. We will always assume that each boundary vertex b_i is either a *source* or a *sink.* Even if b_i is an *isolated* boundary vertex, i.e., a vertex not incident to any edges, we will assign b_i to be a source or a sink. A *planar directed network* $N = (G, x)$ is a planar directed graph G as above together with *strictly positive* real weights $x_e > 0$ assigned to all edges e of G.

For such network N, the *source set* $I \subset [n]$ and the *sink set* $\bar{I} := [n] \setminus I$ of N are the sets such that b_i, $i \in I$, are the sources of N (among the boundary vertices) and the b_j, $j \in \bar{I}$, are the boundary sinks.

If the network N is acyclic, that is it does not have closed directed paths, then, for any $i \in I$ and $j \in \bar{I}$, we define the *boundary measurement* M_{ij} as the finite sum

$$M_{ij} := \sum_{P\,:\, b_i \to b_j} \prod_{e \in P} x_e,$$

where the sum is over all directed paths P in N from the boundary source b_i to the boundary sink b_j, and the product is over all edges e in P.

If the network is not acyclic, we have to be more careful because the above sum might be infinite, and it may not converge. We will need the following definition.

For a path P from a boundary vertex b_i to a boundary vertex b_j, we define its *winding index,* as follows. We may assume that all edges of the network are given by smooth curves; thus the path P is given by a continuous piecewise-smooth curve. We can slightly modify the path and smoothen it around each junction, so that it is given by a smooth curve $f : [0, 1] \to \mathbb{R}^2$, and furthermore make the initial tangent vector $f'(0)$ to have the same direction as the final tangent vector $f'(1)$. We can now define the winding index $wind(P) \in \mathbb{Z}$ of the path P as the signed number of full $360°$ turns the tangent vector $f'(t)$ makes as we go from b_i to b_j (counting counterclockwise turns as positive); see example in Figure 4.1. Similarly, we define the winding index $wind(C)$ for a closed directed path C in the graph.

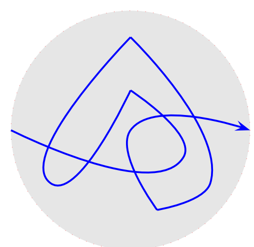


FIGURE 4.1. A path P with the winding index $wind(P) = -1$

Let us give a recursive combinatorial procedure for calculation of the winding index for a path P with vertices $v_1, v_2, \ldots, v_l$. If the path P has no self-intersections, i.e., all vertices v_i in P are distinct, then $wind(P) = 0$. Also for a counterclockwise (resp., clockwise) closed path C without self-intersections, we have $wind(C) = 1$ (resp., $wind(C) = -1$).

Suppose that P has at least one self-intersection, that is $v_i = v_j = v$ for $i < j$. Let C be the closed segment of P with the vertices $v_i, v_{i+1}, \ldots, v_j$, and let P' be the path P with erased segment C, i.e., P' has the vertices. $v_1, \ldots, v_i, v_{j+1}, \ldots, v_l$.

Consider the four edges $e_1 = (v_{i-1}, v_i)$, $e_2 = (v_i, v_{i+1})$, $e_3 = (v_{j-1}, v_j)$, $e_4 = (v_j, v_{j+1})$ in the path P, which are incident to the vertex v (the edges e_1 and e_3 are incoming, and the edges e_2 and e_4 are outgoing). Define the number $\epsilon = \epsilon(e_1, e_2, e_3, e_4) \in \{-1, 0, 1\}$, as follows. If the edges are arranged as e_1, e_2, e_3, e_4 clockwise, then set $\epsilon = 1$; if the edges are arranged as e_1, e_2, e_3, e_4 counterclockwise, then set $\epsilon = -1$; otherwise set $\epsilon = 0$. In particular, if some of the edges e_1, e_2, e_3, e_4 are the same, then $\epsilon = 0$. Informally, $\epsilon = \pm 1$, if the path P does not cross but rather touches itself at the vertex v.

The following claim is straightforward from the definitions.

Lemma 4.2. *We have $wind(P) = wind(P') + wind(C) + \epsilon$.*

If C is a *cycle,* that is a closed path without self-intersections, then by Lemma 4.2

$$wind(P) = \begin{cases} wind(P')+1 & \text{if } C \text{ is a counterclockwise cycle and } \epsilon = 0;\\ wind(P')-1 & \text{if } C \text{ is a clockwise cycle and } \epsilon = 0;\\ wind(P') & \text{if } \epsilon = \pm 1.\end{cases}$$

If $\epsilon = 0$, then we say that C is an *essential cycle.* We can now determine the winding index of P by repeatedly erasing cycles in P (say, always erasing the cycle C with the smallest possible index j) until we get a path without self-intersections. Thus we express the winding index in terms of the erased cycles as $wind(P) := \#\{\text{counterclockwise essential cycles}\} - \#\{\text{clockwise essential cycles}\}$.

Remark 4.3. Note that in general the number of cycles in P is not well-defined. Indeed the number of cycles that we need to erase until we get a path without self-intersections may depend on the order in which we erase the cycles. However the winding index $wind(P)$ is a well-defined invariant of a path in a planar graph. For example, for the path shown on Figure 4.1, we can erase a counterclockwise cycle and then two clockwise cycles. On the other hand, for the same path, we can also erase just one big clockwise cycle to get a path without self-intersections.

Let us now return to boundary measurements. Let N be a planar directed network with graph G as above, which is now allowed to have cycles. Let us assume for a moment that the weights x_e of edges in N are formal variables. For a path P in G with the edges $e_1, \dots, e_l$ (possibly with repetitions), we will write $x_P := x_{e_1} \cdots x_{e_l}$. For a source b_i, $i \in I$, and a sink b_j, $j \in \bar{I}$, we define the *formal boundary measurement* M_{ij}^{form} as the formal power series in the x_e

$$M_{ij}^{\mathrm{form}} := \sum_{P\,:\,b_i \to b_j} (-1)^{wind(P)} x_P\,, \tag{4.1}$$

where the sum is over all directed paths P in N from b_i to b_j.

Recall that a *subtraction-free rational expression* is an expression with positive integer coefficients that can be written with the operations of addition, multiplication, and division (but subtraction is strictly forbidden), or equivalently, it is an expression that can be written as a quotient of two polynomials with positive coefficients. For example, $\frac{x+y/x}{z^2+25y/(x+t)} = \frac{(x^2+y)(x+t)}{xz^2(x+t)+25xy}$ is subtraction-free. We will mostly work with subtraction-free expressions in some variables $x_1, \dots, x_m$ that give well-defined functions on $\mathbb{R}_{\geq 0}^m$. In other words, such expressions can be written as quotients of two positive polynomials such that the denominator has a strictly positive constant term. For example, the above expression does not have this property.

Lemma 4.4. *The formal power series M_{ij}^{form} sum to subtraction-free rational expressions in the variables x_e. These expressions give well-defined functions on $\mathbb{R}_{\geq 0}^{E(G)}$, where $E(G)$ is the set of edges of G.*

The lemma implies that the M_{ij}^{form} sum to rational expressions where we can specialize the formal variables x_e to any nonnegative values and never get zero in the denominator.

Proof. Let us fix a boundary source b_i and a boundary sink b_j. Without loss of generality, we may assume that degrees of the boundary vertices in G are at most 1.

Let us prove the claim by induction on the number of *internal edges* in the graph G, that is the edges connecting pairs of internal vertices. Let u be the internal vertex connected with the boundary vertex b_i by an edge. If u is not incident to an internal edge, then the claim is trivial. Otherwise, we pick an internal edge $e = (v, w)$ incident to u (that is $v = u$ or $w = u$) which is "close" to the boundary of the disk. By this we mean that e is an edge of a face f of the planar graph G attached to the boundary of the disk. Let us add two new boundary vertices b' and b'' on the boundary of the disk and connect them with the vertices v and w by the directed edges $e' = (v, b')$ and $e'' = (b'', w)$ so that the two new edges e' and e'' are inside of the face f. (So e' and e'' do not cross with any edges of G.) Let G' be the planar directed graph obtained from G by removing the edge e and adding the edges e' and e''. Note that b' is a sink and b'' is a source of G'.

The graph G' has less internal edges than the graph G. By induction, all formal boundary measurement for G' sum to subtraction-free rational expressions defined on $\mathbb{R}_{\geq 0}^{E(G')}$. We can specialize some of the edge variables in these expressions to nonnegative numbers and get valid subtraction-free expressions.

Let us show how to express the formal boundary measurements of G in terms of the formal boundary measurements of the graph G'. We will use the notation $M'_{a,b}$ for the formal boundary measurements of the graph G' specialized at $x_{e'} = x_{e''} = 1$, where $a, b \in \{b_i, b_j, b', b''\}$.

Every directed path P in the graph G from the boundary vertex b_i to the boundary vertex b_j breaks into several segments $P_1, e, P_2, e, \ldots, e, P_s$, where the path P_1 is from b_i to v, the paths $P_2, \ldots, P_{s-1}$ are from w to v, and the path P_s is from w to b_j. (If $s = 1$, then $P_1 = P_s = P$ is a path from b_i to b_j, which does not contain the edge e.) Clearly, for $s \geq 2$, the path P_1 corresponds to a path P'_1 in the graph G' from b_i to b', the paths $P_2, \ldots, P_{s-1}$ correspond to paths $P'_2, \ldots, P'_{s-1}$ in G' from b'' to b', and the path P_s corresponds to a path P'_s in G' from b'' to b_j. For $s = 1$, the path P corresponds to a path P' in G' from b_i to b_j.

Clearly, if $s = 1$, then $wind(P) = wind(P')$. For $s \geq 2$, we need to be more careful with winding indices of these paths. Let us consider two cases.

Case 1: $v = u$. We may assume that the boundary vertices are arranged as b_i, b', b'', b_j clockwise. In this case $wind(P) = wind(P'_1) + \cdots + wind(P'_s) - (s - 2)$. Thus M_{ij}^{form} equals

$$M'_{b_i,b_j} + \sum_{s\geq 2} (-1)^s \, M'_{b_i,b'} \, x_e \, (M'_{b'',b'} \, x_e)^{s-2} \, M'_{b'',b_j} = M'_{b_i,b_j} + \frac{M'_{b_i,b'} \, x_e \, M'_{b'',b_j}}{1 + x_e \, M'_{b'',b'}}.$$

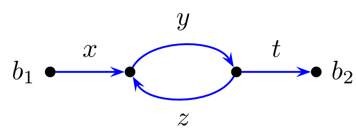
Case 2: $w = u$. We may assume that boundary vertices are arranged as b_i, b'', b', b_j clockwise. In this case $wind(P) = wind(P'_1) + \cdots + wind(P'_s) - (s - 1)$. Let $f = (b_i, v)$ be the edge incident to b_i. Note that in this case, $M'_{b_i,\,a} = x_f M'_{b'',a}$ for any a. We have

$$\begin{aligned} M_{ij}^{\text{form}} &= M'_{b_i,b_j} - \sum_{s\geq 2} (-1)^s \, M'_{b_i,b'} \, x_e \, (M'_{b'',b'} \, x_e)^{s-2} \, M'_{b'',b_j} = \\ &= x_f \, M'_{b'',b_j} - \sum_{s\geq 2} (-1)^s \, x_f \, (M'_{b'',b'} \, x_e)^{s-1} \, M_{b'',b_j} = \frac{x_f \, M'_{b'',b_j}}{1 + x_e \, M'_{b'',b'}}. \end{aligned}$$

In both cases we expressed M_{ij}^{form} as subtraction-free rational expressions in terms of the $M'_{a,b}$. Notice that the denominators in these expressions have the constant term 1. Thus, whenever x_e and $M'_{b'',b'}$ are nonnegative, the denominators are strictly positive and, in particular, non-zero. By induction, we conclude that M_{ij}^{form} is a subtraction-free rational function well-defined on $\mathbb{R}_{\geq 0}^{E(G)}$. □

Definition 4.5. We can now define the *boundary measurements* M_{ij} as the specializations of the formal boundary measurements M_{ij}^{form}, written as subtraction-free expressions, when we assign the x_e to be the positive real weights of edges e in the network N. Thus the boundary measurements M_{ij} are well-defined nonnegative real numbers for an arbitrary network.

Example 4.6. For the network



we have $M_{12}^{\mathrm{form}} = xyt - xyzyt + xyzyzyt - \cdots = xyt/(1+yz)$, which is a subtraction-free rational expression. If all weights of edges are $x = y = z = t = 1$, then the boundary measurement is $M_{12} = 1/(1+1) = 1/2$.

Inverse Boundary Problem. *What information about a planar directed network can be recovered from the collection of boundary measurements M_{ij}? How to recover this information? Describe all possible collections of boundary measurements. Describe transformation of networks that preserve the boundary measurements.*

Let us describe the *gauge transformations* of the weights x_e. Pick a collection of positive real numbers $t_v > 0$, for each internal vertex v in N; and also assume that $t_{b_i} = 1$ for each boundary vertex b_i. Let N' be the network with the same directed graph as the network N and with the weights

$$x'_e = x_e\, t_u {t_v}^{-1}, \tag{4.2}$$

for each directed edge $e = (u, v)$. In other words, for each internal vertex v we multiply by t_v the weights of all edges outgoing from v, divide by t_v the weights of all edges incoming to v. Then the network N' has the same boundary measurements as the network N. Indeed, for a directed path P between two boundary vertices and for an internal vertex v, we have to divide the weight $\prod_{e\in P} x_e$ of P by t_v every time when P enters v and multiply it by t_v every time when P leaves v.

We will see that there are also some local structure transformations of networks that preserve the boundary measurements.

Let us now describe the set of all possible collections of boundary measurements. For a network N with k boundary sources b_i, $i \in I$, and $n-k$ boundary sinks b_j, $j \in \bar{I}$, it will be convenient to encode the $k(n-k)$ boundary measurements M_{ij}, $i \in I$, $j \in \bar{I}$, as a certain point in the Grassmannian Gr_{kn}. Recall that $\Delta_J(A)$ is the maximal minor of a matrix A in the column set J. The collection of all Δ_J, for k-subsets $J \subset [n]$, form projective Plücker coordinates on Gr_{kn}.

Definition 4.7. Let Net_{kn} be the set of planar directed networks with k boundary sources and $n-k$ boundary sinks. Define the *boundary measurement map*

$$Meas : Net_{kn} \to Gr_{kn},$$

as follows. For a network $N \in Net_{kn}$ with the source set I and with the boundary measurements M_{ij}, the point $Meas(N) \in Gr_{kn}$ is given in terms of its Plücker coordinates $\{\Delta_J\}$ by the conditions that $\Delta_I \neq 0$ and

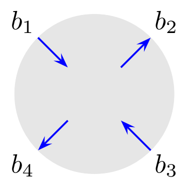
$$M_{ij} = \Delta_{(I\setminus\{i\})\cup\{j\}}/\Delta_I \text{ for any } i \in I \text{ and } j \in \bar{I}.$$

More explicitly, if $I = \{i_1 < \cdots < i_k\}$, then the point $Meas(N) \in Gr_{kn}$ is represented by the *boundary measurement matrix* $A(N) = (a_{ij}) \in \mathrm{Mat}_{kn}$ such that

(1) The submatrix $A(N)_I$ in the column set I is the identity matrix Id_k.
(2) The remaining entries of $A(N)$ are $a_{rj} = (-1)^s M_{i_r,j}$, for $r \in [k]$ and $j \in \bar{I}$, where s is the number of elements of I strictly between i_r and j.

Note that the choice of signs of entries in $A(N)$ ensures that $\Delta_{(I\setminus\{i\})\cup\{j\}}(A(N)) = M_{ij}$, for $i \in I$ and $j \in \bar{I}$. Clearly, we have $\Delta_I(A(N)) = 1$.

Example 4.8. For a network N with four boundary vertices, with the source set $I = \{1, 3\}$, and the sink set $\bar{I} = \{2, 4\}$, we have

$N =$ 

$$A(N) = \begin{pmatrix} 1 & M_{12} & 0 & -M_{14} \\ 0 & M_{32} & 1 & M_{34} \end{pmatrix}.$$

In this case, we have $M_{12} = \frac{\Delta_{23}}{\Delta_{13}}$, $M_{14} = \frac{\Delta_{24}}{\Delta_{13}}$, $M_{32} = \frac{\Delta_{12}}{\Delta_{13}}$, $M_{34} = \frac{\Delta_{14}}{\Delta_{13}}$.

The following two results establish a relationship between networks and total positivity on the Grassmannian.

Theorem 4.9. *The image of the boundary measurement map Meas is exactly the totally nonnegative Grassmannian:*

$$Meas(Net_{kn}) = Gr_{kn}^{\mathrm{tnn}}.$$

This theorem will follow from Corollary 5.6 and Theorem 6.5.

Definition 4.10. Let us say that a *subtraction-free rational parametrization* of a cell $S_{\mathcal{M}}^{\mathrm{tnn}} \subset Gr_{kn}^{\mathrm{tnn}}$ is an isomorphism $f : \mathbb{R}_{>0}^d \to S_{\mathcal{M}}^{\mathrm{tnn}}$ such that

(1) The quotient of any two Plücker coordinates Δ_J/Δ_I, $I, J \in \mathcal{M}$, of the point $f(x_1, \ldots, x_d) \in S_{\mathcal{M}}^{\mathrm{tnn}}$ can be written as a subtraction-free rational expression in the usual coordinates x_i on $\mathbb{R}_{>0}^d$.
(2) For the inverse map f^{-1}, the x_i can be written as subtraction-free rational expressions in terms of the Plücker coordinates Δ_J.

Moreover, we say that such subtraction-free parametrization is *I-polynomial* for given $I \in \mathcal{M}$, if the quotients Δ_J/Δ_I, for $J \in \mathcal{M}$, are given by polynomials in the x_i with nonnegative integer coefficients.

Let G be a planar directed graph with the set of edges $E(G)$. Clearly, we can identify the set of all networks on the given graph G with the set $\mathbb{R}_{>0}^{E(G)}$ of positive real-valued functions on $E(G)$. The boundary measurement map induces the map

$$Meas_G : \mathbb{R}_{>0}^{E(G)}/\{\text{gauge transformations}\} \to Gr_{kn}. \tag{4.3}$$

Theorem 4.11. *For a planar directed graph G, the image of the map $Meas_G$ is a certain totally nonnegative Grassmann cell $S_{\mathcal{M}}^{\mathrm{tnn}}$.*

For a cell $S_{\mathcal{M}}^{\mathrm{tnn}}$, let $I_i \subset [n]$ as the lexicographically minimal base of the matroid $\mathcal{M}$ with respect to the linear order $i < i+1 < \cdots < n < 1 < \cdots < i-1$ on $[n]$. In particular, $I_1 = I(\lambda)$, whenever $S_{\mathcal{M}}^{\mathrm{tnn}} \subset \Omega_\lambda$.

Theorem 4.12. *For any cell $S_{\mathcal{M}}^{\mathrm{tnn}}$, one can find a graph G such that the map $Meas_G$ is a subtraction-free rational parametrization of this cell. Moreover, for $i = 1, \dots, n$, there is an acyclic planar directed graph G with the source set I_i such that $Meas_G$ is an I_i-polynomial parametrization of the cell $S_{\mathcal{M}}^{\mathrm{tnn}}$.*

This theorem will follow from Theorem 6.5.

In the next section we will prove that $Meas(Net_{kn}) \subseteq Gr_{kn}^{\mathrm{tnn}}$. In other words, we will prove that all maximal minors of the boundary measurement matrix $A(N)$ are nonnegative.

5. Loop-erased walks

In this section we generalize the well-known *Lindström lemma* to suit our purposes. The main result of the section is the claim that every maximal minor of the matrix $A(N)$ is given by a subtraction-free rational expression in the x_e. Since our graphs may not be acyclic, we will follow Fomin's approach from the work on *loop-erased walks* [Fom], which extends the Lindström lemma to non-acyclic graphs. The sources and sinks in our graphs may be interlaced with each other, which gives an additional complication. Also one needs to take an extra care with winding indices.

For two k-subsets $I, J \subset [n]$, let $K = I \setminus J$ and $L = J \setminus I$. Then $|K| = |L|$. For a bijection $\pi : K \to L$, we say that a pair of indices (i, j), where $i < j$ and $i, j \in K$, is a *crossing,* an *alignment,* or a *misalignment* of π, if the two chords $[b_i, b_{\pi(i)}]$ and $[b_j, b_{\pi(j)}]$ are arranged with respect to each other as shown on Figure 5.1. Define the *crossing number* $xing(\pi)$ of π as the number of crossings of π.

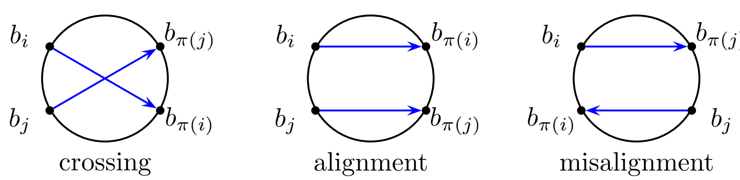


Figure 5.1. Crossings, alignments, and misalignments

Lemma 5.1. *Let I, J be two k-element subsets in $[n]$, $K = I \setminus J$ and $L = J \setminus I$. Also let $r = |K| = |L|$. Then the following identity holds for the Plücker coordinates in Gr_{kn}*

$$\Delta_J \cdot \Delta_I^{r-1} = \sum_{\pi : K \to L} (-1)^{xing(\pi)} \prod_{i \in K} \Delta_{(I \setminus \{i\}) \cup \{\pi(i)\}},$$

where the sum is over all $r!$ bijections $\pi : K \to L$.

Note that, in the case $r = 2$, this identity is equivalent to a 3-term Grassmann-Plücker relation.

Proof. Let us prove this identity for maximal minors $\Delta_J(A)$ of a matrix A. We first show that the identity is invariant under permutations of columns of the matrix. Let us switch two adjacent rows of A with indices a and $a+1$ and correspondingly modify the subsets I and J. Then a minor Δ_M switches its sign if $a, a+1 \in M$; and otherwise the minor and does not change. Also note that the crossing number $xing(\pi)$ changes by ± 1 if $a, a+1 \in K \cup L$ and $\pi^{\pm 1}(a) \neq a+1$; and otherwise $xing(\pi)$ does not change. Considering several cases according to which of the subsets K, L, $I \cap J$, or $[n] \setminus (I \cup J)$ contain the elements a and $a+1$, we verify in all cases that both sides of the identity make the same switch of sign. Using this invariance under permutations of columns we can reduce the problem to the case when $I = [k]$ and $J = [k-r] \cup [k+1, k+r]$. Let us assume that $\Delta_I \neq 0$. Multiplying the matrix A by $(A_I)^{-1}$ on the left, we reduce the problem to the case when A_I is the identity matrix. In this case, $\Delta_I(A) = 1$ and $\Delta_J(A)$ is the determinant of $r \times r$-matrix $B = (b_{ij})$, where $b_{ij} = a_{i+k-r,\, j+k}$, for $i, j \in [r]$. Note that in this case $\Delta_{(I \setminus \{i+k-r\}) \cup \{j+k\}} = (-1)^{r-j} b_{ij}$ and the crossing number equals $xing(\pi) = \binom{r}{2} - inv(\pi)$, where $inv(\pi)$ is number of inversions in π. Thus we have the same determinant $\det(B)$ in the right-hand side. The case when $\Delta_I(A) = 0$ follows by the continuity since we have already proved the identity for the dense set of matrices A with $\Delta_I(A) \neq 0$. □

Let us fix a planar directed network N with graph G and edge weights x_e. The following proposition gives an immanant expression for the maximal minors of the boundary measurement matrix $A = A(N)$; see Definition 4.7.

Proposition 5.2. *Let N be a network with n boundary vertices, including k sources b_i, $i \in I$, and the boundary measurements M_{ij}, $i \in I$, $j \in \bar{I}$. Then the maximal minors of the boundary measurement matrix A are equal to*

$$\Delta_J(A) = \sum_{\pi : K \to L} (-1)^{xing(\pi)} \prod_{i \in K} M_{i,\, \pi(i)}\,,$$

for any k-subset $J \subset [n]$, where the sum is over all bijections $\pi : K \to L$ from $K = I \setminus J$ to $L = J \setminus I$.

For example, for a network as in Example 4.8, we have $\Delta_{24}(N) = M_{12}M_{34} + M_{14}M_{32}$ because both bijections $\pi : \{1,3\} \to \{2,4\}$ have just one misalignment and no crossings, i.e., $xing(\pi) = 0$.

Proof. Let us express both sides of the needed identity in terms of the Plücker coordinates of the point $Meas(N) \in Gr_{kn}$; see Definition 4.7. Then the identity can be reformulated as $\Delta_J/\Delta_I = \sum_{\pi:K \to L} (-1)^{xing(\pi)} \prod_{i \in K} (\Delta_{(I \setminus \{i\}) \cup \{\pi(i)\}}/\Delta_I)$, which is equivalent to Lemma 5.1. □

Proposition 5.2 gives an expression for each maximal minor of A as a certain alternating sum of products of the M_{ij}, which corresponds to the generating function for collections of paths in the graph G that connect the boundary vertices b_i, $i \in K$ with the boundary vertices b_j, $j \in L$. Let us show how to cancel the negative terms in this generating function.

Let $P_1 = (u_1, \dots, u_s)$ and $P_2 = (v_1, \dots, v_t)$ be two paths in the graph G, where u_1, v_1, u_s, v_t are four different boundary vertices. Let $xing(P_1, P_2) \in \{0, 1\}$ be the crossing number of the bijection $\{u_1, v_1\} \to \{u_s, v_t\}$ that sends u_1 to

u_s and v_1 to v_t. We also say that the pair of paths (P_1, P_2) forms a crossing/alignment/misalignment if the corresponding bijection of the boundary vertices has a crossing/alignment/misalignment.

Suppose that the paths P_1 and P_2 have a common vertex $v = u_i = v_j$. Let $e_1 = (u_{i-1}, u_i)$, $e_2 = (u_i, u_{i+1})$, $e_3 = (v_{j-1}, v_j)$, $e_4 = (v_j, v_{j+1})$ be the four edges in these paths incident to the vertex v, and let $\epsilon = \epsilon(e_1, e_2, e_3, e_4)$ be the number defined before Lemma 4.2. Recall that $\epsilon = \pm 1$ if these edges are arranged as e_1, e_2, e_3, e_4 clockwise or counterclockwise, and otherwise $\epsilon = 0$. We say that v is an *essential intersection* of the paths P_1 and P_2 if $\epsilon = 0$. In other words, v is not an essential intersection if the paths P_1, P_2 do not cross but touch each other at the vertex v and they have opposite directions at this vertex.

For P_1, P_2, v as above, we say that the paths $P_1' = (u_1, \dots, u_i, v_{j+1}, \dots, v_t)$ and $P_2' = (v_1, \dots, v_j, u_{i+1}, \dots, u_s)$ are obtained from P_1 and P_2 by *switching their tails* at the vertex v. Clearly, $xing(P_1', P_2') = 1 - xing(P_1, P_2)$ if (P_1, P_2) forms a crossing or an alignment, and $xing(P_1', P_2') = xing(P_1, P_2)$ if (P_1, P_2) forms a misalignment.

Lemma 5.3. *Suppose that P_1, P_2 are two paths as above and P_1', P_2' are obtained by switching their tails at v. Then*

$$(-1)^{xing(P_1,P_2)+wind(P_1)+wind(P_2)} = -(-1)^{xing(P_1',P_2')+wind(P_1')+wind(P_2')}$$

if and only if the vertex v is an essential intersection of P_1 and P_2.

Proof. Let a be the number of turns of the tangent vector to the path P_1 as we go from u_1 to v, let b be the number of turns of the tangent vector as we go from v to u_s. Let c and d be the similar numbers for the path P_2. (The numbers a, b, c, d may not be integers.) Clearly, the $a + b$ contributes to $wind(P_1)$, $c + d$ contributes to $wind(P_2)$, $a+d$ contributes to $wind(P_1') = a+d$, and $c+b$ contributes $wind(P_2') = c + b$. Thus $wind(P_1') + wind(P_2') - wind(P_1) - wind(P_2)$ does not depend on these numbers. It only depends on the cyclic order of the boundary vertices u_1, v_1, u_s, v_t and on the cyclic order of the four edges e_1, e_2, e_3, e_4 incident to the vertex v. If we consider all arrangements of these vertices and edges, we obtain the following table of possible cases (where all entries are given modulo 2):

$\epsilon(e_1, e_2, e_3, e_4)$	0	0	1	1
(P_1, P_2) forms a	crossing/ alignment	misalign.	crossing/ alignment	misalign.
$wind(P_1') + wind(P_2')$ $-wind(P_1) - wind(P_2)$	0	1	1	0
$xing(P_1', P_2') - xing(P_1, P_2)$	1	0	1	0

It is an exercise for the reader to check this table. In all four cases we see that the needed identity holds if and only if $\epsilon(e_1, e_2, e_3, e_4) = 0$. □

Let us say that a planar directed graph G is a *perfect graph* if each internal vertex in G is incident to exactly one incoming edge, or to exactly one outgoing edge; and each boundary vertex has degree 1. In Section 9 we will show that an arbitrary planar directed network can be easily transformed into a network with a perfect graph (even a trivalent perfect graph) without changing the boundary measurements M_{ij}; see Lemma 9.1. For a perfect graph we always have $\epsilon(e_1, e_2, e_3, e_4) = 0$ because at least two of the edges e_1, e_2, e_3, e_4 should coincide. Thus any common vertex v of two paths is their essential intersection, and the formula in Lemma 5.3 always holds.

In the rest of this section we will assume, without loss of generality, that G is a perfect graph. Remark that it is possible to extend the results below to arbitrary planar directed graphs, but the construction will be more complicated.

Let I, J be two k-element subsets in $[n]$, $K = I \setminus J$, and $L = J \setminus I$, as in Proposition 5.2. Let $K = \{k_1, \dots, k_r\}$. We say that a collection of directed paths $\mathcal{Q} = (Q_1, \dots, Q_r)$ is a *non-crossing collection* if

(1) For $i = 1, \dots, r$, the path Q_i starts at the boundary vertex b_{k_i} and ends at the boundary vertex $b_{\pi(k_i)}$, where $\pi : K \to L$ is some bijection.
(2) The paths Q_i have no intersections and no self-intersections, that is all vertices in these paths are different.

Note that there are finitely many non-crossing collections of paths in G. Let $Q_i = (u_{i1}, u_{i2}, \dots, u_{i,m_i})$ be the i-th path in the collection. Define

$$S_{ij}(\mathcal{Q}) := 1 + \sum_C (-1)^{wind(C)} x_C,$$

where the sum is over all nonempty closed directed paths C that start and end at the j-th vertex u_{ij} of Q_i and do not pass through any of the preceding vertices $u_{i1}, \dots, u_{i,j-1}$ in Q_i, and through any vertex of a path $Q_{i'}$, for $i' < i$,

Lemma 5.4. *The sum $S_{ij}(\mathcal{Q})$ is given by a subtraction-free rational expression in the x_e defined on $\mathbb{R}_{\geq 0}^{E(G)}$.*

Proof. We will show that S_{ij} equals to a certain boundary measurement for some other graph G'. Any perfect graph G can be easily transformed into a trivalent graph (that is a graph where every internal vertex has degree 3) by splitting vertices of higher degree into several trivalent vertices; see Section 9. Let us assume, without loss of generality, that G is a trivalent graph. Let e be the edge from Q_i that enters the vertex u_{ij}, let e' be the edge from Q_i that leaves the vertex u_{ij}, and let e'' be the remaining third edge incident to u_{ij}. A closed path C that contributes to S_{ij} cannot use the edge e. Thus e' should be the first edge in C and e'' should be the last edge in C. (It e'' is an outgoing edge from u_{ij} then there are no non-empty paths C and $S_{ij} = 1$.) Let us remove from the graph G all vertices of the paths $Q_{i'}$, for all $i' < i$, and also the vertices $u_{i1}, \dots, u_{i,j-1}$. Let us cut the disk along the segment of the path Q_i between the vertices u_{i1} and u_{ij}. Then add two new boundary vertices b' (source) and b'' (sink) and connect them with the vertex u_{ij} by the edges $f' = (b', u_{ij})$ and $f'' = (u_{ij}, b'')$ so that these edges are arranged as e', e'', f', f'' clockwise or counterclockwise. Let G' be the resulting graph. Then closed paths C from u_{ij} to u_{ij} that contribute to S_{ij} (including the empty path) are in one-to-one correspondence with all directed paths P' in the graph G' from the boundary vertex b' to the boundary vertex b''. Moreover, we have chosen the vertices b' and b'' so that $wind(C) = wind(P')$. Thus the sum S_{ij} equals to the boundary measurement $M'_{b',b''}$ for the graph G' (specialized at $x_{b'} = x_{b''} = 1$). According to Lemma 4.4, this sum is given by a subtraction-free rational expression defined on $\mathbb{R}_{\geq 0}^{E(G)}$. □

Proposition 5.5. *Assume that G is a perfect graph. For k-element subsets $I, J \subset [n]$ and $K = I \setminus J$ and $L = J \setminus I$ as above, the maximal minor $\Delta_J(A)$ of the matrix*

$A = A(N)$ is given by the following subtraction-free rational expression in the x_e defined on $\mathbb{R}_{\geq 0}^{E(G)}$:

$$\Delta_J(A) = \sum_{\mathcal{Q}} \left(\prod_{i=1}^{r} x_{Q_i} \prod_j S_{ij}(\mathcal{Q}) \right), \tag{5.1}$$

where the sum is over all non-crossing collections $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_r)$ of paths joining the boundary vertices b_i, $i \in K$ with the boundary vertices b_j, $j \in L$.

Compare the following argument with the proof of [Fom, Theorem 6.1].

Proof. Let $P = (v_1, \dots, v_m)$ be a directed path in G from a boundary vertex v_1 to a boundary vertex v_m. If P has at least one self-intersection, then find the first self-intersection, that is the pair $i < j$ such that $v_i = v_j$ with the minimal possible index j. Let $P' = (v_1, \dots, v_i, v_{j+1}, \dots, v_m)$ be the path obtained from P by erasing the cycle $(v_i, \dots, v_j)$. If P' still has a self-intersection then again erase the first cycle in P' to get another path P'', etc. Finally, we obtain the path $\mathrm{LE}(P)$ without self-intersections, called the *loop-erased part* of P.

For any path $Q = (u_1, \dots, u_s)$ without self-intersections from the boundary vertex $u_1 = v_1$ to the boundary vertex $u_s = v_m$, all paths P that have the loop-erased part $\mathrm{LE}(P) = Q$ can be obtained from Q by inserting at each vertex u_i (except the initial and the final boundary vertices u_1 and u_s) a closed path C_i (possibly empty and possibly self-intersecting) from u_i to u_i that does not pass through the preceding vertices $u_1, \dots, u_{i-1}$. Clearly, we have $x_P = x_Q \prod_i x_{C_i}$. Moreover, we have $wind(P) = \sum_i wind(C_i)$. This follows from Lemma 4.2 and the fact that $\epsilon = 0$ for a perfect graph.

According to Proposition 5.2 and the definition (4.1) of the formal boundary measurements M_{ij}^{form}, the minor $\Delta_J(A)$ equals to the sum

$$\Delta_J(A) = \sum_{\pi: K \to L} (-1)^{xing(\pi)} \sum_{\mathcal{P}} \prod_{i=1}^{r} (-1)^{wind(P_i)} x_{P_i}$$

where the first sum is over all bijections $\pi : K \to L$, the second sum is over all collections of paths $\mathcal{P} = (P_1, \dots, P_r)$ such that P_i starts at the boundary vertex b_{k_i} and ends at the boundary vertex $b_{\pi(k_i)}$. Note that $xing(\pi) = \sum_{i<j} xing(P_i, P_j)$.

Let $Q_1, \dots, Q_r$ be the loop-erased parts of the paths $P_1, \dots, P_r$. Suppose that there is a pair of indices $i < j$ such that such that the loop-erased part $Q_i = \mathrm{LE}(P_i)$ has a common vertex with the path P_j. (Note that we use the loop-erased part of P_i and the *whole* path P_j.) Let us find the lexicographically minimal such pair (i, j) and find the first common vertex v in the paths Q_i and P_j, that is the common vertex v that has the minimal index in Q_i. Let us find the corresponding position of v inside the path P_i that occurs *before* the inserted closed path C at this vertex. Let $\mathcal{P}' = (P_1, \dots, P_i', \dots, P_j', \dots, P_r)$ be the collection of paths obtained from collection $\mathcal{P}$ by replacing the paths P_i and P_j with the paths P_i' and P_j' obtained by switching their tails at the common vertex v.

Then the map $\mathcal{P} \to \mathcal{P}'$ is an involution on collections of paths $\mathcal{P}$ such that $\mathrm{LE}(P_i) \cap P_j \neq \emptyset$, for some $i < j$. In other words, if the apply the above tail-switching operation to $\mathcal{P}'$ we obtain the original collection $\mathcal{P}$. This statement is proved in [Fom, Proof of Theorem 6.1].

The contribution of $\mathcal{P}'$ to the minor $\Delta_J(A)$ is exactly the same as the contribution of $\mathcal{P}$ with the opposite sign. Indeed, according to Lemma 5.3, the sign of $(-1)^{xing(P_i,P_j)+wind(P_i)+wind(P_j)}$ reverses. Also, for any $l \notin \{i,j\}$, the number $xing(P_i', P_l) + xing(P_j', P_l) - xing(P_i, P_l) - xing(P_j, P_l)$ is even. (This is an exercise for the reader. Check all possible arrangements of 3 directed chords in a circle.) Thus $\mathcal{P} \to \mathcal{P}'$ is a sign-reversing involution. This implies that the contributions to $\Delta_J(A)$ of all such families $\mathcal{P}$ cancel each other.

The surviving terms in $\Delta_J(A)$ correspond to families of paths $\mathcal{P} = (P_1, \dots, P_r)$ such that $\mathrm{LE}(P_i) \cap P_j = \emptyset$, for all $i < j$. That exactly means that the collection $\mathcal{Q} = (Q_1, \dots, Q_r)$ of the loop-erased parts of the P_i is an non-crossing collection and that all closed paths C at the jth vertex of Q_i correspond to the terms in $S_{ij}(\mathcal{P})$. Thus the contribution of all terms with a given non-crossing family $\mathcal{Q}$ is $\prod_i x_{Q_i} \prod_j S_{ij}(\mathcal{Q})$, as needed. □

Since any minor of $A(N)$ is given by a subtraction-free rational expression in the edge weights x_e, we deduce that the minor has a nonnegative value whenever the edge weights x_e are nonnegative. Thus the boundary measurement map *Meas* sends any network into a point in the totally nonnegative Grassmannian.

Corollary 5.6. *We have* $Meas(Net_{kn}) \subseteq Gr_{kn}^{\mathrm{tnn}}$.

6. ⅃-diagrams

In this section we define ⅃-diagrams which are on one-to-one correspondence with totally nonnegative Grassmann cells.

Definition 6.1. For a partition λ, let us define a ⅃*-diagram* D of shape λ as a filling of boxes of the Young diagram of shape λ with 0's and 1's such that, for any three boxes indexed (i', j), (i', j'), (i, j'), where $i < i'$ and $j < j'$, filled with a, b, c, correspondingly, if $a, c \neq 0$ then $b \neq 0$; see Figure 6.1. Note that these three boxes should form a "⅃" shaped pattern.[1] For a ⅃-diagram D, let $|D|$ be the number of 1's it contains. Let ⅃$_{kn}$ be the set of ⅃-diagrams whose shape λ fits inside the $k \times (n-k)$-rectangle.

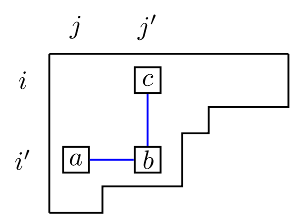


Figure 6.1. ⅃-property: if $a, c \neq 0$ then $b \neq 0$

Figure 6.2 shows an example of ⅃-diagram. Here dots in boxes of the Young diagram indicate that they are filled with 1's, and empty boxes are assumed to be filled with 0's. Let us draw the *hook* for each box with a dot, i.e., two lines going

[1] The letter "⅃" should be pronounced as [le], because it is the mirror image of "L" [el]. We follow English notation for drawing Young diagrams on the plane. A reader who prefers another notation may opt to use one of the following alternative terms instead of ⅃-diagrams: L-diagram, Γ-diagram, ⅂-diagram, V-diagram, Λ-diagram, <-diagram, or >-diagram.

to the right and down from the dotted box. The Ⅎ-property means that every box of the Young diagram located at an intersection of two lines should contain a dot.

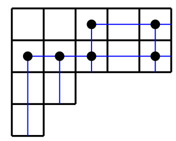


FIGURE 6.2. A Ⅎ-diagram D of shape $\lambda = (5, 5, 2, 1)$ with $|D| = 6$

For a Young diagram filled with 0's and 1's, let us say that that a 0 is *blocked* if there is a 1 somewhere above it in the same column. For example, the diagram shown on Figure 6.2 has three blocked 0's: two in the first column and one in the second column. The Ⅎ-property can be reformulated in terms of blocked 0's, as follows. For each blocked 0, all entries to the left and in the same row as this 0 are also 0's.

Remark 6.2. One can use this observation to recursively construct Ⅎ-diagrams. Suppose that we have a Ⅎ-diagram D of shape λ whose last column contains d boxes and b blocked 0's. Let $\tilde{D}$ be the Ⅎ-diagram of shape $\tilde{\lambda}$ obtained from D by removing the last column and the b rows (filled with all 0's) that contain these blocked 0's in the last column. The shape $\tilde{\lambda}$ of this diagram is obtained from λ by removing b rows of maximal length and removing the last column. Then $\tilde{D}$ can be any Ⅎ-diagram of shape $\tilde{\lambda}$. Thus an arbitrary Ⅎ-diagram D as above with prescribed 0's and 1's in the last column is constructed by picking an arbitrary Ⅎ-diagram $\tilde{D}$ as above and inserting rows filled with all 0's in the positions corresponding to the blocked zeros and then inserting the last column.

Definition 6.3. A Γ-*graph* is a planar directed graph G satisfying the conditions:

1. The graph G is drawn inside a closed boundary curve in $\mathbb{R}^2$.
2. G contains only vertical edges oriented downward and horizontal edges oriented to the left.
3. For any internal vertex v, the graph G contains the line going down from v until it hits the boundary (at some boundary sink) and the line going to the right from v until it hits the boundary (at some boundary source).
4. All pairwise intersections of such lines should also be vertices of G.
5. The graph may also contain some number of isolated boundary vertices, which are assigned to be sinks or sources.

In other words, a Γ-graph G is obtained by drawing several Γ-shaped hooks inside the boundary curve. A Γ-*network* is a network with a Γ-graph.

Note that for an arbitrary Γ-network N there is a unique gauge transformation of edge weights (4.2) that transforms the weights of all vertical edges into 1.

Figure 6.3 shows an example of Γ-graph. We displayed boundary sources by black vertices and boundary sinks by white vertices.

For a Ⅎ-diagram D of shape λ, define Ⅎ-*tableaux* T as nonnegative real-valued functions T on boxes (i, j) of the Young diagram of shape λ such that and $T(i, j) > 0$ if and only if the box (i, j) of the diagram D is filled with a 1.

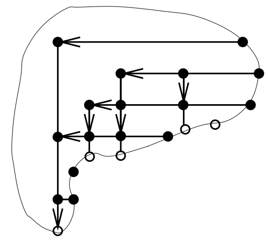


FIGURE 6.3. A Γ-graph

There is a simple correspondence between ⅃-tableaux of shape λ that fit inside the rectangle $(n-k)^k$ and Γ-networks with k boundary sources and $n-k$ boundary sinks modulo gauge transformations. Let T be a ⅃-tableau of shape $\lambda \subseteq (n-k)^k$. The boundary of the Young diagram of λ gives the lattice path of length n from the upper right corner to the lower left corner of the rectangle $(n-k)^k$. Let us place a vertex in the middle of each step in the lattice path and mark these vertices by $b_1, \dots, b_n$ as we go downwards and to the left. The vertices b_i, $i \in I$, corresponding to the vertical steps in the lattice path will be the sources of the network and the remaining vertices b_j, $j \in \bar{I}$, corresponding to horizontal steps will be the sinks. Notice that the source set I is exactly the set $I(\lambda)$ as defined in Section 2.1. Then connect the upper right corner with the lower left corner of the rectangle by another path so that together with the lattice path they form a closed curve containing the Young diagram in its interior. For each box (i,j) of the Young diagram such that $T(i,j) \neq 0$, draw an internal vertex in the middle of this box and draw the line that goes downwards from this vertex until it hits a boundary sink and another line that goes to the right from this vertex until it hits a boundary source. As we have already mentioned in Section 6, the ⅃-property means that any intersection of such lines should also be a vertex; cf. Figure 6.2. Orient all edges of the obtained graph to the left and downwards. Finally, for each internal vertex v drawn in the middle of the box (i,j) assign the weight $x_e = T(i,j) > 0$ to the horizontal edge e that enters v (from the right). Also assign weights $x_e = 1$ to all vertical edges of the network. Let us denote the obtained network N_T. It is not hard to see that any Γ-network (with the weights of all vertical edges equal to 1) comes from a ⅃-tableau in this fashion. We leave it as an exercise for the reader to rigorously prove this claim.

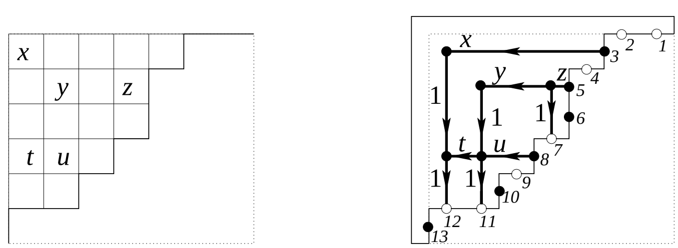


FIGURE 6.4. A ⅃-tableau T and the corresponding Γ-network N_T

Example 6.4. Figure 6.4 gives an example of a ⅃-tableau and the corresponding Γ-network. In the ⅃-tableau only nonzero entries are displayed. In the Γ-network we marked the boundary vertices $b_1, \dots, b_n$ just by the numbers $1, \dots, n$. The dotted lines indicate the boundary of the rectangle $(n-k)^k$; they are not edges of the Γ-network. In this example, $n = 13$ and the source set is $I = \{3, 5, 6, 8, 10, 13\}$.

For a Ⅎ-diagram $D \in \text{Ⅎ}_{kn}$, let $\mathbb{R}_{>0}^D \simeq \mathbb{R}_{>0}^{|D|}$ be the set of Ⅎ-tableaux T associated with D. The map $T \mapsto N_T$ gives the isomorphism

$$\mathbb{R}_{>0}^D \simeq \mathbb{R}_{>0}^{E(G)}/\{\text{gauge transformations}\}$$

between the set of Ⅎ-tableaux T with fixed Ⅎ-diagram D and the set of Γ-networks (modulo gauge transformations) with the fixed graph G corresponding to the Ⅎ-diagram D as above. The boundary measurement map *Meas* (see Definition 4.7) induces the map

$$Meas_D : \mathbb{R}_{>0}^D \to Gr_{kn}, \quad Meas_D : T \mapsto Meas(N_T).$$

Recall Definition 4.10 of a subtraction-free parametrization.

Theorem 6.5. *For each Ⅎ-diagram $D \in \text{Ⅎ}_{kn}$, the map $Meas_D$ is a subtraction-free parametrization a certain certain totally nonnegative Grassmann cell $S_{\mathcal{M}}^{\text{tnn}} = Meas_D(\mathbb{R}_{>0}^{\text{tnn}}) \subset Gr_{kn}^{\text{tnn}}$. This gives a bijection between Ⅎ-diagrams $D \in \text{Ⅎ}_{kn}$ and all cells $S_{\mathcal{M}}^{\text{tnn}}$ in Gr_{kn}^{tnn}. The Ⅎ-diagram D has shape λ if and only if $S_{\mathcal{M}}^{\text{tnn}} \subset \Omega_\lambda$. The dimension of $S_{\mathcal{M}}^{\text{tnn}}$ equals to $|D|$. Moreover, for D of shape λ, the map $Meas_D$ is I-polynomial, where $I = I(\lambda)$.*

Theorem 6.5, together with Corollary 5.6, implies Theorem 4.9. Moreover, it implies Theorem 4.12. Indeed, the parametrization $Meas_D$ is I_1-polynomial, where $I_1 = I(\lambda)$. The claim for other bases I_i follows from the cyclic symmetry; see Remark 3.3. In other words, take the Ⅎ-diagram corresponding to a cell $S_{\mathcal{M}}^{\text{tnn}}$ as above, but assuming that the boundary vertices are ordered as $b_i < \cdots < b_n < b_1 < \cdots < b_{i-1}$. It gives an I_i-polynomial parametrization of $S_{\mathcal{M}}^{\text{tnn}}$.

7. Inverting the boundary measurement map

In this section we prove Theorem 6.5 by constructing the bijective map $Gr_{kn}^{\text{tnn}} \to \{\text{Ⅎ-tableaux}\}$, which is inverse to the boundary measurement map.

The construction is based on the following four lemmas. Let $A = (a_{ij})$ be a $k \times n$-matrix in I-echelon form; see Section 2.1. Let A_{d+1} be the first column-vector of A that can be expressed as a linear combination of the previous column-vectors: $A_{d+1} = (-1)^{d-1} x_1\, A_1 + \cdots + x_{d-2}\, A_{d-2} - x_{d-1}\, A_{d-1} + x_d\, A_d$. In other words, d is the maximal integer such that $[d] \subseteq I$. (Here we exclude the trivial case $k = n$ when A should be the identity matrix. But we allow $d = 0$ when the first column of A is zero.) Then A_i is the ith coordinate vector, for $i = 1, \dots, d$, and $A_{d+1} = ((-1)^{d-1}x_1, ..., x_{d-2}, -x_{d-1}, x_d, 0, \dots, 0)^T$, that is the matrix A has the following form:

$$A = \begin{pmatrix} 1 & \cdots & 0 & 0 & (-1)^{d-1}x_1 & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & -x_{d-1} & * & \cdots & * \\ 0 & \cdots & 0 & 1 & x_d & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & * & \cdots & * \end{pmatrix}. \tag{7.1}$$

Recall that $\mathrm{Mat}_{kn}^{\mathrm{tnn}}$ is the set of $k\times n$-matrices of rank k with nonnegative maximal minors $\Delta_J \geq 0$.

Lemma 7.1. *We have $\Delta_{(I\setminus\{i\})\cup\{d+1\}}(A) = x_i$. Thus the condition $A \in \mathrm{Mat}_{kn}^{\mathrm{tnn}}$ implies that $x_i \geq 0$ for $i = 1, \ldots, d$.*

Proof. The matrix $A_{(I\setminus\{i\})\cup\{d+1\}}$ is obtained from the identity matrix A_I by skipping its ith column and inserting the column $v_{d+1} = (\pm x_1, \ldots, -x_{d-1}, x_d, 0, \ldots, 0)^T$ in dth position. □

Lemma 7.2. *Let $r \in [d]$ be an index such that $x_r = 0$ and there exists $i < r$ such that $x_i \neq 0$. Then the condition $A \in \mathrm{Mat}_{kn}^{\mathrm{tnn}}$ implies that $a_{rj} = 0$ for all $j > d$. In other words, the r-row of A has only one nonzero entry $a_{rr} = 1$.*

We will call an entry $x_r = 0$ of the vector $(x_1, \ldots, x_d)$ satisfying the condition in this lemma a *blocked zero.*

Proof. For $j \in I$ we have $a_{rj} = 0$ because the j-th column of A has only one nonzero entry $a_{sj} = 1$ for some $s > d$. Suppose that $j \notin I$. Then $\Delta_{(I\setminus\{r\})\cup\{j\}} = (-1)^t\, b_{rj} \geq 0$, where $t := |I \cap [r+1, j-1]|$. On the other hand, $\Delta_{(I\setminus\{i,r\})\cup\{d+1,j\}} = (-1)^{t+1}\, x_i\, a_{rj} \geq 0$. Since by Lemma 7.1 $x_i > 0$, we deduce that $(-1)^{t+1}\, a_{rj} \geq 0$. This implies that $a_{rj} = 0$. □

Lemma 7.3. *Assume that the r-row of A has only one nonzero entry $a_{rr} = 1$ for some $r \in [d]$. Let $B = (b_{ij})$ be the $(k-1)\times(n-1)$-matrix obtained from A by removing the r-row and the r-column and inverting signs of the entries a_{ij} for $i = 1, \ldots, r-1$ and $j \geq d+1$. Then $A \in \mathrm{Mat}_{kn}^{\mathrm{tnn}}$ if and only if $B \in \mathrm{Mat}_{k-1,n-1}^{\mathrm{tnn}}$.*

Moreover, the maximal minors of the matrices A and B are equal to each other. More explicitly, $\Delta_J(A) = 0$ if $r \notin J$, and $\Delta_J(A) = \Delta_{\tilde{J}\setminus\{r\}}(B)$ if $r \in J$, where $\tilde{J}$ means that we decrease elements $> r$ in J by 1.

Proof. The equality of the minors is straightforward; it implies the first claim. □

Lemma 7.4. *Assume that there are no blocked zeros, that is $x_1 = \cdots = x_s = 0$ and $x_{s+1}, x_{s+2}, \ldots, x_d > 0$, for some $s \in [0, d]$. Let $C = (c_{ij})$ be the $k\times(n-1)$-matrix whose first d columns are the first coordinate vectors (as in the matrix A) and the remaining entries are*

$$c_{i,\,j-1} = \begin{cases} a_{ij} & \text{if } i \in [s] \cup [d+1, k], \\ \frac{a_{ij}}{x_i} + \frac{a_{i+1,\,j}}{x_{i+1}} & \text{if } i \in [s+1, d-1], \\ \frac{a_{dj}}{x_d} & \text{if } i = d, \end{cases}$$

for $j = d+2, \ldots, n$. Then $A \in \mathrm{Mat}_{kn}^{\mathrm{tnn}}$ if and only if $C \in \mathrm{Mat}_{k,n-1}^{\mathrm{tnn}}$.

Moreover, if we fix a totally nonnegative cell $S_{\mathcal{M}}^{\mathrm{tnn}} \subset Gr_{kn}^{\mathrm{tnn}}$ and require that (the coset of) the matrix A belongs to $S_{\mathcal{M}}^{\mathrm{tnn}}$, then we can write all maximal minors $\Delta_J(C)$ as subtraction-free rational expressions in terms of the minors $\Delta_K(A)$, $K \in \mathcal{M}$. On the other hand, we can write the minors $\Delta_K(A)$ as nonnegative integer polynomials in terms of the minors $\Delta_J(C)$ and the x_i, $i \in [s+1, d]$.

This gives a bijective correspondence between totally nonnegative cells $S_{\mathcal{M}}^{\mathrm{tnn}} \subseteq Gr_{kn}^{\mathrm{tnn}}$ that can contain a matrix A of this form and totally nonnegative cells $S_{\mathcal{M}'}^{\mathrm{tnn}} \subseteq Gr_{k,n-1}^{\mathrm{tnn}}$ that can contain a matrix C of this form.

We will prove this lemma in Section 8. Note that the matrix C is still in echelon form. Let us illustrate this lemma by an example.

Example 7.5. Let

$$A = \begin{pmatrix} 1 & 0 & -x_1 & y \\ 0 & 1 & x_2 & z \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & \frac{y}{x_1} + \frac{z}{x_2} \\ 0 & 1 & \frac{z}{x_2} \end{pmatrix},$$

where $x_1 > 0$ and $x_2 > 0$. Then $\Delta_{12}(C) = 1$, $\Delta_{13}(C) = \frac{\Delta_{14}(A)}{\Delta_{13}(A)}$, $\Delta_{23}(C) = \frac{\Delta_{34}(A)}{\Delta_{13}(A)\cdot\Delta_{23}(A)}$. On the other hand, $\Delta_{12}(A) = 1$, $\Delta_{13}(A) = x_2$, $\Delta_{23}(A) = x_1$, $\Delta_{14}(A) = x_2\,\Delta_{13}(C)$, $\Delta_{24}(A) = x_1\,(\Delta_{13}(C) + \Delta_{23}(C))$, $\Delta_{34}(A) = x_1\,x_2\,\Delta_{23}(C)$.

Proof of Theorem 6.5. Let us prove the theorem, together with the additional claim that the map $\mathbb{R}^D_{>0} \to \mathrm{Mat}_{kn}$ given by $T \mapsto A(N_T)$ produces $k\times n$-matrices in echelon form. The proof is by induction on n. The cases when $k = n$ or $k = 0$ are trivial, which provides the base of induction. Assume that $k \in [n]$. Assume by induction that the theorem is valid for all $Gr^{\mathrm{tnn}}_{k'n'}$ with $n' < n$.

Let A be a $k \times n$-matrix in I-echelon form that represents a point in $S^{\mathrm{tnn}}_{\mathcal{M}} \subseteq \Omega_\lambda \cap Gr^{\mathrm{tnn}}_{kn}$, where $I = I(\lambda)$. Let us find the integer d and the real numbers $x_1, \dots, x_d$ as above in this section; see (7.1). According to Lemma 7.1, we have $x_i \geq 0$, for $i = 1, \dots, d$. Note that we can uniquely determine the number d and the set of indices i with $x_i \neq 0$ from the matroid $\mathcal{M}$, because these numbers are certain maximal minors of A; see Lemma 7.1.

Suppose that there are $b > 0$ blocked zeros in the vector $(x_1, \dots, x_d)$. For each blocked zero $x_r = 0$, all entries in the rth row of A are zero, except $a_{rr} = 1$. Let A' be the $(k-b) \times (n-b)$-matrix obtained from A by skipping the rth row and the rth column, for each blocked zero $x_r = 0$, and inverting signs of some entries, as in Lemma 7.3. Namely, we need to invert the sign of a_{ij}, $i \leq d < j$, if and only if there is an odd number of blocked zeros $x_r = 0$ with $r > i$. According Lemma 7.3, the maximal minors of A are equal to the corresponding maximal minors of A' (or to zero). Thus the matroid $\mathcal{M}'$ associated with A' can be uniquely constructed from the matroid $\mathcal{M}$; and vise versa, if we know the matroid $\mathcal{M}'$ and the positions of blocked zeros, then we can uniquely reconstruct the matroid $\mathcal{M}$. In particular, the Schubert cell $\Omega_{\lambda'}$ of A' corresponds to the Young diagram of shape λ' obtained by removing b (longest) rows with $n-k$ boxes from the Young diagram of λ.

Note that A' can be any I'-echelon matrix, where $I' = I(\lambda')$, that represents a cell $S^{\mathrm{tnn}}_{\mathcal{M}'} \subset \Omega_{\lambda'}$ such that the vector $(x'_1, \dots, x'_{d-b})$ with $x'_i = \Delta_{(I'\setminus\{i\})\cup\{d-b+1\}}(A')$ has no blocked zeros. By the induction hypothesis, we already know that the map $T' \mapsto A(N_{T'})$ is a bijection between Ⅎ-tableaux T' of shape λ' such that the last column of T' contains no blocked zeros and the set of matrices A' as above. Moreover, it gives a bijection between Ⅎ-diagrams D' corresponding to such tableaux and cells $S^{\mathrm{tnn}}_{M'}$ containing such matrices A'; and the map $T' \mapsto A(N_{T'})$ gives a subtraction-free parametrization for each cell $S^{\mathrm{tnn}}_{\mathcal{M}'}$.

Let T' be the Ⅎ-tableau such that $A(N_{T'}) = A'$ and D' be its Ⅎ-diagram. Let T be the Ⅎ-tableau (and D be its Ⅎ-diagram) obtained from T' (reps., from D) by inserting b rows filled with all 0's in the positions corresponding to blocked zeros in $(x_1, \dots, x_d)$. Then we have $A(N_T) = A$. Indeed, the network N_T is obtained from $N_{T'}$ by inserting b isolated sources in the positions corresponding to the blocked zeros. Thus, according to Definition 4.7, its boundary measurement matrix $A(N_T)$ is obtained from $A' = A(N_{T'})$ by inserting, for each blocked zero x_r, a row and a column in the rth positions with a single nonzero entry $a_{rr} = 1$, and switching the signs of some entries a_{ij}. (The signs are switched because we insert additional elements r into I. These switches are exactly the same as in Lemma 7.3.) Note that all ⅃-diagrams and all ⅃-tableaux of shape λ with given entries in the last column are of this form; see Remark 6.2. This implies that $T \mapsto A(N_T)$ gives subtraction-free parametrizations with needed properties for the cells $S^{\mathrm{tnn}}_{\mathcal{M}}$ with some blocked zeros.

Let us now suppose that $(x_1, \dots, x_d)$ contains no blocked zeros, that is $x_1 = \cdots = x_s = 0$ and $x_{s+1}, \dots, x_d > 0$. Let C be the matrix obtained from A as in Lemma 7.4. The matrix C represents a point in $S^{\mathrm{tnn}}_{\mathcal{M}''} \subset \Omega_{\lambda''} \cap Gr^{\mathrm{tnn}}_{k,n-1}$. According to Lemma 7.4, the matroid $\mathcal{M}''$ is uniquely determined by the matroid $\mathcal{M}$, and vise versa $\mathcal{M}$ is uniquely determined by $\mathcal{M}''$ and positions of zeros in $(x_1, \dots, x_d)$. In this case, the Young diagram of λ'' is obtained from the Young diagram of λ by removing the last column (with d boxes).

Note that $S^{\mathrm{tnn}}_{\mathcal{M}''}$ can be any cell in $\Omega_{\lambda''}$ and A'' can be the echelon representative of any point in such cell. Again, by the induction hypothesis, we already know that $T'' \mapsto A(N_{T''})$ gives a subtraction-free parametrization for cells $S^{\mathrm{tnn}}_{\mathcal{M}''} \subset \Omega_{\lambda''}$ and that this map induces a bijection between all ⅃-diagrams D'' of shape λ'' and these cells. Let T'' be the ⅃-tableau such that $A(N_{T''}) = C$, and D'' be its ⅃-diagram. Let T be the ⅃-tableau obtained from T'' by inserting the last column filled with $(x_1, \dots, x_d)$, and D be the ⅃-diagram of T, i.e., D is obtained from D'' by inserting the last row filled with $(0, \dots, 0, 1, \dots, 1)$ (s zeros followed by $d-s$ ones). According to Remark 6.2, any ⅃-diagram D and any ⅃-tableau T of shape λ without blocked zeros in the last column have this form.

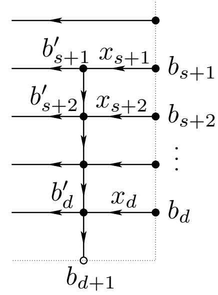


FIGURE 7.1. A piece of the Γ-network N_T

We claim that for this ⅃-tableau T, we have $A(N_T) = A$. Indeed, the network $N_{T''}$ is obtained from N_T by removing the $d - s$ internal vertices $b'_{s+1}, \dots, b'_d$ adjacent to the boundary sources $b_{s+1}, \dots, b_d$, removing the vertical edges incident to the vertices b'_i, merging $d - s$ pairs of horizontal edges, erasing the weights $x_{s+1}, \dots, x_d$, removing the sink b_{d+1}, and shifting the labels of the boundary vertices b_i, $i > d + 1$, by one; see Figure 7.1. The boundary measurements M_{ij} of the network N_T are obtained from the boundary measurements of M''_{ij} of $N_{T''}$ by $M_{i,j+1} = x_i(M''_{ij} + M''_{i+1,j} + \cdots + M''_{d,j})$ for $i \in [s + 1, d]$, $j \geq d + 1$; $M_{i,j+1} = M''_{ij}$ for $i \notin [s + 1, d]$, $j \geq d + 1$; and $M_{i,d+1} = x_i$ for $i \in [k]$; see Figure 7.1. Thus the boundary measurement matrix $\tilde{A} = (\tilde{a}_{ij})$ of N_T is obtained from the matrix $C = A(N_{T''})$, by inserting the column $(x_1, \dots, x_d, 0, \dots, 0)^T$ in the $(d+1)$st position and changing other entries as

$$\tilde{a}_{i,j+1} = \begin{cases} x_i(c_{ij} - c_{i+1,j} + \cdots \pm c_{dj}) & \text{if } i \in [s+1, d], \\ c_{ij} & \text{otherwise,} \end{cases}$$

for $j \geq d+1$; and of course $\tilde{a}_{ij} = c_{ij} = \delta_{ij}$, for $j \leq d$. Notice that the $C \mapsto \tilde{A}$ is the inverse of the transformation $A \mapsto C$ from Lemma 7.4. Thus $A(N_T)$ is the original matrix A, as needed. Now Lemma 7.4 implies that the map $T \mapsto A(N_T)$ gives needed subtraction-free parametrizations for the cell $S_{\mathcal{M}}^{\mathrm{tnn}}$ without blocked zeros.

Finally, note that the network N_T is acyclic and its source set is $I = I(\lambda)$. Thus all boundary measurements M_{ij} are nonnegative polynomials in the edge weights. By Proposition 5.5 all maximal minors $\Delta_J(A(N_T))$ are given nonnegative polynomials and $\Delta_I(A(N_T)) = 1$. This implies the last claim of Theorem 6.5 about I-polynomiality of the map $Meas_D$. □

We can describe the map $Gr_{kn}^{\mathrm{tnn}} \to \{$Ⅎ-tableaux$\}$ implicitly constructed in the above proof, which is inverse to the boundary measurement map, via the following recursive procedure. This procedure transforms points in the totally nonnegative part of a Schubert cell $\Omega_\lambda \cap Gr_{kn}^{\mathrm{tnn}}$ into Ⅎ-tableaux T of shape λ. It inserts nonnegative real numbers into boxes of the Young diagram of shape λ starting with the rightmost column, then filling the next available rightmost column, etc. The procedure uses a variable matrix A of variable dimensions.

Procedure. Map from $\Omega_\lambda \cap Gr_{kn}^{\mathrm{tnn}}$ to Ⅎ-tableaux of shape λ.

(1) Take the $k \times n$-matrix A in echelon form representing a point in $\Omega_\lambda \cap Gr_{kn}^{\mathrm{tnn}}$.
(2) For the matrix A, find the integer d and real numbers $x_1, \dots, x_d$ as in (7.1). Insert the nonnegative real numbers $x_1, \dots, x_d$ (see Lemma 7.1) into the empty boxes of the rightmost available column of T, skipping the boxes of T which are already filled with 0's.
(3) Let $B = \{r \mid x_r = 0 \text{ and } x_i \neq 0 \text{ for some } i < r\}$ be the set of blocked indices.
(4) If $B \neq \emptyset$, then for each index $r \in B$, invert the sign of entries a_{ij}, $i < r$, $j \geq d+1$, in the matrix A, remove the rth row and the rth column from A (see Lemma 7.3), and insert zeros in all boxes of T to the left of the blocked zero x_r.
(5) If the obtained matrix has 0 rows, then stop. Otherwise go to step (2).
(6) If $B = \emptyset$, then transform the matrix A into the $(n-1) \times k$-matrix as in Lemma 7.4.
(7) If the obtained matrix has dimensions $k' \times n'$ with $k' = n'$ then stop. Otherwise, go to step (2).

8. Lemma on tails

In this section we prove Lemma 7.4 essential in the proof of Theorem 6.5.

Let us identify a sequence $(v_1, \dots, v_n) \in (\mathbb{R}^k)^n$ of k-vectors with the $k \times n$-matrix with the column vectors v_i. We say that such sequence is *totally nonnegative* if all maximal $k \times k$-minors of the corresponding matrix are nonnegative and at least one of these minors is strictly positive.

Definition 8.1. For a sequence $u = (u_1, \dots, u_m) \in (\mathbb{R}^k)^m$, we define the *$r$-tail* $\mathrm{Tail}_r(u)$ of u as the set of sequences $w = (w_1, \dots, w_r) \in (\mathbb{R}^k)^r$ such that the concatenation $(u, v) := (u_1, \dots, u_m, w_1, \dots, w_r)$ of u and w is totally nonnegative:

$$\mathrm{Tail}_r(u) := \{w \in (\mathbb{R}^k)^r \mid (u, w) \text{ is totally nonnegative}\}.$$

The set $\mathrm{Tail}_r(u)$ comes equipped with a *stratification,* that is a subdivision into the disjoint union of strata defined as follows. We say that $w, w' \in \mathrm{Tail}_r(u) \subset (\mathbb{R}^k)^r$ are in the same *stratum* of $\mathrm{Tail}_r(u)$ if the the corresponding $k\times(m+r)$-matrices (u,w) and (u,w') are in the same matroid strata, see Section 2.3, that is the maximal minor $\Delta_I(u,w)$ is nonzero if and only if the maximal minor $\Delta_I(u,w')$ is nonzero, for any k-element subset $I\subseteq[m+r]$.

Definition 8.2. Let u and v be two sequences of vectors in $\mathbb{R}^k$, which are allowed to have different lengths. We say that the sequences u and v are *tail-equivalent,* and write $u\sim_{\mathrm{tail}} v$, if, for any $r\geq 1$, we have

(1) $\mathrm{Tail}_r(u)=\mathrm{Tail}_r(v)$;
(2) the stratifications of $\mathrm{Tail}_r(u)$ and $\mathrm{Tail}_r(v)$ are the same;
(3) for any stratum S of $\mathrm{Tail}_r(u)$, there is a subtraction-free rational expression for each maximal minor $\Delta_I(v,w)$ in terms of the nonzero maximal minors $\Delta_J(u,w)$, for $w\in S$; and vise versa the minors $\Delta_J(u,w)$ can be written as subtraction-free rational expressions in terms of the $\Delta_I(v,w)$ on each stratum of $\mathrm{Tail}_r(v)$.

Note that condition (3) actually implies conditions (1) and (2).

The following lemma is the main technical tool in our proof of Theorem 6.5

Lemma 8.3. *Let $u_1,\dots,u_d$ be a linearly independent set of vectors in $\mathbb{R}^k$. Then the following two sequences are tail-equivalent:*

$$(u_1,\dots,u_d)\sim_{\mathrm{tail}}(v_1,\dots,v_{d+1}),$$

where $v_1=u_1$, $v_2=u_1+u_2$, $v_3=u_2+u_3,\dots,v_d=u_{d-1}+u_d$, $v_{d+1}=u_d$.

Proof. For three subsets $I\subseteq[d]$, $J\subseteq[d+1]$, and $K\subseteq[r]$ with the total cardinality $|I|+|J|+|K|=k$, let $\Delta_{I,J,K}=\Delta_{I,J,K}(u,v,w)$ denote the determinant of the $k\times k$-matrix with columns $\{u_i\}_{i\in I}$, $\{v_j\}_{j\in J}$, $\{w_l\}_{l\in K}$, where the vectors u_i, v_j, w_l are taken in the order that agrees with the total order

$$v_1<u_1<v_2<u_2<v_3<\cdots<u_{d-1}<v_d<u_d<v_{d+1}<w_1,\cdots<w_r.$$

In other words, the $\Delta_{I,J,K}$ are the maximal $k\times k$-minors of the $k\times(2d+1+r)$-matrix formed by these column vectors.

We can write each minor $\Delta_{\emptyset,J,K}$ as a sum the minors $\Delta_{I',\emptyset,K'}$ by replacing each column vector v_j in $\Delta_{\emptyset,J,K}$ with the sum $u_{j-1}+u_j$ and expanding the result by the linearity. Thus the nonnegativity of all minors $\Delta_{I,\emptyset,K}$ implies that all minors $\Delta_{\emptyset,J',K'}$ are also nonnegative. Moreover, if we know which minors $\Delta_{I,\emptyset,K}$ are strictly positive, then we can determine which minors $\Delta_{\emptyset,J',K'}$ are strictly positive. In other words, we have $\mathrm{Tail}_r(u)\subseteq\mathrm{Tail}_r(v)$ and the stratification of $\mathrm{Tail}_r(u)$ refines the stratification of $\mathrm{Tail}_r(v)$.

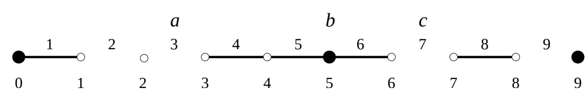
The proof of the opposite claim is a little bit more elaborate. Let us fix a stratum of $\mathrm{Tail}_r(v)$. We will show that, for w in this stratum, all minors $\Delta_{I,J,K}$ can be written as subtraction-free rational expressions in terms of the nonzero minors $\Delta_{\emptyset,J',K'}$. In particular, this would imply that all minors $\Delta_{I,\emptyset,K}$ are nonnegative and we can determine which of these minors are strictly positive. Thus we will get $\mathrm{Tail}_r(v)\subseteq\mathrm{Tail}_r(u)$ and deduce that the stratification of $\mathrm{Tail}_r(v)$ refines the stratification of $\mathrm{Tail}_r(u)$.

Let us obtain subtraction-free rational expressions for the $\Delta_{I,J,K}$ in terms of the $\Delta_{\emptyset,J',K'}$ by induction. Clearly, we get such expressions when $I=\emptyset$. Let us assume that I is nonempty. If $|J|=d$ or $d+1$, then $\Delta_{I,J,K}=0$, because any vector u_i and any d vectors v_j are linearly dependent. This provides the base of the induction. Let us assume by induction we have already determined which of the minors $\Delta_{I',J',K'}$ are nonzero and wrote them as subtraction-free rational expressions for all triples of subsets I', J', K' such that $|J'| > |J|$.

Let us show that either $\Delta_{I,J,K} = 0$ or $\Delta_{I,J,K} = \Delta_{I',J',K'}$ for some triple (I', J', K') with $|J'| > |J|$, or else there is a triple of indexes (a, b, c) such that $1 \leq a \leq b < c \leq d+1$; $b \in I$; $a, c \notin J$; and the collection of $|I| + |J| + 2$ vectors v_a, v_c and u_i, v_j, for $i \in I$, $j \in J$, is linearly independent.

Let us give a criterion when a subsequence S of vectors in $u_1, \ldots, u_d, v_1, \ldots, v_{d+1}$ is linearly independent. We can graphically present such collection S of vectors by the graph G_S on the set $\{0, \ldots, d+1\}$ with some marked vertices where G_S has the edge $(j-1, j)$ for each $v_j \in S$, G_S has the marked vertex i for each $u_i \in S$, and the vertices 0 and $d+1$ are always marked. Then S is a linearly independent collection of vectors if and only if each connected component of the graph G_S has at most one marked vertex.

The following figure shows the graph G_S for $d = 8$ and $S = (u_5, v_1, v_4, v_5, v_6, v_8)$. The vertex labels are displayed below the graph and the edge labels are displayed above the graph. The vector u_5 corresponds to the marked vertex 5 and the vectors v_1, v_4, v_5, v_6, v_8 correspond to edges of G_S labelled $1, 4, 5, 7, 8$. The marked vertex 5 belongs to the connected component of G_S with the edges labelled $4, 5, 6$. For this connected component, we have $(a, b, c) = (3, 5, 7)$.



Now let S be the collection of vectors u_i, v_j, for $i \in I$ and $j \in J$. If S is linearly dependent, then $\Delta_{I,J,K} = 0$. So we can assume that S is linearly independent. Let us pick any element $b \in I$ and find the connected component of G_S that contains the marked vertex b. Let a, c be as above, i.e., a, c are the integers such that $a \leq b < c$, $[a+1, c-1] \subset J$, and $a, c \notin J$.

If $a = 0$, that is $[1, b] \subseteq J$, then $\Delta_{I,J,K} = 0$ because the vectors $v_1, \ldots, v_b, u_b$ are linearly dependent. (In this case the connected component of G_S contains two marked points 0 and b.) If $a = 1$, that is $1 \notin J$ and $[2, b] \subset J$, then the vectors $v_2, \ldots, v_b, u_b$ in S can be expressed in term of the vectors $v_1 = u_1, v_2, \ldots, v_b$ by a unimodular linear transformation. Thus we can replace the columns $v_2, \ldots, v_b, u_b$ in the minor $\Delta_{I,J,K}$ with the columns $v_1, \ldots, v_b$ without affecting the value of the minor. So we have $\Delta_{I,J,K} = \Delta_{I \setminus \{b\},\, J \cup \{1\},\, K}$, where we have already expressed the minor in the right-hand side by the induction hypothesis. Thus we can assume that $a \geq 2$. Analogously, we eliminate the cases $c = d+2$ and $c = d+1$ and assume that $c \leq d$.

Suppose that the vertex $a-1$ belongs to a connected component of G_S with a marked vertex a', that is we have $a' \in I$ and $[a'+1, a-1] \in J$ for some $a' \leq a-1$ (or just $[1, a-1] \in J$ when $a' = 0$). In this case we can express the sequences of vectors $u_{a'}, v_{a'+1}, \ldots, v_{a-1}$ and $v_{a+1}, \ldots, v_b, u_b$ (that belong to S) in terms of the consecutive sequences of vectors $u_{a'}, \ldots, u_{a-1}$ and $u_a, \ldots, u_b$ by unimodular transformations, then join these two sequences into a single sequence and unimodularly transform it again into $v_{a'+1}, \ldots, v_b, u_b$. The graph of the new sequence is obtained by merging two connected components of G_S with marked points a' and b into a single connected connected component. The new sequence of vectors is obtained from the old by skipping $u_{a'}$ and adding v_a. This transformation will not affect the value of the minor: $\Delta_{I,J,K} = \Delta_{I\setminus\{a'\},J\cup\{a\},K}$, when $a' \geq 0$. In the case when $a' = 0$, we get $\Delta_{I,J,K} = \Delta_{I\setminus\{b\},J\cup\{a\},L}$. In all cases, the minor $\Delta_{I,J,K}$ is reduced to a minor which has already been expressed subtraction-freely by the induction hypothesis.

So we can assume that the vertex $a-1$ does not belong to a connected component of G_S with a marked vertex. Similarly, we can eliminate the case when the vertex c belongs to a connected component with a marked point. Now the graph obtained from G_S by adding the edges $(a-1, a)$ and $(c-1, c)$ still has at most one marked vertex in each connected component. By the above criterion this means that the set of vectors $\{u_i, v_j \mid i \in I,\, j \in J\} \cup \{v_a, v_c\}$ is linearly independent.

If $\Delta_{I,J,K} \neq 0$ then the set of vectors $B = \{u_i, v_j, w_l \mid i \in I,\, j \in J,\, l \in K\}$ forms a basis in $\mathbb{R}^k$, and thus the set of $k+2$ vectors $B \cup \{v_a, v_c\}$ linearly spans $\mathbb{R}^k$. In this case (according to the exchange axiom) we can complete the linearly independent subset $\{u_i, u_j \mid i \in I, j \in J\} \cup \{v_a, v_c\}$ of the latter set to a basis of $\mathbb{R}^k$. Thus there exist a pair of elements $x, y \in K$ such that $(B \setminus \{w_x, w_y\}) \cup \{v_a, v_c\}$ is a basis of $\mathbb{R}^k$, or equivalently, $\Delta_{I,J\cup\{a,c\},K\setminus\{x,y\}} \neq 0$. It follows that if $\Delta_{I,J\cup\{a,c\},K\setminus\{x,y\}} = 0$ for all pairs $x, y \in K$ then $\Delta_{I,J,K} = 0$. By the induction hypothesis we have already determined which of the minors $\Delta_{I,J\cup\{a,c\},K\setminus\{x,y\}}$ are nonzero. Thus we either deduce that $\Delta_{I,J,K} = 0$ or find a pair $x < y$ with nonzero (and thus strictly positive) minor $\Delta_{I,J\cup\{a,c\},K\setminus\{x,y\}}$.

We can now assume that there exist five indexes $a \leq b < c$ and $x < y$ such that $b \in I$; $a, c \in [d] \setminus J$; $x, y \in [r]$; and $\Delta_{I,J\cup\{a,c\},K\setminus\{x,y\}} > 0$. For a subword s in the word $abcxy$, let $[s]$ be a shorthand for the triple of subsets I', J', K' such that I' contains b if and only if s contains the letter b, J' contains a or c if and only if s contains the corresponding letter, K' contains x or y if and only if s contains the corresponding letter, and all other entries in I', J', K' are the same as I, J, K. For example, $\Delta_{[bxy]} = \Delta_{I,J,K}$ and $\Delta_{[abc]} = \Delta_{I,J\cup\{a,c\},K\setminus\{x,y\}}$.

Let us write the following 3-term Grassmann-Plücker relations:

$$\Delta_{[bxy]} \cdot \Delta_{[abc]} + \Delta_{[aby]} \cdot \Delta_{[bcx]} = \Delta_{[abx]} \cdot \Delta_{[bcy]}\,, \tag{8.1}$$

$$\Delta_{[axy]} \cdot \Delta_{[abc]} + \Delta_{[aby]} \cdot \Delta_{[acx]} = \Delta_{[abx]} \cdot \Delta_{[acy]}\,, \tag{8.2}$$

$$\Delta_{[cxy]} \cdot \Delta_{[abc]} + \Delta_{[acy]} \cdot \Delta_{[bcx]} = \Delta_{[acx]} \cdot \Delta_{[bcy]}\,. \tag{8.3}$$

By the induction hypothesis, for each minor in the above equations, except $\Delta_{[bxy]} = \Delta_{I,J,K}$, we have already proved its nonnegativity, determined if it is zero or strictly positive, and found a subtraction-free rational expression in terms of the minors $\Delta_{\emptyset,J',K'}$. We also know that $\Delta_{[abc]} > 0$.

Suppose that $\Delta_{[acx]} = 0$. Replacing the column $v_a = u_{a-1} + u_a$ in this minor by the sum of two columns u_{a-1} and u_a, we get $\Delta_{[acx]} = \Delta_{(I\setminus\{b\})\cup\{a-1\},J\cup\{c\},K\setminus\{y\}} + \Delta_{I,J\cup\{c\},K\setminus\{y\}} = 0$. The both summands in the right-hand size are nonnegative by the induction hypothesis and thus they should be zero. In particular, the second summand is $\Delta_{[bcx]} = \Delta_{I,J\cup\{c\},L\setminus\{y\}} = 0$. Similarly, replacing the column v_c in $\Delta_{[acx]}$ by the sum of u_{c-1} and u_c, we get $\Delta_{[acx]} = \Delta_{[abx]} + \Delta_{(I\setminus\{b\})\cup\{c\},J\cup\{a\},L\setminus\{y\}} = 0$ and thus $\Delta_{[abx]} = 0$. So the second term in the left-hand side and the term in the right-hand size of (8.1) are both zero, which implies that $\Delta_{[bxy]} \cdot \Delta_{[abc]} = 0$ and thus $\Delta_{[bxy]} = 0$. Similarly, when $\Delta_{[acy]} = 0$, we deduce that $\Delta_{[aby]} = \Delta_{[bcy]} = 0$ and thus again $\Delta_{[bxy]} = 0$. We obtain that, if $\Delta_{[acx]} = 0$ or $\Delta_{[acy]} = 0$, then $\Delta_{I,J,K} = \Delta_{[bxy]} = 0$.

We can now assume that $\Delta_{[acx]} > 0$ and $\Delta_{[acy]} > 0$. Let us multiply both sides of equations (8.2) and (8.3) and divide the result by the nonzero expression $\Delta_{[acx]} \cdot \Delta_{[acy]}$. Then subtract the resulting equation from (8.1) and finally divide it by $\Delta_{[abc]} > 0$. We obtain the following subtraction-free expression for the minor $\Delta_{[bxy]}$:

$$\Delta_{[bxy]} = \frac{\Delta_{[axy]} \cdot \Delta_{[cxy]} \cdot \Delta_{[abc]}}{\Delta_{[acx]} \cdot \Delta_{[acy]}} + \frac{\Delta_{[axy]} \cdot \Delta_{[bcx]}}{\Delta_{[acx]}} + \frac{\Delta_{[aby]} \cdot \Delta_{[cxy]}}{\Delta_{[acy]}}.$$

Since all minors in the right-hand side have already been expressed subtraction-freely from the minors $\Delta_{\emptyset,J',K'}$ we finally obtain a subtraction-free rational expression for $\Delta_{I,J,K} = \Delta_{[bxy]}$. This finishes the proof of lemma. □

Example 8.4. Let us consider two matrices

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & z_1 & t_1 \\ 0 & 1 & 1 & 0 & z_2 & t_2 \\ 0 & 0 & 1 & 1 & z_3 & t_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & z_1 & t_1 \\ 0 & 1 & 0 & z_2 & t_2 \\ 0 & 0 & 1 & z_3 & t_3 \end{pmatrix}.$$

Suppose that all maximal minors $\Delta_I(A)$ are nonnegative. According to Lemma 8.3, all maximal minors $\Delta_J(B)$ should also be nonnegative and, if we prescribe which minors $\Delta_I(A)$ are zero and which are strictly positive, then we can express each minor $\Delta_J(B)$ as a subtraction-free rational expression in terms of the $\Delta_I(A)$. The only nontrivial case is $\Delta_{245}(B) = z_3 t_1 - z_1 t_3$.

Let us show how to express the minor $\Delta_{245}(B)$. We have $\Delta_{345}(A) = z_1 \geq 0$, $\Delta_{145}(A) = -z_2 \geq 0$, $\Delta_{125}(A) = z_3 \geq 0$, $\Delta_{346}(A) = t_1 \geq 0$, $\Delta_{146}(A) = -t_2 \geq 0$, $\Delta_{126}(A) = t_3 \geq 0$. If $\Delta_{235}(A) = z_1 - z_2 + z_3 = 0$ or $\Delta_{236}(A) = t_1 - t_2 + t_3 = 0$, then either $z_1 = z_2 = z_3 = 0$ or $t_1 = t_2 = t_3 = 0$. In both cases $\Delta_{245}(B) = 0$. If both minors $\Delta_{235}(A)$ and $\Delta_{236}(A)$ are nonzero, then

$$\Delta_{235}(B) = \frac{\Delta_{256}(A) \cdot \Delta_{356}(A)}{\Delta_{235}(A) \cdot \Delta_{236}(A)} + \frac{\Delta_{256}(A) \cdot \Delta_{345}(A)}{\Delta_{235}(A)} + \frac{\Delta_{126}(A) \cdot \Delta_{356}(A)}{\Delta_{236}(A)}.$$

Indeed, this equality can be written as

$$z_3 t_1 - z_1 t_3 = \frac{(z_2 t_3 - z_3 t_2 - z_1 t_3 + z_3 t_1)\,(z_1 t_2 - z_2 t_1 - z_1 t_3 + z_3 t_1)}{(z_1 - z_2 + z_3)\,(t_1 - t_2 + t_3)} + \\ + \frac{(z_2 t_3 - z_3 t_2 - z_1 t_3 + z_3 t_1)\, z_1}{z_1 - z_2 + z_3} + \frac{t_3\,(z_1 t_2 - z_2 t_1 - z_1 t_3 + z_3 t_1)}{t_1 - t_2 + t_3}.$$

We leave the verification of the last identity as an exercise for the reader.

Lemma 8.5. *For two tail-equivalent sequences of k-vectors $(v_1, \dots, v_r)$, $(v'_1, \dots, v'_s)$ and a k-vector v, the sequences $(v, v_1, \dots, v_r)$ and $(v, v'_1, \dots, v'_s)$ are tail-equivalent.*

Proof. Follows form the cyclic symmetry; see Remark 3.3. The maximal minors of the $(r+1+m)$-matrix $(v, v_1, \dots, v_r, w_1, \dots, w_m)$ are equal to the corresponding maximal minors of $(v_1, \dots, v_r, w_1, \dots, w_m, (-1)^{k-1} v)$. □

Let us now prove Lemma 7.4.

Proof of Lemma 7.4. We have $x_1, \dots, x_s = 0$ and $x_i > 0$ for $i \in [s+1, d]$. Let $v_1, \dots, v_{d+1}$ be the first column vectors of the matrix A rescaled by the positive factors: $v_i = A_i$ for $i \in [s] \cup \{d+1\}$, and $v_i = x_i\, A_i$ for $i \in [s+1, d]$. Then $v_{d+1} = v_d - v_{d-1} + v_{d-2} - \cdots + (-1)^{d-s-1} v_{s+1}$.

Also let $u_1, \dots, u_d$ be the linearly independent vectors given by $u_i = v_i$ for $i \in [s]$, and $u_i = v_i - v_{i-1} + v_{i-2} - \cdots + (-1)^{i-s-1} v_{s+1}$ for $i \in [s+1, d]$. Equivalently, these two sequences of vectors are related to each other, as follows: $v_i = u_i$ for $i \in [s]$, $v_i = u_i + u_{i-1}$ for $i \in [s+1, d]$, and $v_{d+1} = u_d$. Lemmas 8.3 and 8.5 imply that $(u_1, \dots, u_d)$ and $(v_1, \dots, v_{d+1})$ are tail-equivalent.

Let $\tilde{A} = (v_1, \dots, v_{d+1}, A_{d+2}, \dots, A_n)$ and $\tilde{C} = (u_1, \dots, u_d, A_{d+2}, \dots, A_n)$. The matrix $\tilde{A}$ is obtained from the matrix A by multiplying the columns with indices $j \in [s+1, d]$ by the factors x_j. Also note that the matrix $\tilde{C}$ is obtained from C by first multiplying it on the left the unipotent upper-triangular matrix (t_{ij}) (which preserves all maximal minors) and then multiplying the rows by indices $i \in [s+1, d]$ by the factors x_i. Here $t_{ij} = (-1)^{i-j}$ if $s+1 \le i \le j \le d$, and $t_{ij} = \delta_{ij}$ otherwise. Thus any maximal minor $\Delta_I(\tilde{A})$ equals $\Delta_I(A)$ times some square-free product of the x_j, $j \in [s+1, d]$, and $\Delta_J(\tilde{C}) = \Delta_J(C) \cdot \prod_{i \in [s+1,d]} x_j$.

Now the tail-equivalence $(u_1, \dots, u_d) \sim_{\text{tail}} (v_1, \dots, v_{d+1})$ implies that, assuming that A belongs to a fixed cell $S_{\mathcal{M}}^{\text{tnn}}$, any maximal minor of C can be written as a subtraction-free rational expression in the nonzero minors of A; and vise versa minors of A can be expressed subtraction-freely in terms of the minors of C and the x_i. Moreover, note that the minors of $\tilde{A}$ can be written just as sums of some minors of $\tilde{C}$; see the beginning of proof of Lemma 8.3. Thus a minor of A can be written as a sum of some minors of C multiplied by some products of the x_i. □

9. Perfection through trivalency

In this section we show how to simplify the structure of a planar directed network without changing the boundary measurements M_{ij}.

First of all, if the network N contains any internal sources or sinks, then we can remove them with all their incident edges. Indeed, a directed path connecting two boundary vertices can never pass through an internal source or sink.

If N contains an internal vertex of degree 2, then we can remove this vertex, glue two incident edges e_1 and e_2 into a single edge e with weight $x_e = x_{e_1} x_{e_2}$.

If N contains a boundary vertex b_i, say, a boundary source of degree $\neq 1$, with incident edges $e_1, \dots, e_d$, then we can pull these edges away from b_i, create a new internal vertex b_i' connected with the source b_i by an edge of weight 1 (directed from b_i to b_i'), and reattach the edges e_i to b_i'. If $d = 0$, that is b_i was an isolated vertex in N, then we create an additional loop at the new vertex b_i', so that b_i' has degree 3.

Now suppose that N contains an internal vertex v of degree > 3. Let $e_1, \dots, e_d$ be the edges incident to v in the clockwise order. If two adjacent edges e_i and e_{i+1} have the same orientation, say, towards the vertex v, then we can again pull these two edges away from the vertex v by creating a new vertex v' and a new edge from v' to v with weight 1 and attaching the edges e_i and e_{i+1} to the vertex v'. This transformation does not change the boundary measurements of the network.

However, if the orientations of the edges $e_1, \dots, e_d$ alternate, then we cannot make the above reduction. In this case we can do the following transformation of the network. Remove the vertex v and create a new cycle C with vertices $v_1, \dots, v_d$ and, say, the clockwise orientation of edges. Assign weights 1 to all edges of the cycle. Reattach the edges $e_1, \dots, e_d$ to the new vertices $v_1, \dots, v_d$. Multiply by 2 the weights of the edges e_i which are oriented towards from the cycle; see Figure 9.1.

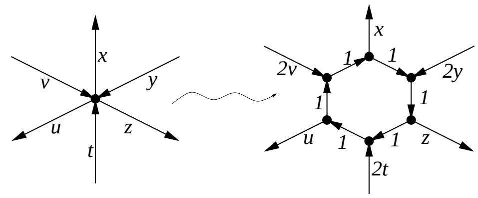


FIGURE 9.1. Blowing up a vertex into a cycle with trivalent vertices

Lemma 9.1. *The above transformation of networks does not change the boundary measurements M_{ij}.*

Proof. First, let us assign variable weights x_i to the edges (v_i, v_{i+1}) of the new cycle. Let P be a directed path in the network N that passes through a vertex v. This path arrives to v through some edge e_i and leaves through e_j. In the new network consider the infinite family of paths that go exactly as the path P with the only exception that upon arrival to C through the edge e_i they can make several turns around C before the departure through the edge e_j. We may assume that $i < j$. (Otherwise cyclicly shift the labels). The contribution of a new path from this family to the corresponding boundary measurement will be the same as the weight of P times the additional factor $(-1)^s\, 2\, x_i \cdots x_{j-1}(x_1 \cdots x_d)^s$, where s is the number of full turns around C the path makes. Indeed, we get the factor $(-1)^s$ because every new turn changes the winding index by 1; the factor 2 comes from the rescaled weight of the incoming edge e_i in the network; and the other factors correspond to the edges of the cycle C the new path passes through. The total contribution of these paths to the boundary measurement is the weight of P times $2\, x_i \cdots x_{j-1} \sum_{s\geq 0}(-1)^s(x_1 \cdots x_d)^s = 2\, x_i \cdots x_{j-1}/(1 + x_1 \cdots x_n)$. If the path P passes through the vertex v several times, then for each passage of P through v we will get a similar factor in the new network. If we now specialize the weights x_i to 1 the additional factors become $2/(1+1) = 1$. Thus the boundary measurement do not change. □

Thus a planar network can be transformed, without changing the boundary measurements, into a network with only trivalent internal vertices which are neither sources nor sinks. In other words, the obtained network has only 2 types of internal vertices: the vertices with two incoming edges and one outgoing edge, and the vertices with one incoming edge and two outgoing edges; see Figure 9.2. Such trivalent networks belong to a more general class of networks, defined as follows.

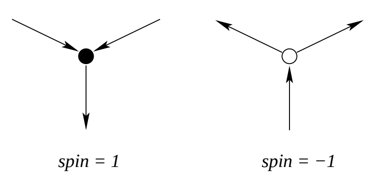


FIGURE 9.2. Two types of trivalent vertices in a perfect network

Definition 9.2. A *perfect network* N is a planar directed network, in the sense of Definition 4.1, such that

(1) Each internal vertex v either has exactly one incident edge directed away from v (and all other edges directed towards v), or has exactly one incident edge directed towards v (all other edges directed away from v).
(2) All boundary vertices b_i in N have degree 1.

The previous discussion implies the following claim.

Proposition 9.3. *Any planar directed network can be transformed without changing the boundary measurements into a perfect network with trivalent internal vertices.*

For an internal vertex v of degree $\deg(v) \neq 2$ in a perfect network N, define the number $col(v) = col_N(v) \in \{1, -1\}$, called the *color* of v, such that, if a vertex v has exactly one outgoing edge then $col(v) = 1$, and if v has exactly one incoming edge then $col(v) = -1$. (If v has degree 2, then we can assign $col(v)$ in any way.) We will display internal vertices with $col(v) = 1$ in black color, and vertices with $col(v) = -1$ in white color.

Lemma 9.4. *For a perfect network N, we have $\sum col(v)\cdot(\deg(v)-2) = k-(n-k)$, where k is the number of sources and $n-k$ is the number of sinks in N, the sum is over internal vertices v.*

Proof. Write two numbers at both ends of each directed edge $e = (v_1, v_2)$ in the network N: a "1" at the target v_2 and a "-1" at v_1. Clearly, the total sum of all written numbers is 0. The sum of written numbers at each internal vertex v equals $col(v) \cdot (\deg(v) - 2)$. On the other hand, at each boundary source we have a "-1" and at each boundary sink we have a "1." □

10. Forget the orientation

In this section we show that it does not really matter how edges are directed in a perfect network.

Theorem 10.1. *Let $N = (G, x)$ and $N' = (G', x')$ be two perfect networks with k sources and $n - k$ sinks such that:*

(1) *The graphs G and G' are isomorphic as undirected graphs.*
(2) *Each internal vertex v of degree $\deg(v) \neq 2$ has the same color $col_N(v) = col_{N'}(v)$ in the networks N and N'.*
(3) *If e is an edge that has the same orientation in N and in N', then $x_e = x'_e$. If N contains an edge $e = (u, v)$ and N' contains the same undirected edge in the opposite orientation $e' = (v, u)$, then $x_e = (x'_{e'})^{-1}$.*

Then the boundary measurement map Meas maps the networks N and N' to the same point $Meas(N) = Meas(N')$ in the Grassmannian Gr_{kn}.

In other words, if we switch directions of some edges in a perfect network N at the same time inverting their weights (as shown in Figure 10.1) so that the colors of internal vertices are preserved, then the boundary measurement $Meas(N) \in Gr_{kn}$ will not change.

Figure 10.1. A switch of edge direction

Example 10.2. Figure 10.2 shows two perfect networks obtained from each other by such switch of edge directions. Their boundary measurement matrices are $A(N) = (1, x + y)$ and $A(N') = ((x + y)^{-1}, 1)$. Indeed, we have $M_{12} = x + y$ in N, and $M'_{21} = y^{-1}/(1 + x\, y^{-1}) = (x + y)^{-1}$ in N'. These two matrices represent the same point in the Grassmannian $Gr_{1,2} = \mathbb{RP}^1$.

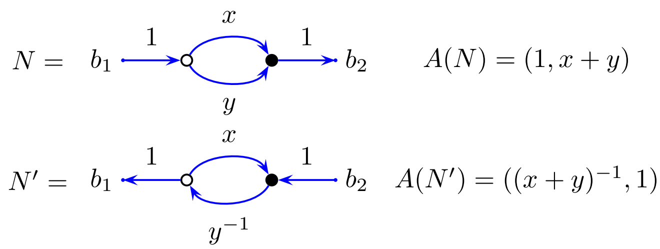


FIGURE 10.2. Two networks that map to the same point in the Grassmannian

Theorem 10.1 means that the maximal minors of the boundary measurement matrices for two networks N and N' obtained by such switches of orientations of edges are related to each other, as follows.

Corollary 10.3. *Let N and N' be two perfect networks satisfying conditions of Theorem 10.1. Let I' be the source set of N'. Then the minor $\Delta_{I'}(A(N))$ is nonzero and, for any k-subset $J \subset [n]$, we have*

$$\Delta_J(A(N')) = \frac{\Delta_J(A(N))}{\Delta_{I'}(A(N))}.$$

We now proceed to proving Theorem 10.1 by first checking two special cases: switching edges in a closed directed cycle and switching edges in a path between two boundary vertices.

Remark 10.4. Perfect networks N have the following important property. If we pick a directed closed cycle C in N, then, for any vertex v in C, the edges that are incident to v and do not belong to C are either all directed towards v or all directed away from v. Thus, for any directed path that hits C at some point and later departs from C, the arrival and departure points should be different from each other. The same property holds for a directed path joining two boundary vertices: another path cannot arrive and depart from it at the same point.

Lemma 10.5. *Let N be a perfect network. Let N' be the network obtained from N by switching the directions of all edges in a closed directed cycle C and inverting their weights. Then the network N' has the same boundary measurements M_{ij} as the network N.*

Proof. Let $v_1, \dots, v_d$ be the vertices in the cycle C, and let $x_i = x_{(v_i, v_{i+1})}$, $i = 1, \dots, d$, be the weights of edges in the cycle. (Here we assume that $v_{d+1} = v_1$.) Consider a collection $\mathcal{P}$ of directed paths connecting two boundary vertices in N that are identical outside the cycle C, but every time when paths enter C they can make any number of turns around C. Suppose that paths from this collection enter C through the vertex v_i and leave through the vertex v_j. Note that $i \neq j$ because the network N is perfect; see Remark 10.4. We may assume that $i < j$. (Otherwise cyclicly shift the labels.) Since paths can make any number of turns around C, this passage through C contributes the factor $x_i x_{i+1} \cdots x_{j-1} \sum_{s\geq 0} (-1)^s (x_1 \cdots x_d)^s =$

$x_i \cdots x_{j-1}/(1+x_1\cdots x_n)$ to the sum of terms $\sum_{P\in\mathcal{P}}(-1)^{wind(P)}\prod_{e\in P} x_e$ from the corresponding boundary measurement. For each passage of paths in $\mathcal{P}$ through C we get a similar factor. Similarly, we have the collection of paths $\mathcal{P}'$ in the network N' identical to paths in $\mathcal{P}$ outside of C. Since we switch directions of the cycle C and invert its weights in the network N', for each passage through C with i and j as above, we now get the factor $x_{i-1}^{-1}x_{i-2}^{-1}\cdots x_1^{-1}x_d^{-1}\cdots x_j^{-1}/(1+x_1^{-1}\cdots x_n^{-1})$. This factor is actually equal to $x_i\cdots x_{j-1}/(1+x_1\cdots x_n)$. Thus the total contributions of paths from $\mathcal{P}$ and $\mathcal{P}'$ to the boundary measurements in N and N', respectively, are the same. This implies that all boundary measurements in N and N' are the same.

Compare this argument with the proof of Lemma 9.1. □

We will need the following matrix identities. All matrices below are square matrices of size m. Let E_{ij} denote the the matrix whose (i,j)th entry is 1 and all other entries are 0. Let $U := \sum_{i\le j} E_{ij}$ be the upper-triangular and $L := \sum_{i>j} E_{ij}$ be the strictly lower-triangular matrices filled with 1's in all allowed places. Also let $E = \sum_i E_{i,\,i+1}$ be the superdiagonal matrix filled with 1's, and let $X = (x_{ij})$ be the matrix filled with the formal variables x_{ij}.

Lemma 10.6. *For the $m\times m$-matrices X, U, L, E as above, define the matrices $A = (a_{ij}) := U + UXU + UXUXU + \cdots$, $B = (b_{ij}) := L - LXL + LXLXL - \cdots$, $C = (c_{ij}) := (1-E-X) + (1-E-X)\,B\,(1-E-X)$, $P = (p_{ij}) := 1 + (1-E-X)\,B$, $Q = (q_{ij}) := 1 + B\,(1-E-X)$, whose coefficients are power series in the x_{ij}. Then the following identities hold*

1. $A^{-1} = 1 - E - X$.
2. $C = a_{1m}^{-1}\,E_{m1}$.
3. $p_{ij} = \delta_{im}\,a_{1m}^{-1}\,a_{1j}$.
4. $q_{ij} = \delta_{j1}\,a_{1m}^{-1}\,a_{im}$.
5. $b_{ij} = (a_{1j}\,a_{im} - a_{1m}\,a_{ij})\,a_{1m}^{-1}$, *for any $i,j\in[m]$.*

Here 1 *denotes the identity matrix and δ_{ij} is the Kronecker delta.*

Proof. (1) We have $A = U + UXU + UXUXU + \cdots = U\,(1 - X\cdot U)^{-1}$, thus $A^{-1} = (1 - X\cdot U)\,U^{-1} = U^{-1} - X$. Notice that $U^{-1} = 1 - E$. Thus $A^{-1} = 1 - E - X$.

(3), (4), (5) Let us show that these follow from (1) and (2). We have $P = C\cdot A = a_{1m}^{-1}E_{m1}\cdot A$, which implies (3). Similarly, $Q = A\cdot C = a_{1m}^{-1}A\cdot E_{m1}$, which implies (4). Also $B = A\cdot C\cdot A - A = a_{1m}^{-1}\,(A\cdot E_{m1}\cdot A) - A$. Thus $b_{ij} = a_{im}(a_{1m})^{-1}a_{1j} - a_{ij}$, which implies (5).

(2) The matrix C expands as the alternating sum

$$C = 1 - E - X + L - EL - XL - EL - LX + LXL + ELX + LXL + \cdots,$$

of all words w in the alphabet $\{L, X, E\}$ such that w has the form $XLXLX\cdots$ or $LXLXL\cdots$, where the initial and/or the final letter "X" can be replaced by the letter "E"; and the sign of w is equal to $(-1)^{xe}$, where xe is the total number of occurrences of letters "X" and "E" in w.

We need to show that the matrix C has only one nonzero entry $c_{m1} = a_{1m}^{-1}$ in the lower left corner.

Suppose that $i < m$. Let $C = C' + C''$, where $C' = (c'_{ij}) := -(EL) + (EL)X + (EL)E + (EL)XL - (EL)XLX - (EL)XLE + \cdots$ is the matrix given by the alternating sum of all words w starting with "EL"; and $C'' = (c''_{ij}) := 1 - E - X + L - XL - LX + XLX + XLE - LXL + \cdots$ is given by the alternating sum of the remaining words starting with a letter "X" or "L" (plus $1-E$). For a fixed index $i<m$, the contribution of a word $w=EL\tilde{w}$ to c'_{ij} equals to the contribution of $(E_{i,i+1})(E_{i+1,i}+\sum_{j<i}E_{i+1,j})\,\tilde{w}=\tilde{w}+L\tilde{w}$. Thus the contribution of a word w from C' equals to the sum of contributions of two words $\tilde{w}$ and $L\tilde{w}$ from C'' (obtained from w by erasing one or two initial letters). Notice that that both words $\tilde{w}$ and $L\tilde{w}$ come with signs opposite to the sign of w. Also note that any word in C'' is of the form $\tilde{w}$ or $L\tilde{w}$. Thus all terms from c'_{ij} cancel all terms from c''_{ij}, implying that $c_{ij}=c'_{ij}+c''_{ij}=0$.

In case when $j>1$ we can use the mirror image of the above argument to show that $c_{ij}=0$. In this case we need consider letters in the end of a word w. Thus $c_{ij}=0$ unless $(i,j)=(m,1)$.

In remains to prove that $c_{m1}=a_{1m}^{-1}$, or equivalently $a_{1m}\cdot c_{m1}=1$. We can express a_{1m} as the sum

$$a_{1m}=1+\sum x_{i_1,j_1}+\sum_{j_1\le i_2} x_{i_1,j_1}x_{i_2,j_2}+\sum_{j_1\le i_2,\,j_2\le i_3} x_{i_1j_1}x_{i_2j_2}x_{i_3j_3}+\cdots,$$

over $i_1,j_1,i_2,j_2,\dots\in[m]$ such that $j_1\le i_2,\ j_2\le i_3,\cdots$. Note that any word w that starts or ends with an "E" makes no contribution to c_{m1}. Thus c_{m1} is given by the alternating sum

$$c_{m1}=1-\sum x_{k_1,l_1}+\sum_{l_1>k_2} x_{k_1,l_1}x_{k_2,l_2}-\sum_{l_1>k_2,\,l_2>k_3} x_{k_1l_1}x_{k_2l_2}x_{k_3l_3}+\cdots,$$

over $k_1,l_1,k_2,l_2,\dots\in[m]$ such that $l_1>k_2,\ l_2>k_3,\ \cdots$.

Let us use the involution principle to prove the equality $a_{1m}\cdot c_{m1}=1$. The product $a_{1m}\cdot c_{m1}$ can be written as the sum of terms $x_{i_1,j_i}\cdots x_{i_rj_r}(-1)^s x_{k_1,l_1}\cdots x_{k_s,l_s}$ over pairs of sequences $p=((i_1,j_1,i_2,j_2,\dots,i_r,j_r),(k_1,l_1,k_2,l_2,\dots,k_s,l_s))$ such that $j_1\le i_2,\ j_2\le i_3,\cdots$ and $l_1>k_2,\ l_2>k_3,\ \cdots$. For $r+s\ge 1$, let us define the map ι from the set of such pairs of sequences to itself by

$$\iota(p):=\begin{cases}((i_1,j_1,\dots,i_r,j_r,k_1,l_1),(k_2,l_2,\dots,k_s,l_s)) & \text{if } j_r\le k_1 \text{ or } r=0,\\ ((i_1,j_1,\dots,i_{r-1},j_{r-1})(i_r,j_r,k_1,l_1,\dots,k_s,l_s)) & \text{if } j_r>k_1 \text{ or } s=0.\end{cases}$$

Then ι is an involution, that is $(\iota)^2=id$. It preserves the monomial corresponding to p and switches its sign. Thus all terms in the product $a_{1m}\cdot c_{m1}$, except the constant term 1, cancel each other. This implies the needed identity. □

Lemma 10.7. *Let N be a perfect network, and let P_0 be a directed path in N from the boundary vertex b_{i_0} to the boundary vertex b_{j_0}. Let N' be the network obtained from N by switching the directions of all edges in P and inverting their weights. Let M_{ij} and M'_{ij} be the boundary measurements of the networks N and N', respectively. Then $M_{i_0,j_0}\ne 0$, and the boundary measurements M'_{ij} can be expressed through the boundary measurements M_{ij}, as follows:*

1. *If $(i,j)=(j_0,i_0)$, then $M'_{j_0,i_0}=M_{i_0,j_0}^{-1}$.*
2. *If $i=j_0$ and $j\ne i_0$, then $M'_{j_0,j}=M_{i_0,j}/M_{i_0,j_0}$.*
3. *If $i\ne j_0$ and $j=i_0$, then $M'_{i,i_0}=M_{i,j_0}/M_{i_0,j_0}$.*
4. *If $i\ne j_0$ and $j\ne i_0$, then $M'_{ij}=\Delta_{(I\setminus\{i_0,i\})\cup\{j_0,j\}}(A(N))/M_{i_0,j_0}$, where I is the source set of the network N.*

This implies that the boundary measurement map Meas maps the networks N and N' to the same point $Meas(N)=Meas(N')$ in the Grassmannian.

Proof. The measurement M_{i_0,j_0} is nonzero because there is at least one path from b_{i_0} to b_{j_0} in the network N, e.g., the path P_0. Let $P_0 = (b_{i_0}, v_1, v_2, \dots, v_m, b_{j_0})$. We may assume that the path P_0 has no self-intersections, because we can get rid of all self-intersections using Lemma 10.5. We may also assume that the weights of all edges in P_0 are 1. Indeed, we can apply the same gauge transformation (4.2) to the networks N and N' that transforms the weights of all edges of P_0, expect a single edge $e_0 = (b_{i_0}, v_1)$, into 1's. Since the weight of e_0 produces same factors in both sides of all identities (1)–(4), we may assume that this weight in also 1.

Let us use the notation $b_{\overline{1}}, \dots, b_{\overline{m}}$ for the vertices $v_1, \dots, v_m$ of the path P_0. For $i, j \in [n] \cup \{\overline{1}, \dots, \overline{m}\}$, let

$$\tilde{M}_{ij} := \sum_{P: b_i \to b_j} (-1)^{wind(P)} \prod_{e \in P} x_e$$

be the generating function for all directed paths in N from b_i to b_j that have no common points with the path P_0 (expect the first and the last point in P). Note that all paths P should lie in one of the two networks on which the path P_0 subdivides the network N. Thus $\tilde{M}_{ij}$ are the boundary measurements for these smaller networks. We have $\tilde{M}_{ii} = 0$, because the network is perfect. Let us also define the $m \times m$-matrix $X = (x_{ab})$ such that $x_{ab} = \tilde{M}_{\overline{a},\overline{b}}$ if $a < b$, $x_{ab} = -\tilde{M}_{\overline{a},\overline{b}}$ if $a > b$, and $x_{ab} = 0$ if $a = b$.

Any path P' in N from b_{i_0} to b_{j_0} first goes along the path P_0; then P' may depart from P_0 at some vertex v_{k_1} and then arrive to P_0 at v_{l_1}; then P' may depart from P_0 again at v_{k_2} and arrive at v_{l_2}; etc. Let $k_1, l_1, k_2, l_2, \dots, k_s, l_s$ be the indices of these departure and arrival points. Then we have $l_1 < k_2,\ l_2 < k_3,\ \cdots$, because P' coincides with P_0 on the segments from v_{l_i} to $v_{k_{i+1}}$. The total contribution to M_{i_0,j_0} of all paths P' with given departure and arrival points is $x_{k_1,l_1} \cdots x_{k_s,l_s}$. Indeed, each path P' breaks into segments between the departure and arrival points, which give the factors x_{k_i,l_i}. The extra factor -1 in the case when $k_i > l_i$ accounts for an extra cycle in P' that we get in this case because the path P' bumps into itself at the arrival point v_{l_i}. This shows that

$$M_{i_0,j_0} = 1 + \sum x_{k_1,l_1} + \sum_{l_1 \le k_2} x_{k_1,l_1}\, x_{k_2,l_2} + \sum_{l_1 \le k_2,\, l_2 \le k_3} x_{k_1,l_1}\, x_{k_2,l_2}\, x_{k_3,l_3} + \cdots,$$

where the sum is over departure-arrival sequences $k_1, l_1, k_2, l_2, \dots, k_s, l_s \in [m]$ such that $l_i \le k_{i+1}$, for $i \in [s-1]$.

The boundary measurement M'_{j_0,i_0} in the network N' is given by a similar expression where we need to sum over departure-arrival sequences such that $l_i > k_{i+1}$ for $i \in [s-1]$ and we need to switch the signs of all x_{ab}.

Let A, B, C, P, Q be the matrices as in Lemma 10.6. Then $M_{i_0,j_0} = a_{1m}$ and $M'_{j_0,i_0} = c_{m1}$. According to Lemma 10.6(2), we have $c_{m1} = a_{1m}^{-1}$. Thus $M'_{j_0,i_0} = M_{i_0,j_0}^{-1}$.

Similarly, we can express other boundary measurements of the networks N and N' in terms of these matrices. For $i \ne i_0$ and $j \ne j_0$, we have

$$M_{i_0,j} = \sum_{c=1}^{m} a_{1c}\, \hat{M}_{\overline{c},j}, \qquad M_{i,j_0} = \sum_{c=1}^{m} \hat{M}_{i,\overline{c}}\, a_{cm},$$

$$M'_{j_0,j} = \sum_{c=1}^{m} p_{mc}\, \hat{M}_{\overline{c},j}, \qquad M'_{i,i_0} = \sum_{c=1}^{m} \hat{M}_{i,\overline{c}}\, q_{c1}.$$

According to parts (3) and (4) of Lemma 10.6, we have $p_{mc} = a_{1c}/a_{1m}$ and $q_{c1} = a_{cm}/a_{1m}$. Thus $M'_{j_0,j} = M_{i_0,j}/M_{i_0,j_0}$ and $M'_{i,i_0} = M_{i,j_0}/M_{i_0,j_0}$.

We also have

$$M_{i,j} = \epsilon \cdot \hat{M}_{i,j} + \delta \cdot \sum_{c,d\in[m]} \hat{M}_{i,\overline{c}}\, a_{cd}\, \hat{M}_{\overline{d},j}, \quad M'_{i,j} = \epsilon' \cdot \hat{M}_{i,j} + \delta' \cdot \sum_{c,d\in[m]} \hat{M}_{i,\overline{c}}\, b_{cd}\, \hat{M}_{\overline{d},j},$$

where $(\epsilon,\delta,\epsilon',\delta') = (1,0,1,0)$ if the cords $[b_{i_0}, b_{j_0}]$, $[b_i, b_j]$ form a crossing (see Figure 5.1), $(\epsilon,\delta,\epsilon',\delta') = (1,1,-1,1)$ if the cords form an alignment, and $(\epsilon,\delta,\epsilon',\delta') = (-1,1,1,1)$ if the cords form a misalignment. By Lemma 10.6(5), $b_{cd} = (a_{1d}a_{cm} - a_{1m}a_{cd})\, a_{1m}^{-1}$. According to Proposition 5.2, the minor $\Delta_{(I\setminus\{i_0,i\})\cup\{j_0,j\}}(A(N))$ is equal to $M_{i_0\,j}M_{i,j_0} - M_{i_0,j_0}M_{ij}$ the above cords form an crossing, to $M_{i_0,j_0}M_{ij} - M_{i_0\,j}M_{i,j_0}$ if the cords form an alignment, or to $M_{i_0,j_0}M_{ij} + M_{i_0\,j}M_{i,j_0}$ if the cords form a misalignment. In all three cases, we get $M'_{ij} = \Delta_{(I\setminus\{i_0,i\})\cup\{j_0,j\}}(A(N))/M_{i_0,j_0}$, as needed, which proves (4).

Finally, note that the proved relations (1)–(4) mean that $\frac{\Delta_J(A(N'))}{\Delta_{I'}(A(N'))} = \frac{\Delta_J(A(N))}{\Delta_{I'}(A(N))}$ for all k-subsets $J \subset [n]$ such that $|J \setminus I'| = 1$, where $I' = (I \setminus \{i_0\}) \cup \{j_0\}$ is the source set of the network N'. Since the $k(n-k)$ quotients of the Plücker coordinates $\frac{\Delta_J}{\Delta_{I'}}$, for all such J's, form a coordinate system on $Gr_{kn} \setminus \{\Delta_{I'} = 0\}$, we deduce that the matrices $A(N)$ and $A(N')$ represent the same point in the Grassmannian Gr_{kn}, as needed. □

Proof of Theorem 10.1. Let H be the subset of edges of G whose orientations are switched. For any internal vertex v, the fact that the color $col(v)$ is preserved implies that H contains zero or exactly two edges adjacent to v. Thus H is the disjoint union of several cycles and/or paths connecting pairs of boundary vertices. According to Lemmas 10.5 and 10.7 we can switch the orientations of edges in these cycles and paths one by one without changing the boundary measurements. □

11. Plabic networks

In this section we will define new weights y_f assigned to faces f of a network, which are obtained from the edge weights x_e by a simple transformation. Then we define plabic graphs and networks which are no longer directed.

For a planar graph G (directed or undirected) drawn inside a disk, let $V := V(G)$ be the set of its internal vertices, $E := E(G)$ be the set of its edges, and $F := F(G)$ be the set of its *faces,* that is the regions on which the edges subdivide the disk. Let us say that a connected component of G is *isolated* if it does not contain a boundary vertex; let c be the number of such isolated components. The Euler formula says that $|V| - |E| + |F| = 1 + c$. If $c = 0$, then all faces are simply connected. If $c \geq 1$, then there are some faces which are not simply connected, because they contain isolated components inside of them. Clearly, if we can always remove isolated components from a directed network without affecting the boundary measurements.

Lemma 11.1. *Suppose that G is a planar directed graph without isolated components. Then the space of directed networks with given graph G, modulo the gauge transformations, is isomorphic to*

$$\mathbb{R}_{>0}^E/\{\textit{gauge transformations}\} \simeq \mathbb{R}_{>0}^{|E|-|V|} = \mathbb{R}_{>0}^{|F|-1}.$$

Proof. In this case all gauge transformations (4.2) are nontrivial. Indeed, if we have a gauge transformation $x'_e = x_e\, t_u t_v^{-1}$ such that $x'_e = x_e$ for all edges e, then we should have $t_u = 1$ for all vertices u adjacent to the boundary vertices b_i, then we should have $t_v = 1$ for all vertices v adjacent to the vertices u, etc. Thus $t_v = 1$ for all internal vertices v. This implies the first isomorphism. The Euler formula $|V| - |E| + |F| = 1$ implies the second equality. □

Let $f \in F(G)$ be a face in a planar directed network $N = (G, x)$. The exterior boundary of f consists of some edges $e_1, \dots, e_k$. If f is not simply connected then it has one or several holes corresponding to isolated components inside f. Let $e_{k+1}, e_{k+2}, \dots, e_l$ be the edges in these holes. (Note that the same edge might occur twice in this sequence.) Let us assume that the exterior boundary of f is oriented clockwise and the boundaries of all holes are counterclockwise. Let $I_f^+ \subseteq [l]$ be the index set of edges e_i whose orientations in the graph G agree with the orientation of the boundary of f, and let $I_f^- = [l] \setminus I^+$ be the index set of edges whose orientations disagree. Then we define the *face weight* y_f of the face f in N as

$$y_f := \prod_{e_i \in I_f^+} x_{e_i} \cdot \prod_{e_j \in I_f^-} x_{e_j}^{-1},$$

see Figure 11.1. Note that, if we switch directions of some edges inverting their weights (as shown on Figure 10.1), then the face weights y_f will not change. There is one relation for the faces weights, namely, $\prod_{f \in F} y_f = 1$. Indeed, this product includes exactly one x_e and exactly one x_e^{-1} for each edge $e \in E$.

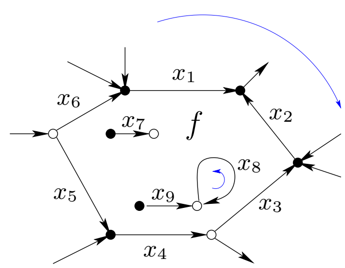


FIGURE 11.1. A face with weight $y_f = (x_1\, x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1} x_6)\, (x_7\, x_7^{-1})\, (x_8^{-1} x_9^{-1} x_9)$

Let $\mathbb{R}_{>0}^{F-1} \simeq \mathbb{R}_{>0}^{|F|-1}$ be the set of $(y_f)_{f \in F} \in \mathbb{R}_{>0}^F$ such that $\prod y_f = 1$.

Lemma 11.2. *For a planar directed graph G, the map $(x_e)_{e \in E} \mapsto (y_f)_{f \in F}$ defined as above gives the isomorphism*

$$\mathbb{R}_{>0}^E / \{\textit{gauge transformations}\} \simeq \mathbb{R}_{>0}^{F-1}.$$

Proof. Let us first assume that G has no isolated components. We prove the claim by induction on $|F|$. By Lemma 11.1 we already know that both sides have the same dimension. Thus it is enough to show that the kernel of the map consists of a single point. Let $(x_e)_{e \in E} \in \mathbb{R}_{>0}^E$ be a point that maps to $(y_f)_{f \in F}$ with all $y_f = 1$. We need to show that one can transform all edge weights x_e into 1's by the gauge transformations. Let us pick a face $f_0 \in F$ adjacent to the boundary of the disk. The boundary of f_0 contains a segment (b_i, b_{i+1}) of the boundary of the disk, and a path $P = (b_i, v_1, \dots, v_k, b_{i+1}]$. Applying the gauge transformations at the vertices

$v_1, \dots, v_k$, we can transform all weights of the $k+1$ edges in the path P, except a single edge, into 1's. But, since we have $y_{f_0} = 1$ and y_{f_0} is the product of weights of edges in P (some of them might be inverted), we deduce that the weight of the last edge in P should also be 1. We can now apply the induction hypothesis to the smaller graph G', which is obtained from G by removing the face f_0 and replacing the segment (b_i, b_{i+1}) of the boundary of the disk with the path P. By induction, all weights of edges in G' can be transformed into 1's by the gauge transformations at the internal vertices of the graph G', that is the vertices in $V \setminus \{v_1, \dots, v_k\}$. Thus all weights x_e of edges in G are now transformed into 1's, as needed.

In the general case, we prove the claim by induction on the number c of isolated components. We have already established the base case $c = 0$. Suppose that $c \geq 1$. Let G' be the graph obtained from G by removing an isolated component G'' located in some face f of G. Let F', E', F'', E'' be the face and edge sets of these two graphs, and also let $\tilde{E}'' \subset E''$ be the set of internal edges of G''. The face weights for G' and G'' are the same as for the graph G with a single exception: the face weight y_f for G is obtained by multiplying the corresponding weight for G' by the product of all face weights for G''. By induction, we have $\mathbb{R}_{>0}^{E'}/\{\text{gauge transformations}\} \simeq \mathbb{R}_{>0}^{F'-1}$. If G'' has no faces, i.e., it is a tree, then $\mathbb{R}_{>0}^{E}/\{\text{gauge transformations}\} \simeq \mathbb{R}_{>0}^{E'}/\{\text{gauge transformations}\} \simeq \mathbb{R}_{>0}^{F'-1} = \mathbb{R}_{>0}^{F'-1}$. Otherwise, G'' breaks into several disjoint subgraphs $G_1, \dots, G_k$ (connected by paths), each of which is formed by a cycle with a small graph inside, so that each face of G'' belong to one of these graphs G_i. Applying the induction hypothesis to each graph G_i, we deduce that $\mathbb{R}_{>0}^{\tilde{E}''}/\{\text{gauge transformations}\} \simeq \mathbb{R}_{>0}^{F''-k}$. Since the weights of the boundary edges in G'' can be arbitrary, we get $\mathbb{R}_{>0}^{E''}/\{\text{gauge transformations}\} \simeq \mathbb{R}_{>0}^{F''}$. Thus $\mathbb{R}_{>0}^{E}/\{\text{gauge transformations}\} \simeq \mathbb{R}_{>0}^{F'-1} \times \mathbb{R}_{>0}^{F''} \simeq \mathbb{R}_{>0}^{F-1}$, as needed. □

Remark 11.3. As we have already mentioned, isolated components do not affect the boundary measurements. The reason that we are considering graphs that might have isolated components will be clear below, when we define certain transformations of graphs. Even if an original graph does not have isolated components, after performing several moves we might create such components.

The boundary measurement map $Meas_G : \mathbb{R}_{>0}^{E}/\{\text{gauge transformations}\} \to Gr_{kn}$, (4.3), now transforms into the map $\mathbb{R}_{>0}^{F-1} \to Gr_{kn}$. Below we will use the face weights y_f instead of the edge weights x_e. With these weights we no longer need to care about gauge transformations, and we no longer need to invert the weights when we switch edge directions; see Figure 10.1.

We can define the boundary measurements of a directed network N purely in terms of the face weights y_f without using the edge weights x_e. A directed path P without self-intersections that connects two boundary vertices b_i and b_j subdivides the disk into two parts: the part that is on the right side of P and the part on left of P (as we go from b_i to b_j). We define $wt(P, y)$ as the product of the weights y_f for the faces in the right part of P. Also, for a clockwise (resp., counterclockwise) closed cycle C, define $wt(C, y)$ as the product of the y_f for the faces f inside (resp., outside) of C. Now, for an arbitrary path P from b_i to b_j, we can remove cycles $C_1, \dots, C_k$ and reduce it to a path $\tilde{P}$ without self-intersections. In this case, define $wt(P, y) = wt(\tilde{P}) \cdot \prod C_i$.

Lemma 11.4. *For any directed path P, we have $wt(P, y) = \prod_{e \in P} x_e$.*

Proof. It is enough to prove this equality of a path P without self-intersections and also prove that $wt(C) = \prod_{e\in C} x_e$ for a closed cycle C. In all cases, the product of the y_f for all faces f in the right side of P (resp., inside/outside of C) includes exactly one weight x_e and exactly one weight $(x_e)^{-1}$ for all edges e in the corresponding areas expect for the edges e that belong to P (resp., to C). The remaining terms give the needed product of edge weights. □

Thus the (formal) boundary measurements of a network can be defined as

$$M_{ij} = \sum_{P:b_i\to b_j} (-1)^{wind(P)} wt(P, y),$$

cf., (4.1).

Definition 11.5. A *planar bicolored graph,* or simply a *plabic graph* is a planar (undirected) graph G, defined as in Definition 4.1 but without orientations of edges, such that each boundary vertex b_i is incident to a single edge, together with a function $col : V \to \{1, -1\}$ on the set V of internal vertices. As before, we will display vertices with $col(v) = 1$ in black color, and vertices with $col(v) = -1$ in white.

A *plabic network* $N = (G, y)$ is plabic graph G together with positive real weights $y_f > 0$ assigned to faces f of G such that $\prod y_f = 1$.

A *perfect orientation* of a plabic graph or network is a choice of orientation of its edges such that each internal vertex v with $col(v) = 1$ is incident to exactly one edge directed away from v; and each v with $col(v) = -1$ is incident to exactly one edge directed towards v. A plabic graph or network is called *perfectly orientable* if it has a perfect orientation.

Let us say that a plabic graph or network has *type* (k, n) if its has n boundary vertices and $k + (n - k) = \sum_{v\in V} col(v)\,(\deg(v) - 2)$.

Remark 11.6. One can think about plabic graphs as some kind of "Feynman diagrams," where the black and white vertices represent certain "elementary particles" of two types and edges represent "interactions" between these particles.

According to Lemma 11.2, plabic networks with a choice of a perfect orientation correspond to perfect networks modulo gauge transformation. Theorem 10.1 says that two perfect networks N and N' that correspond to two orientations of the same plabic network should map into the same point $Meas(N) = Meas(N') \in Gr_{kn}$. Lemma 9.4 says a perfect orientation of a plabic graph of type (k, n) should have k sources and $n - k$ sinks. Thus the boundary measurement map $Meas$ gives a well defined map

$$\tilde{Meas} : \{\text{perfectly orientable plabic networks of type } (k, n)\} \to Gr^{\text{tnn}}_{kn}.$$

For a perfectly orientable plabic graph G, we have the induced map on the set $\mathbb{R}^{F(G)-1}_{>0}$ of plabic networks with the given graph G:

$$\tilde{Meas}_G : \mathbb{R}^{F(G)-1}_{>0} \to Gr^{\text{tnn}}_{kn}.$$

Note that not any plabic network is perfectly orientable. For example, a plabic network that contains an isolated component with a single vertex is not perfectly orientable.

For a plabic graph G of type (k, n) and a perfect orientation $\mathcal{O}$ of G, let $I_{\mathcal{O}} \subset [n]$ be the k-element source set of this orientation. Define the *matroid* of G as the set of the k-subsets $I_{\mathcal{O}}$ for all perfect orientations:

$$\mathcal{M}_G := \{I_{\mathcal{O}} \mid \mathcal{O} \text{ is a perfect orientation of } G\}.$$

Proposition 11.7. *For any perfectly orientable plabic graph G, the collection $\mathcal{M} = \mathcal{M}_G$ is a totally nonnegative matroid. The boundary measurement map $\tilde{Meas}_G$ sends plabic networks with the graph G into the totally nonnegative Grassmann cell $S_{\mathcal{M}}^{\text{tnn}}$ associated with this matroid:*

$$\tilde{Meas}_G(\mathbb{R}_{>0}^{F(G)-1}) \subseteq S_{\mathcal{M}}^{\text{tnn}}.$$

In Section 16 we will prove that this inclusion is actually the equality; see Corollary 16.5.

Remark 11.8. This proposition gives way to combinatorially determine the totally nonnegative Grassmann cell corresponding to a planar network without any calculations of the boundary measurements. Indeed, first transform a network into a perfect network (see Section 9), then transform it into a plabic network (forget orientations of edges but remember colors of vertices), and calculate the matroid $\mathcal{M}_G$.

Proof of Proposition 11.7. □

Example 11.9. Figure 11.2 shows five perfect orientations of a plabic graph G. Recording their source sets, we obtain the following matroid with five bases $\mathcal{M}_G = \{\{1,4\}, \{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}\}$.

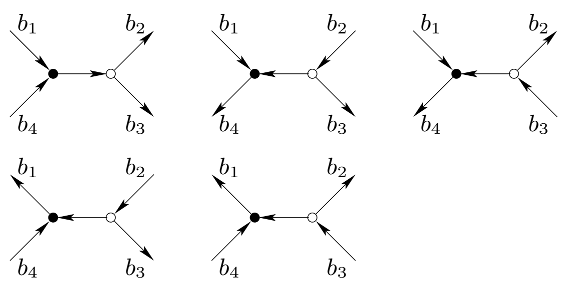


FIGURE 11.2. Perfect orientations of a plabic graph

Let us give two additional combinatorial descriptions of the matroid $\mathcal{M}_G$ in terms of paths and in terms of matchings.

Let us fix a perfect orientation $\mathcal{O}$ of G and let $I = I_{\mathcal{O}}$. Define the *path matroid* $\mathcal{M}_G^p$ as the set of k-element subsets $J \subset [n]$ such that the boundary vertices $\{b_i \mid i \in I \setminus J\}$ can be connected with the boundary vertices $\{b_j \mid j \in J \setminus I\}$ by a family of pairwise noncrossing directed paths in the graph G with edge orientation $\mathcal{O}$.

Let us say that a plabic graph G is *bipartite* if any edge in G joins two vertices of different colors (assuming that the colors of all boundary vertices are white). Note that we can easily make any plabic graph bipartite by inserting vertices of different color in the middle of unicolored edges. A *partial matching* in such graph G is a subset M of edges such that each internal vertex is incident to exactly one edge in

M. (But the boundary vertices can be incident to one or zero edges.) Let $I_M \subseteq [n]$ be the set of indices i such that b_i belongs an edge from M. Define the *matching matroid*

$$\mathcal{M}_G^m := \{I_M \mid M \text{ is a partial matching of } G\}.$$

Lemma 11.10. *For any plabic graph G, we have $\mathcal{M}_G^p = \mathcal{M}_G$. Also, if G is a bipartite plabic graph, then $\mathcal{M}_G^m = \mathcal{M}_G$.*

Proof. Any perfect orientation $\mathcal{O}'$ is obtained from the fixed perfect orientation $\mathcal{O}$ by switching edge directions in a family of noncrossing directed paths between boundary vertices or closed cycles. This implies that $\mathcal{M}_G^p = \mathcal{M}_G$. For a perfect orientation $\mathcal{O}$ in bipartite plabic graph, let M be the set of edges in G directed from a black vertex to a white vertex. The map $\mathcal{O} \mapsto M$ is a bijection between perfect orientations and partial matchings, which implies that $\mathcal{M}_G^m = \mathcal{M}_G$. □

12. Transformations of plabic networks

In this section we define several local transformation of plabic networks. In all transformations below, we change a small fragment in a network and sometimes change weights of adjacent faces. The weights of remaining faces are not changed. We will call the first three transformations (M1)–(M3) the *moves,* and the next three transformations (R1)–(R3) the *reductions.* We think of the moves as invertible transformations of networks, which we can perform in both directions. On the other hand, we will perform reductions only in one direction in order to simplify the structure of a network. Essentially, the only nontrivial transformation of networks is the *square move;* all other moves and reductions play an auxiliary role.

(M1) SQUARE MOVE. If a network has a square formed by four trivalent vertices whose colors alternate as we go around the square, then we can switch colors of these four vertices and transform the weights of adjacent faces as shown on Figure 12.1. In other words, if y_0 is the weight of the face inside the square and y_1, y_2, y_3, y_4 are weights of the four adjacent faces, then we transform these weights as follows

$$y_0' = y_0^{-1},\ y_1' = \frac{y_1}{1+y_0^{-1}},\ y_2' = y_2\,(1+y_0), \quad y_3' = \frac{y_3}{1+y_0^{-1}},\ y_4' = y_4\,(1+y_0). \tag{12.1}$$

In the case when some of the four areas marked by y_1, y_2, y_3, y_4 in Figure 12.1 belong to the same face (connected outside of the shown fragment of the network), say, if $y_1 = y_2$ are in the same face, then its weight changes to $y_1 \frac{1}{1+y_0^{-1}}(1+y_0) = y_1 y_0$.

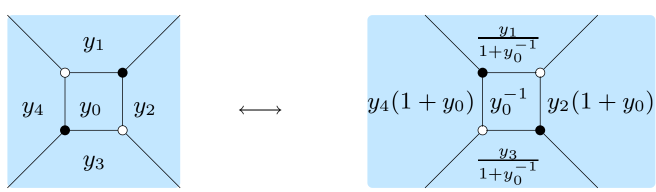


Figure 12.1. Square move

(M2) UNICOLORED EDGE CONTRACTION/UNCONTRACTION. If a network contains an edge with two vertices of the same color, then we can contract this edge into a single vertex with the same color; see Figure 12.2. The face weights y_f are not

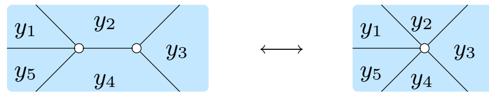


FIGURE 12.2. Unicolored edge contraction

changed. On the other hand, we can also uncontract a vertex into an edge with vertices of the same color.

(M3) MIDDLE VERTEX INSERTION/REMOVAL. If a network contains a vertex of degree 2, then we can remove this vertex and glue the incident edges together; see Figure 12.3. The face weights y_f are not changed. On the other hand, we can always insert a vertex (of any color) in the middle of any edge.

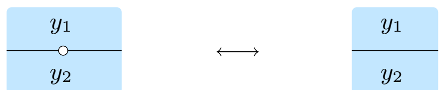


FIGURE 12.3. Middle vertex insertion/removal

(R1) PARALLEL EDGE REDUCTION. If a network contains two trivalent vertices of different colors connected by a pair of parallel edges, then we can remove these vertices and edges, glue the remaining pair of edges together, and transform the face weights as shown on Figure 12.4. If $y_1 = y_2$ correspond to the same face, then we change its weight to $y_1 \frac{1}{1+y_0^{-1}}(1+y_0) = y_1 y_0$.

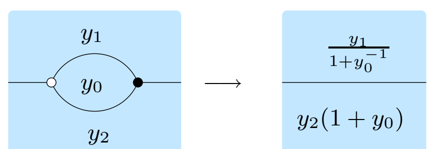


FIGURE 12.4. Parallel edge reduction

(R2) LEAF REDUCTION. If a network contains a vertex (leaf) u incident to a single edge $e = (u, v)$ which in turn is incident to edges $e_1, \dots, e_k$, $k \geq 2$, such that $col(u) = -col(v)$, then we can remove the vertex u together with the edge e, disconnect the edges $e_1, \dots, e_k$, and assign the color equal to $col(u)$ to all newly created vertices of the edges $e_1, \dots, e_k$; see Figure 12.5. If this operation joins several faces $f_1, \dots, f_l$ into a single face f, then its weight y_f should be the product of weights of $f_1, \dots, f_l$. (Note that is it possible all faces were already connected outside the shown fragment so that this transformation does not reduce the number of faces. In this case it creates new isolated components.)

(R3) DIPOLE REDUCTION. If a network contains an isolated component C that consists of a pair of vertices of different colors connected by an edge, then we can remove C from the network; see Figure 12.6. The edge weights are not changed.

If forget about face weights in the above moves and reductions we obtain corresponding transformations of plabic graphs.

Let us say that two plabic networks (or graphs) are *move-equivalent* if they can be obtained from each other by moves (M1)–(M3). Similarly, two plabic networks

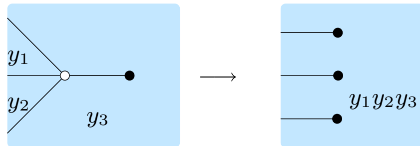


Figure 12.5. Leaf reduction

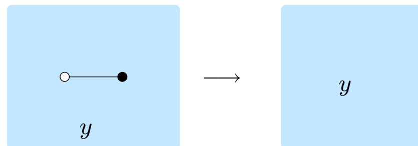


Figure 12.6. Dipole reduction

(graphs) are *move-reduction-equivalent* if they can be transformed into the same network (graph) by moves (M1)–(M3) and reductions (R1)–(R3).

Theorem 12.1. *Let N and N' be two perfectly orientable plabic networks of the same type (k, n). Then the boundary measurement map $\tilde{Meas}$ maps them into the same point $\tilde{Meas}(N) = \tilde{Meas}(N')$ in the Grassmannian Gr_{kn} if and only if the networks N and N' are move-reduction-equivalent.*

We will prove this theorem in Section 16. In one direction this claim can be verified by direct calculation.

Lemma 12.2. *Suppose that a plabic network N' is obtained from N by performing a move* (M1)–(M3) *or a reduction* (R1)–(R3). *Then N' is perfectly orientable if and only if N is perfectly orientable. If this is the case, then $\tilde{Meas}(N) = \tilde{Meas}(N')$.*

Proof. It is quite easy to check in all six cases that a perfect orientation of N gives a perfect orientation of N', and vise versa. The only nontrivial transformation of networks is the square move (M1). Let us check that the boundary measurement map is invariant under this transformation. Let us pick two perfect orientations of the networks N and N', say, the orientations whose parts in the transformed fragment are shown on Figure 12.7 and which are are identical everywhere else. The transformation of face weights in the square move corresponds to the following transformations of weights of the four edges that form the square:

$$x_1' = \frac{x_3x_4}{x_2 + x_1x_3x_4},\ x_2' = x_2 + x_1x_3x_4,\ x_3' = \frac{x_2x_3}{x_2 + x_1x_3x_4},\ x_4' = \frac{x_1x_3}{x_2 + x_1x_3x_4},$$

where we assume that the remaining edge weights are not changed.

Then for the both oriented network fragments shown on Figure 12.7, sums over paths from u_i to v_j in N and N' are the same: $x_2 + x_1x_3x_4 = x_2'$, $x_1x_3 = x_2'x_4'$, $x_3x_4 = x_1'x_2'$, $x_3 = x_3' + x_1'x_2'x_4'$. Thus all boundary measurements in the both networks should be the same, implying $\tilde{Meas}(N) = \tilde{Meas}(N')$.

Similarly, for the the parallel edge reduction (R1), let us pick a perfect orientation of edges as in Figure 12.8. The transformation of face weights in this move correspond to the transformation of edge weights given by $x_1' = x_1(x_2 + x_3)x_4$. Clearly, this transformation does not change the boundary measurements.

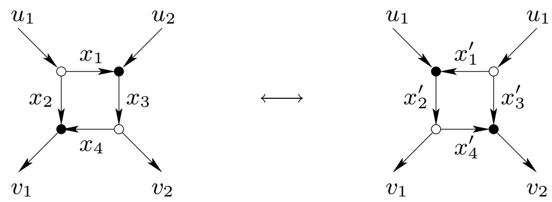


FIGURE 12.7. Square move in directed networks

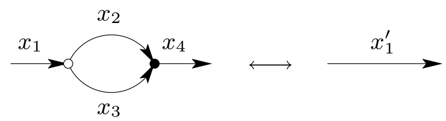


FIGURE 12.8. Parallel edge reduction in directed networks

For the remaining moves and reductions is it clear that $\tilde{Meas}(N) = \tilde{Meas}(N')$. □

Remark 12.3. Figure 12.7 shows just one of several possible square moves in *directed* networks. If we pick another perfect orientation of the edges (inverting the edge weights x_i and x'_i whenever we switch orientations), then we get another legitimate square move.

There are several special kinds of networks to which we can easily transform any plabic network using moves (M2)–(M3) and reductions (R1)–(R3).

Loop removal: We can remove all loops (i.e., edges whose both ends are the same vertices) from a network. By uncontracting some edges we can make all loops attached to trivalent vertices. Then we apply the loop reduction shown on Figure 12.9. This reduction follows from parallel edge reduction (R1). Indeed, insert an additional vertex of different color to the loop using (M3), then uncontract this vertex into an edge using (M2), and apply parallel edge reduction (R1). Let us call a network without loops *loopless.*

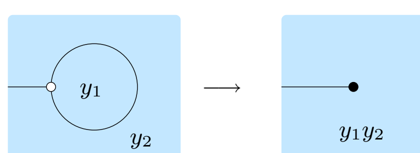


FIGURE 12.9. Loop reduction

Leaf removal: We can also easily get rid of all leaves in a plabic network, except the leaves connected to boundary vertices. (Let us call these special unremovable leaves the *boundary leaves.*) Indeed, if a leaf u is attached to a vertex v of the same color then we can just contract the edge (u, v) using move (M2). If colors of u and v are different, then we can remove the edge (u, v) using leaf reduction (R3) if $\deg(v) \geq 3$. (If $\deg v = 2$ then we can remove v using (R2) and if $\deg(v) = 1$ then we can remove it by dipole reduction (R3).) Then similarly treat all newly formed leaves, etc. Let us call a network *leafless* if it has no leaves, except the boundary leaves.

Contraction: Any plabic network can be transformed into a network that has no unicolored edges, no non-boundary leaves, and no vertices of degree 2. Indeed, first remove all non-boundary leaves, then contract all unicolored edges, then remove all new vertices of degree 2, then contract all new unicolored edges, etc. Let us call such networks *contracted*.

Making the graph trivalent: On the other hand, we can first get rid of non-boundary leaves and vertices of degree 2, and then uncontract all vertices of degree > 3 by replacing them with trivalent trees. We obtain a network with all trivalent internal vertices (expect boundary leaves). Let us call such plabic networks *trivalent.*

Corollary 12.4. *Any plabic network can be transformed (without changing the boundary measurements) into a loopless contracted network. On the other hand, it can also be transformed into a loopless trivalent network.*

Definition 12.5. Let us say that a plabic network (or graph) is *reduced* if it has no isolated connected components and there is no network/graph in its move-equivalence class to which we can apply a reduction (R1) or (R2). A *leafless* reduced network/graph is a reduced network/graph without non-boundary leaves.

We will see that reduced networks are exactly the networks without isolated component with the minimal possible value of $E - V = F - c - 1$ in its move-reduction-equivalence class, where V, E, F, c as in Section 11.

Note that if a network has no isolated components then all its move-equivalent networks have no isolated components, so there is no chance to apply dipole reduction (R3). In many cases it will be more convenient to use leafless reduced networks, to which we can easily transform any reduced network by the leaf removal procedure as described above. An arbitrary reduced network can obtained from a leafless one by uncontracting vertices into trees of the same color, then maybe inserting vertices of different color in the middle of new edges, then maybe uncontracting some of them into trees, etc. So that we can grow a bicolored tree of special kind at each vertex.

Since we can never perform a leaf reduction (R2) in a leafless graph, we obtain the following claim.

Lemma 12.6. *A leafless plabic graph without isolated components is reduced if and only if it is impossible to transform it by the moves* (M1)–(M3) *into a graph where we can perform parallel edge reduction* (R1).

The next claim is the main result on reduced plabic graphs.

Theorem 12.7. *Let G be a reduced plabic graph. Then G is perfectly orientable and the map $\tilde{Meas}_G : \mathbb{R}_{>0}^{F(G)-1} \to S_{\mathcal{M}}^{\mathrm{tnn}}$ gives a subtraction-free rational parametrization of the corresponding totally nonnegative Grassmann cell $S_{\mathcal{M}}^{\mathrm{tnn}}$. In particular, the dimension of $S_{\mathcal{M}}^{\mathrm{tnn}}$ equals $|F(G)| - 1$.*

For any cell $S_{\mathcal{M}}^{\mathrm{tnn}}$ there is a reduced plabic graph G such that $\tilde{Meas}_G$ is a parametrization of $S_{\mathcal{M}}^{\mathrm{tnn}}$.

Any two different parametrizations $\tilde{Meas}_G$ and $\tilde{Meas}_{G'}$ of the same cell $S_{\mathcal{M}}^{\mathrm{tnn}}$ can be obtained from each other by the moves (M1)–(M3).

We will prove this theorem in Section 16.

Remark 12.8. For a non-reduced plabic graph G without isolated components, the map $\tilde{Meas}_G$ is either undefined (when G is not perfectly orientable) or this map is not injective. Indeed, if we can do a reduction (possibly after performing some moves), then we can decrease the number of needed parameters.

13. Trips in plabic graphs

In this section we give a criterion when a plabic graph is reduced and describe move-equivalence classes of reduced graphs. The results of this section can be related to Thurston's work [Thu] on triple diagrams (see Remark 14.5).

For an (undirected) plabic graph G, a *trip* is a directed path T in G such that

(1) T either joins two boundary vertices (*one-way trip*) or it is a closed cycle that contains none of the boundary vertices (*round-trip*).
(2) If T arrives to an internal vertex v with incident edges $e_1, \dots, e_d$ (in the clockwise order) though the edge e_i, then it should leave v through the edge $e_{i-col(v)}$. (Here indices i in e_i are taken modulo d.) In other words, T obeys the following "rules of the road": turn right at a black vertex, and turn left at a white vertex; see Figure 13.1.

Figure 13.1. Rules of the road for trips in plabic graphs

Note that these trips in *undirected* plabic graphs have nothing to do with paths in directed networks that we used in the definition of the boundary measurements.

If two trips pass along the same edge in the same directions then they should be identical. If a trip passes along the same edge twice in the same direction then it is a round-trip. Thus each edge of G belongs to exactly two trips or to one trip with a self-intersection at this edge.

Each plabic graph G with n boundary vertices defines the *trip permutation* $\pi_G \in S_n$ such that $\pi_G(i) = j$ whenever the trip that starts at the boundary vertex b_i ends at the boundary vertex b_j.

The following claim is established by direct examination.

Lemma 13.1. *Let a plabic graph G' is obtained from G by one of the moves* (M1)–(M3). *Then $\pi_G = \pi_{G'}$. In other words, each one-way trip in G is transformed into a one-way trip in G' with the same end points. Also each closed trip in G is transformed into a closed trip in G'.*

Note that reductions (R1)–(R2) (unlike the moves) change the trip permutation.

For an edge e with vertices of different colors, we say that two trips (resp., one trip) that pass(es) through the edge e in two different directions have/has an *essential intersection* (resp., *essential self-intersection*) at this edge e. All other (self)-intersections are called *inessential.* Note that in an inessential (self)-intersection the trips do not cross but rather touch each other. We can can always remove an inessential (self)-intersection by performing edge contraction/uncontraction moves (M2).

Let us say that two trips $T_1 \neq T_2$ in G have a *bad double crossing* if they have two essential intersections at edges e_1 and e_2 such that both trips T_1 and T_2 are directed from e_1 to e_2.

The following theorem gives a criterion when a plabic graph is reduced.

Theorem 13.2. *Let G be a leafless plabic graph without isolated connected components. Then G is reduced if and only if the following conditions hold*

1. *G has no round-trips.*
2. *G has no trips with essential self-intersections.*
3. *G has no pair of trips with a bad double crossing.*
4. *If $\pi_G(i) = i$ then G has a boundary leaf attached to the boundary vertex b_i.*

Note that condition (1) implies that G contains no isolated connected components.

Definition 13.3. A *decorated permutation* $\pi^{:} = (\pi, col)$ is a permutation $\pi \in S_n$ together with a coloring function *col* from the set of fixed points $\{i \mid \pi(i) = i\}$ to $\{1, -1\}$. That is a decorated permutation is a permutation with fixed points colored in two colors.

Suppose for a moment that we have already established Theorem 13.2. Then we can decorate the trip permutation π_G of a *reduced* plabic graph G by coloring each fixed point $\pi_G(i) = i$ to the color $col(i) := col(v)$, where is v is the boundary leaf attached to the boundary vertex b_i. This gives the *decorated trip permutation* $\pi^{:}_G$ of G.

Theorem 13.4. *Let G and G' be two reduced plabic graphs with the same number of boundary vertices. Then the following claims are equivalent:*

1. *G can be obtained from G' by moves* (M1)–(M3).
2. *These two graphs have the same decorated trip permutation $\pi^{:}_G = \pi^{:}_{G'}$.*

Since any reduced plabic graph can be transformed into a leafless graph by moves (M2) and (M3), it is enough to prove Theorem 13.4 for leafless reduced graphs.

We will also need the following auxiliary claim.

Lemma 13.5. *Let G be a reduced plabic graph such that π_G has no fixed points. Let $i < j$ be two indices such that $\pi_G(i) = j$ or $\pi_G(j) = i$ and there is no pair $i', j' \in [i+1, j-1]$ such that $\pi_G(i') = j'$. Then one can transform G by moves* (M1)–(M3) *into a graph with a square face that is attached to the boundary interval $[b_i, b_{i+1}]$ and has two other internal vertices u and v; see Figure 13.2.*

Moreover, if $\pi_G(i) = i+1$ and $\pi_G(i+1) = i$, and if G is leafless and has no vertices of degree 2, then the boundary vertices b_i and b_{i+1} are connected in G by an edge.

Note that, for any pair $i < j$ such that $\pi(i) = j$ or $\pi(j) = i$, either this pair itself satisfies the condition of Lemma 13.5 or there is another pair inside the interval $[i, j]$ that satisfies the condition of this lemma.

Proof of Theorems 13.2 and 13.4. We will prove Theorem 13.2, Theorem 13.4, and Lemma 13.5 all together by induction on the number faces in G.

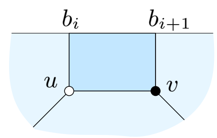


FIGURE 13.2. Square boundary face

Let us assume that G is a leafless plabic graph with 1 face and without isolated connected components. Then G consists of trees attached to the boundary vertices. Since G cannot have non-boundary leaves, that means the G contains only of boundary leaves attached to all boundary vertices. All such graphs are reduced because it is impossible to further reduce them and clearly they satisfy conditions in Theorem 13.2. Theorem 13.4 and Lemma 13.5 are also trivial in this case. This gives the base of induction.

Let us now assume that G is a leafless plabic graph with > 1 faces and without isolated components. By the induction hypothesis we have already established Theorems 13.2, and 13.4, and Lemma 13.5 for all graphs with fewer number of faces than in G.

If a graph G is not reduced then after performing some moves (M1)–(M3) we should be able to reform a parallel edge reduction (R1); see Lemma 12.6. It is easy to see that right before the reduction one of the conditions (1)–(4) in Theorem 13.2 fails. Indeed, suppose that we get a pair of parallel edges between two vertices u and v of different color. Then contract all unicolored edges and consider several cases: if $\deg(u) = 2$ or $\deg(v) = 2$ then (1) fails; if $\deg(u) = 3$ or $\deg(v) = 3$ then (2) or (4) fails; if $\deg(u), \deg(v) > 3$ then (3) fails. Note that moves (M1)–(M3) can never remove a failed condition (1)–(4). Since moves (M1)–(M3) are invertible, that means that in the original graph G we also get a failed condition (1)–(4). This proves Theorem 13.2 in one direction.

Let us prove Theorem 13.2 in the other direction. Suppose that one the conditions (1)–(4) in Theorem 13.2 fails. Let us show that it is possible transform the graph G by the moves (M1)–(M3) to a graph where we can perform reduction (R1).

In all cases a segment of a problematic trip T_1 (or a pair segments in a pair of trips T_1 and T_2) surrounds an area A that consists of some faces in G. Let us assume that A is the area between closest essential (self)-intersections so that there are no other essential (self)-intersections of the T_i's inside A. Let us show that we can always undo all inessential (self)-intersections of the T_i's using moves (M2). Let us first contract all unicolored edges in G. Let v be a vertex, say, with $col(v) = 1$ and with incident edges $e_1, \dots, e_d$ (in the clockwise order). Suppose we have an inessential (self)-intersection of the trip(s) T_i at v. That means that one of these trips arrives to v through the edge e_j and then leaves through e_{j-1} (according to the "rules of the road") and then (the same or the other) trip arrives to v through e_l and leaves through e_{l-1}. Note that all edges $e_j, e_{j-1}, e_l, e_{l-1}$ should be different. (Otherwise we get an essential intersection.) We might get some other pairs of edges at the vertex v corresponding to other passages through this vertex. Since all these pairs of edges (e_j, e_{j-1}), (e_l, e_{l-1}), ... are disjoint, we can always uncontract the vertex v into a trivalent tree such that all these segments of paths no longer intersect. This argument shows that we may assume that the area A is homeomorphic to a disk.

Let us remove all vertices of degree 2 from G and contract all unicolored pairs of edges on the boundary of the disk A so that colors of vertices now alternate as we go around the disk A. Remind that we assume that G is leafless.

Suppose that the area A contains only one face. Let us consider several cases.

I. Suppose that A is surrounded by a round-trip, say, a clockwise round-trip. If there is a white vertex v on the boundary of A, then all edges incident to v should lie *inside* A. Thus there is a tree attached to v, which is impossible because we assume that G is leafless. So v should have degree 2, which is again impossible because have removed all such vertices from G. That mean that there is no white vertices on the boundary of A. Thus there is only one black vertex on the boundary of A. So we get a loop; see Figure 12.9. This loop can be transformed into a pair of parallel edges. Then we can apply reduction (R1), as needed. Similarly, if A is surrounded by a counterclockwise round-trip, there is no black vertices on its boundary and thus A is again formed by a loop.

II. Suppose that A is surrounded by a segment of a trip with an essential self-intersection, say, a clockwise segment. Again in this case there is no white vertices on the boundary A, except the white vertex that belongs to the essential self-intersection. So the boundary of A has at most two vertices, one black and one white. That means that we either get a loop or get a pair of parallel edges. In both cases, we can perform reduction (R1).

III. The case when the area A is surrounded by a segment of the trip that starts and ends at the same boundary vertex b_i, is exactly the same as the case II.

IV. Suppose that A is surrounded by a pair of segments S_1 and S_2 of two trips between a bad double crossing; cf. Figure 13.3. Assume that the segment S_1 is directed clockwise, then S_2 is directed counterclockwise. The same argument as above shows that there are no white vertices on S_1 and similarly there are no black vertices on S_2. That means that S_1 consists of a single black vertex and S_2 consists of a single white vertex. Again the boundary of A is formed by a pair of parallel edges, so that we can perform reduction (R1).

Let us now assume that A has more that two faces. Let $\tilde{b}_1, \dots, \tilde{b}_r$ be the vertices on the boundary of A (in the clockwise order) that have incident edges *inside* the area A. Using uncontractions (M2), we can transform the vertices $\tilde{b}_i$ into trivalent vertices, i.e., for each $\tilde{b}_i$ there is now exactly one incident edge that lies inside of A. Then we obtain a smaller a plabic graph $\tilde{G}$ (with at least one trip) inside the area A. If $\tilde{G}$ is not reduced then so is G. Assume that $\tilde{G}$ is reduced. By the induction hypothesis we have already established all needed claims for the graph $\tilde{G}$.

Let $\pi_{\tilde{G}} \in S_r$ be the trip permutation of the graph $\tilde{G}$. Note that $\tilde{G}$ has no boundary leaves, because we assumed that G is leafless. Thus the trip permutation $\pi_{\tilde{G}}$ has no fixed points.

If A is given by one segment of a trip, then all $\tilde{b}_i$'s have the same color (all white if the segment is clockwise, and all black if the segment is counterclockwise). For each pair $\tilde{b}_i$ and $\tilde{b}_{i+1}$, the graph G has one vertex of the opposite color between these two vertices. By the induction hypothesis Lemma 13.5 holds for the graph $\tilde{G}$. So we can transform the graph $\tilde{G}$ to a graph $\tilde{G}'$ that has a square face f attached to some boundary segment $[\tilde{b}_l, \tilde{b}_{l+1}]$. Furthermore, in the case of a trip with an essential self-crossing we may assume that the boundary segment $[\tilde{b}_l, \tilde{b}_{l+1}]$ does not contain this self-crossing. (Just label the vertices $\tilde{b}_i$ so that the self-crossing is between $\tilde{b}_r$ and $\tilde{b}_1$.) In the graph G, the face f includes 5 vertices (because there is one extra vertex of G between $\tilde b_l$ and $\tilde b_{l+1}$). Note that, as we go around f, the colors of the 5 vertices cannot change more than 4 times. That means that we can always merge at least two vertices of f together by move (M2). If the colors of vertices change less than 4 times, that means that we can transform a graph G into a graph where we can perform a reduction (R1). If the colors change exactly 4 times, then (after some uncontractions for vertices that are not trivalent) we can perform a square move for the face f. In the obtained graph G', the corresponding face now lies outside the area A' formed by the corresponding trip. Thus the number of faces in A' is strictly less than the number of faces in A. Then we can repeatedly apply the same procedure until we get a graph where a problematic trip surrounds exactly one face. This case was already considered above.

If the area A is given by two segments S_1 (clockwise) and S_2 (counterclockwise) of two trips with a bad double crossing, then the $\tilde b_i$ that belong to S_1 are white and the $\tilde b_j$ that belong to S_2 are black; see Figure 13.3. Suppose that there are two vertices $\tilde b_i$ and $\tilde b_j$ that both lie on the side S_1 or both lie on S_2 such that they are connected by a trip in $\tilde G$. We may assume that they are closest such vertices, so that the condition of Lemma 13.5 holds. Thus again we will can transform $\tilde G$ to a graph that contain a square face attached to a segment in S_1 or in S_2. Then we can decrease the number of faces inside A as above. Otherwise all trips in $\tilde G$ that start at S_1 should end at S_2, and vise versa. (In particular, both S_1 and S_2 contain the same number of vertices.) Let $\tilde b_s$ be the last vertex in S_2 and be the first vertex in $\tilde b_{s+1}$; see Figure 13.3.

If $l = \pi_{\tilde G}(s+1) \neq s$ or $l = \pi_{\tilde G}^{-1}(s+1) \neq s$, then the pair $(l, s+1)$ satisfies condition of Lemma 13.5. Thus again we can transform the adjacent face adjacent to $[\tilde b_l, \tilde b_{l+1}]$ into a square and then apply square move (M1) and reduce the number of faces inside of A, as above.

In the remaining case we have $\pi_{\tilde G}(s) = s+1$ are $\pi_{\tilde G}(s+1) = s$. According to the second part of Lemma 13.5, in this case the vertices $\tilde v_s$ and $\tilde v_{s+1}$ are connected by an edge e. Thus the graph G contains a square face below the edge e, so again we can perform a square move (M1) at this face and reduce the number of faces inside A. Thus in all cases we can repeatedly decrease the number of faces in A until we get an area with one face. This finishes the inductive step in the proof of Theorem 13.2.

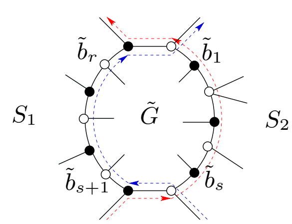


FIGURE 13.3. A bad double crossing with a graph $\tilde G$ inside

Let us now prove Theorem 13.4 and Lemma 13.5. In one direction Theorem 13.4 is straightforward because the moves (M1)–(M3) never change the decorated permutation $\pi^:_G$. Let us assume that G and G' two reduced graphs such that $\pi^:_G = \pi^:_{G'}$. Also assume that $|F(G)| \geq |F(G')|$. We will show that the graphs G and G' can be obtained from each other by the moves (M1)–(M3). By Theorem 13.2 (which is already proved for G and G') all fixed points in π_G should correspond to boundary leaves in the graphs G and G', which should be located in the same positions and should have same colors (given by the decoration in the decorated permutation $\pi^{:}_G$). Thus, without loss of generality, we may assume that π_G has no fixed points.

Let us pick a pair $i < j$ satisfying the condition of Lemma 13.5. Let us assume that $\pi_G(i) = j$. (The other case is completely analogous.) By Theorem 13.2 the trip T in G from b_i to b_j has no essential self-crossings, and we can eliminate inessential self-crossings by the moves (M2), as above. This trip T subdivides the graph G into two smaller graphs G_1 and G_2, where G_1 is the graph containing the boundary segment $(b_i, b_{i+1}, \dots, b_j)$. We may assume that the graph G satisfies the property that G_1 has the smallest possible number of faces for all graphs in the move-equivalence class of G. (Otherwise transform G to such graph by the moves.) Let us remove vertices of degree 2 from G and contract unicolored edges in T.

Let π_{G_1} be the trip permutation for the graph G_1. The trip permutation π_{G_1} contains no fixed points. (They would correspond to boundary leaves in G_1, but we assume that G is leafless.) If there is a trip in G_1 that starts and ends at T, then using the same argument as above, we can transform G_1 into a graph that has a square face attached to a segment of T, then apply the square move at this face, and decrease the number of faces in G_1, which contradicts to our assumption of minimality of G_1. This means that all trips in G_1 that start at a (black) vertex on T should end at one of the vertices $b_{i+1}, \dots, b_{j-1}$, and vise versa. Let $\tilde b_{i+1}, \dots, \tilde b_{j-1}$ be the black vertices on T as we go from b_i to b_j.

If the trip of G_1 that starts at b_{i+1} ends at a vertex $\tilde b_l \neq \tilde b_{i+1}$ or the trip that ends at b_{i+1} starts at a vertex $\tilde b_l \neq \tilde b_{i+1}$, again by Lemma 13.5 we can transform G_1 to a graph that has a square face attached to $[\tilde b_{l-1}, \tilde b_l]$, then perform a square move, and thus reduce the number of faces in G_1, which contradicts to minimality of G_1. Thus vertices b_{i+1} and $\tilde b_{i+1}$ are connected by 2 trips in both ways. By Lemma 13.5, this implies that b_{i+1} and $\tilde b_{i+1}$ should be connected by an edge. Let us remove this edge and apply the same argument again to show that the vertices b_{i+2} and $\tilde b_{i+2}$ are also connected an edge in G, etc. There should be a white vertex in T between two adjacent black vertices $\tilde b_l$, $\tilde b_{l+1}$ in T. If needed, we can also insert white vertices between b_i and $\tilde b_{i+1}$ and between $\tilde b_{j-1}$ and b_j, and then make all white vertices in T trivalent. (So now G might contain one or two leaves.) Therefore we may assume that the trip T in the graph G goes along the boundary and has the form as shown in Figure 13.4. Note that if T has this boundary form, then the trip permutation π_{G_2} of the graph G_2 that lies outside of this boundary strip is uniquely determined by the trip permutation π_G of G.

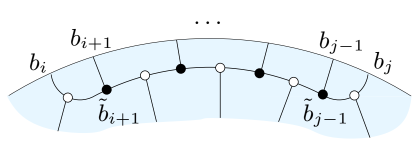


FIGURE 13.4. Boundary trip from b_i to b_j

We can also transform by the moves the second graph G' (with $\pi_{G'} = \pi_G$) to the form where the trip T' from b_i to b_j has exactly the same boundary form as in Figure 13.4. Thus its part G'_2 outside the boundary strip has exactly the same trip permutation $\pi_{G'_2} = \pi_{G_2}$ as G_2. By induction hypothesis, this implies that G_2 and G'_2 are move-equivalent. Then G and G' are move equivalent. This finishes the inductive step for Theorem 13.4.

In order to prove Lemma 13.5, notice that in the above boundary trip from b_i and b_j shown on Figure 13.4 the boundary segment $[b_i, b_{i+1}]$ is adjacent to a square face. This proves the first claim of Lemma 13.5.

Now assume that $\pi_G(i) = i+1$ and $\pi_G(i+1) = i$. Again we can transform G so that the trip T from b_i to b_{i+1} has the boundary form as above. Let us remove leaves and vertices of degree 2. The trip T contains at most one vertex — a white vertex v. If T contain a vertex v, then the other trip T_2 from b_{i+1} to b_i goes away from T at this vertex and then returns back to T at the same vertex v. Let A be the area surrounded by the part of T_2 between two visits of v. Applying the same argument as above in proof of Theorem 13.2 (the case of trip with a self-crossing), we can reduce the number of faces inside A one by one until there is only one face left. Now all other (possible) vertices on the boundary of A can be contracted into a single black vertex. So the graph contains a pair of parallel edges and we can perform reduction (R1), which is impossible because we assume that G is reduced. Thus there is no vertex v on the trip T, that is b_i and b_{i+1} are connected by an edge e. By performing moves (M1)–(M3) we can only insert middle vertices into e and grow some trees at these vertices. But since we assume that the original graph G is leafless and has no vertices of degree 2, the edge e should be present in G. This proves the second claim of Lemma 13.5. This finishes the inductive proof of Theorem 13.2, Theorem 13.4, and Lemma 13.5. □

Let us say that a *singleton* is an isolated connected component with a single vertex and no edges.

Lemma 13.6. *Any plabic graph G can be transformed by moves* (M1)–(M3) *and reductions* (R1)–(R3) *into a reduced plabic graph possibly together with some singletons.*

Proof. Let us first apply the leaf removal procedure to G. If the obtained graph has a non-singleton isolated component, then there is a round-trip T in this component. We can decrease the number of faces inside T and then apply a reduction as in the above proof. Repeatedly applying reductions we end up with a reduced graph possibly together with some singletons. □

14. Alternating strand diagrams

One can transform reduced plabic graphs into the following objects.

Definition 14.1. An *alternating strand diagram* consists of n directed curves, called the *strands,* which are drawn inside a disk and connect pairs of the boundary vertices $b_1, \dots, b_n$ such that the following conditions hold:

(1) For any boundary vertex b_i, there is exactly one strand that enters b_i and exactly one strand that leaves b_i.
(2) No three strands can intersect at the same point.
(3) There is a finite number of pairwise intersection points of the strands. All intersection points are transversal, i.e., the tangent vectors to the strands at the intersection points are independent.

(4) Let S be a strand, and let $v_1 = b_j, v_2, \ldots, v_l = b_j$ be all points on C where it intersects with other strands $S_1, \ldots, S_l$ as it goes from v_i to v_j. (The same strand might occur several times in this sequence.) Then the orientations of the strands $S_1, \ldots, S_l$ at the points $v_1, \ldots, v_l$ alternate. In other words, if, say, S_1 is oriented at v_1 from left to write (with respect to S) then S_2 is oriented at v_2 from right to left, S_3 is oriented from left to right, etc.

(5) A strand has no self-intersections, except the case when the strand is a loop (clockwise or counterclockwise) attached to a boundary vertex b_i.

(6) If two strands have two intersection points u and v, then one of these strands is oriented from u to v and the other is oriented from v to u.

Needless to say that alternating strand diagrams are considered modulo homotopy.

Each alternating strand diagram D has the *decorated strand permutation* $\pi^{:}_D$ such that $\pi^{:}_D(i) = j$ whenever D has a strand from b_i to b_j and if there is a counterclockwise (resp., clockwise) loop attached to b_i, then the fixed point i is colored in black $col(i) = 1$ (resp., white $col(i) = -1$).

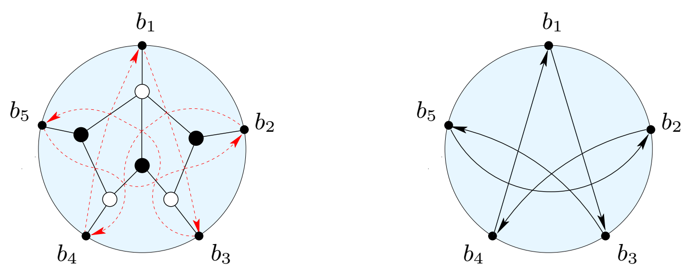


FIGURE 14.1. A reduced plabic graph and the corresponding alternating strand diagram

The right side of Figure 14.1 displays an example of alternating strand diagram D with the strand permutation $\pi_D(i) = i + 2 \pmod 5$. Note that the diagram where the points b_i and $b_{\pi(i)}$ are connected straight chords is not an alternating strand diagram.

Faces of an alternating strand diagram are the regions on which the strands subdivide the disk. There are 3 types of such faces: *clockwise* (whose boundary is directed clockwise), *counterclockwise* (whose boundary is directed counterclockwise), and *alternating* (where directions of the strands alternate when we go around the face. All *boundary faces,* i.e., faces adjacent to a segment of the boundary of the disk, are alternating. For a clockwise or counterclockwise face, all its adjacent faces are alternating. On the other hands, an alternating face is adjacent to both clockwise and counterclockwise faces (in an alternating order).

A reduced plabic graph G can be easily transformed into an alternating strand diagram D_G as follows:

(1) Remove all non-boundary leaves and contract all unicolored edges in G.

(2) Draw a dot in the middle of each edge connecting two internal vertices, and also draw dots at the boundary vertices $b_1, \ldots, b_n$.

(3) For any black internal vertex v, connect the dots $d_1, \ldots, d_l$ (in clockwise order) on the incident edges by new edges $(d_2, d_1), \ldots, (d_l, d_{l-1}), (d_1, d_l)$ oriented counterclockwise.

(4) Similarly, for any white internal vertex v, connect the adjacent dots $d_1, \dots, d_l$ by new edges $(d_1, d_2), \dots, (d_{l-1}, d_l), (d_l, d_1)$ oriented clockwise.
(5) We obtain a new directed graph where all internal vertices (internal dots) have degree 4 and all boundary vertices b_i have degree 2. The strands are directed paths in this graph connecting the boundary vertices b_i and intersecting each other at the dots.

Theorems 13.2 and 13.4 (upon some observation of Figure 14.1) imply the following result.

Corollary 14.2. *The diagram $D = D_G$ constructed from a reduced plabic graph G as above is an alternating strand diagram. The map $G \mapsto D_G$ is a bijection between reduced plabic graphs (without non-boundary leaves and unicolored edges) and alternating strand diagrams. Trips in G are transformed into strands of D. Thus diagram D has the same decorated strand permutation $\pi^:_D = \pi^:_G$ as the decorated trip permutation of G.*

Black (resp., white) vertices in G correspond to counterclockwise (resp., clockwise) faces of D. Faces of G correspond to alternating faces of D.

Two alternating strand diagrams have the same decorated strand permutation if and only if they can be obtained from each other by the moves shown in Figure 14.2.

In alternating strand diagrams weights are assigned only to alternating faces. These weights are transformed as shown on Figure 14.2, where the subtraction-free transformation $(y_0, \dots, y_5) \mapsto (y'_0, \dots, y'_5)$ is given by (12.1).

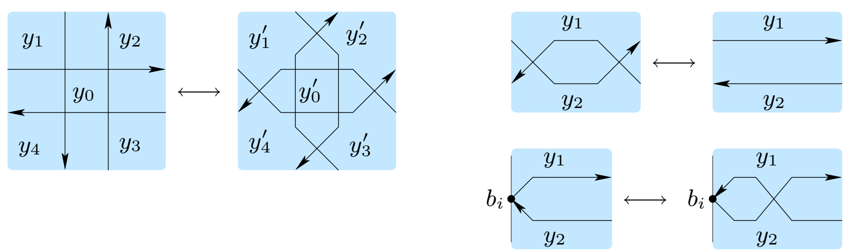


FIGURE 14.2. Moves of alternating strand diagrams

A special kind of alternating strand diagrams corresponds to Thurston's triple diagrams [Thu]. Triple diagrams defined below are Thurston's minimal triple diagrams. Note that this definition is slightly different from the one given in [Thu].

Definition 14.3. Consider a disk with $2n$ boundary vertices $b_1, b'_1, b_2, b'_2, \dots, b_n, b'_n$ (in the clockwise order) on its boundary. A *triple diagram* is a diagram with n directed strands drawn inside the disk such that

(1) Each strand starts a boundary vertex b_i and ends at another vertex b'_j. For each boundary vertex b_i (resp., b'_i) there is exactly one strand starting (resp., ending) at this vertex.
(2) Only triple intersections of strands are allowed inside the disk such that the directions of the six rays at this point alternate (as we go around the intersection point).
(3) Strands have no self-intersections.
(4) If two strands intersect each other at two points u and v, then one of these strands is directed from u to v and the other strand is directed from v to u.

The *strand permutation* $\pi_T \in S_n$ of a triple diagram T is given by $\pi_T(i) = j$ whenever T contain a strand from b_i to b'_j.

The map $T \mapsto D$ from triple diagrams to alternating strand diagrams is quite simple: Slightly deform each triple crossing of the strands in T and replace it by 3 simple crossings so that the boundary of newly created triangle is oriented clockwise, then merge all pairs of boundary vertices b_i and b'_i.

We will call faces of a triple diagram the *chambers.* A triple diagram has two types of chambers — the *dark chambers* whose boundary is oriented counterclockwise, and the *light chambers* with clockwise boundary. Figure 14.3 below shows a triple diagram with dark chambers colored in a darker shade.

Let us transform a triple diagram T into an alternating strand diagram D and then into a plabic graph G, as above. We have the following correspondences:

$\{$dark chambers of $T\} \leftrightarrow \{$counterclockwise faces of $D\} \leftrightarrow \{$black vertices of $G\}$
$\{$triple crossings in $T\} \leftrightarrow \{$clockwise faces of $D\} \leftrightarrow \{$white vertices of $G\}$
$\{$light chambers of $T\} \leftrightarrow \{$alternating faces of $D\} \leftrightarrow \{$faces of $G\}$.

Figure 14.4 shows the plabic graph G associated with the triple diagram T on Figure 14.3.

The following claim is straightforward.

Lemma 14.4. *The above map identifies triple diagrams with alternating strand diagrams D such that all clockwise faces of D are triangles and D has no clockwise loops attached to boundary vertices. Equivalently, triple diagrams correspond to reduced plabic graphs (without non-boundary leaves and unicolored edges) such that all white vertices are trivalent and all boundary leaves are black.*

Remark 14.5. Thurston [Thu] proved that any two triple diagrams with the same strand permutation can be obtained from each other by certain moves. These moves can be related to the moves of alternating strand diagrams on Figure 14.2.

Thurston [Thu, Theorem 1] showed that, for each permutation $\pi \in S_n$, there is a triple diagram T with strand permutation $\pi_T = \pi$. Let us give another construction of a triple diagram T with a given strand permutation π, which is different from the construction in [Thu]. Note that both constructions are quite simple.

We will arrange the vertices $b_1, b'_1, b_2, b'_2, \dots, b_n, b'_n$ on the x-axis in the xy-coordinate plane. We will draw a triple diagram in the half-space below the x-axis. Each strand will be a continuous curve $(x(t), y(t))$, $t \in [0,1]$ such that $x(t)$ is monotonically increasing or monotonically decreasing function. Let us call such special triple diagrams *monotone.*

For a strand S from b_i to b'_j, we say that S is *rightward* if $i \leq j$ (because it is directed from left to right), and we say that S is *leftward* if $i > j$. Let S_i denotes the strand starting at b_i and S'_j denotes the strand ending at b'_j (so each strand has two labels).

We will draw all strands in T by adding little pieces to them as we go from left to right. Start with the vertices b_1 and b'_1. If $\pi(1) = 1$, then draw a (very short) strand from b_1 to b'_1. Otherwise draw initial segments of the strands S_1 and S'_1 attached to the vertices b_1 and b'_1, so that we get two strands with loose ends. Then proceed to the pair of vertices b_2 and b'_2. If $\pi(2) = 2$ then draw a short strand from b_2 to b'_2. If $\pi(2) = 1$ then attach the loose end of S'_1 to b_2. If $\pi(1) = 2$ then draw the initial segment of S_2, add a triple crossing of the strands S_1, S'_1, S_2 and attach the loose end of S_1 to b'_2. Otherwise draw two initial segments of S_2 and S'_2, so that we get 4 loose ends of strands. Then proceed to the pair of vertices b_3, b'_3, etc. Note that a each moment we have some number of loose strands with alternating directions (right, left, right, left, ...) as we list them from the bottom. So between an adjacent pair of rightward loose strands there is exactly one leftward loose strand, and vise versa. This means we can always switch two adjacent rightward (resp., leftward) loose strands by adding a triple crossing. Suppose now that we process the pair of vertices b_i, b'_i. If $\pi(i) = i$, we just draw a short strand $S_i = S'_i$. If $\pi(i) = j < i$, then we extend the (already drawn) loose end of S'_j all the way up by these "adjacent transpositions" and connect it with b_i. Similarly, if $\pi^{-1}(i) = j < i$, then we extend the loose end of S_j all the way up and connect it with b'_i. Otherwise we just add two new strands S_i and S'_i with loose ends. When we finish processing all boundary vertices, all loose ends should be attached to the corresponding vertices, and we obtain a triple diagram. Notice that any pair of rightward strands (or a pair of leftwards strands) will not intersect more than once. (A leftward and a rightward strands can intersect many times but this is not prohibited.) Figure 14.3 shows a monotone triple diagram obtained by this procedure for the permutation $\pi = 24513$.

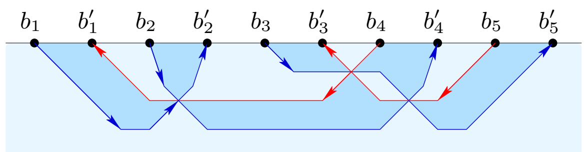


FIGURE 14.3. A triple diagram T with strand permutation $\pi_T = 24513$

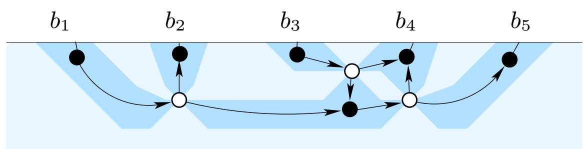


FIGURE 14.4. The plabic graph G corresponding to the triple diagram T (with a perfect orientation of edges)

Lemma 14.6. *For each permutation $\pi \in S_n$ there is a monotone triple diagram T with strand permutation $\pi_T = \pi$.*

Corollary 14.7. *For any decorated permutation $\pi^:$ there is a reduced plabic graph G with trip permutation $\pi^:_G = \pi$.*

Remark 14.8. Note that monotone triple diagrams are quite similar to *double wiring diagrams* of Fomin-Zelevinsky [FZ1]. Such triple diagrams are obtained by superimposing two usual wiring diagrams formed by all rightwards strands and by all leftward strands. These two wiring diagrams are shown in blue and red colors on Figure 14.3. Actually, double wiring diagrams from [FZ1] are exactly the monotone triple diagrams in the case when $n = 2m$ and $\pi : [m] \to [m + 1, 2m]$, $\pi : [m + 1, 2m] \to [m]$.

We will need the following property of monotone triple diagrams constructed above. For a permutation $\pi \in S_n$, we say that $i \in [n]$ is an *anti-exceedance* of π if $\pi^{-1}(i) > i$. Let $I(\pi) \subset [n]$ be the set of all anti-exceedances of π.

Recall Proposition 11.7 that combinatorially describes the matroid corresponding to a perfectly orientable plabic graph G as the matroid $\mathcal{M}_G$ of source sets of perfect orientations of G.

Lemma 14.9. *Let T be a monotone triple diagram with strand permutation π, and let G be the associated reduced plabic graph. Then G is perfectly orientable and $I = I(\pi)$ is the lexicographically minimal base of the matroid $\mathcal{M}_G$.*

Any other base of $\mathcal{M}_G$ is obtained from I by replacing some entries $i_1, \dots, i_s \in I$ with some $j_1, \dots, j_s$ such that $j_1 > i_1, \dots, j_s > i_s$.

Proof. Define the *nose* of a dark chamber C in T as its rightmost point, i.e., the point with the maximal x-coordinate. The monotonicity of a triple diagram implies that each dark chamber has a unique nose. Notice that $i \in I(\pi)$ if and only if b_i' is not the nose of the dark chamber adjacent to the boundary segment $[b_i, b_i']$. In the triple diagram shown of Figure 14.3, the elements of $I(\pi) = \{1, 3\}$ correspond to the targets of the leftward strands (shown in red).

Let us direct edges in of the plabic graph G, as follows. For each black vertex u of G (which corresponds to a dark chamber C in T) the only outgoing edge from u goes to the white vertex at the nose of C. Each white vertex v of G (triple crossing in T) is adjacent to three dark chambers C_1, C_2, C_3 such that v is the nose of C_1. Then the 3 edges incident to v are directed away from C_1 and towards C_2 and C_3; see Figure 14.4. That means that this orientation of edges is perfect, which proves perfect orientability of G.

Notice that, for this orientation of edges, the boundary source set I is exactly the index set $I(\pi)$ of the "noseless" boundary vertices b_i'.

Any other perfect orientation of G is obtained from the constructed one by switching edge directions in a family of noncrossing directed paths joining pairs of boundary vertices; see Lemma 11.10. A directed path $P : b_i \to b_j$ in this digraph G correspond to a sequence of dark chambers $C_1, \dots, C_l$ in T such that C_1 and C_l are adjacent to the boundary segments $[b_i, b_i']$ and $[b_j, b_j']$, and C_{i+1} is adjacent to the nose of C_i, for $i \in [l-1]$. For each i, the nose C_{i+1} is located strictly to the right of the nose of C_i. That means that $j > i$ for any directed path $P : b_i \to b_j$.

In other words, when we switch to any other perfect orientation of G, we replace some sources $i_1, \dots, i_s$ by other sources $j_1, \dots, j_s$ such that $j_1 > i_1, \dots, j_s > i_s$. This implies that $I = I(G)$ is the lexicographically minimal source set of a perfect orientation of G, i.e., the lexicographically minimal base of $\mathcal{M}_G$, as needed. □

Corollary 14.10. *Any reduced plabic graph is perfectly orientable.*

Proof. Let G be a reduced plabic graph. We may assume that it has no boundary leaves, so that the trip permutation π_G has no fixed points. Let G' be a reduced plabic graph coming from a monotone triple diagram as above such that G' has the same trip permutation $\pi_{G'} = \pi_G$. By Lemma 14.9, G' is reduced and by Theorem 13.4 G' is move-equivalent to G. Since the moves never change perfect orientability, the graph G is also perfectly orientable. □

Remark 14.11. Note that not all plabic graphs are perfectly orientable. For example, if a a graph has a singleton (isolated component with a single vertex), then it is not perfectly orientable. Essentially this is the only obstruction for perfect orientability. Indeed, any graph can be transformed into a reduced graph possibly together with some singletons; see Lemma 13.6. Such graph is perfectly orientable if and only if it has no singletons.

15. Mutations of dual graphs

16. From matroids to decorated permutations

Definition 16.1. A *Grassmann necklace* is a sequence $\mathcal{I} = (I_1, \dots, I_n)$ of subsets $I_r \subseteq [n]$ such that, for $i \in [n]$, if $i \in I_i$ then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, for some $j \in [n]$; and if $i \notin I_r$ then $I_{i+1} = I_i$. (Here indices i are taken modulo n.) In particular, we have $|I_1| = \cdots = |I_n|$.

Such necklaces are in bijection with decorated permutations. For a Grassmann necklace $\mathcal{I}$, we construct the decorated permutation $\pi^{:}(\mathcal{I}) = (\pi, col)$ such that

(1) if $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, $j \neq i$, then $\pi(i) = j$;
(2) if $I_{i+1} = I_i$ and $i \notin I_i$ then $\pi(i) = i$ is colored in black $col(i) = 1$;
(3) if $I_{i+1} = I_i$ and $i \in I_i$ then $\pi(i) = i$ is colored in white $col(i) = -1$;

where the indices i are taken modulo n. Notice that as we transform I_1 to I_2, then to I_3, and so on until get get back to I_1, we can remove an element i at most once (at the i-th step); and thus we can add an element j at most once. This shows that π is indeed a permutation in S_n. Note that black fixed points of $\pi^{:}$ are exactly the elements $i \in [n]$ that belong to none of the subsets $I_1, \dots, I_n$ and white fixed points are exactly the elements $j \in [n]$ that belong to all subsets $I_1, \dots, I_n$.

Let us describe the inverse map from decorated permutations to Grassmann necklaces. For a decorated permutation $\pi^{:} = (\pi, col)$ of size n, we say that $i \in [n]$ is an *anti-exceedance* of $\pi^{:}$ if $\pi^{-1}(i) > i$ or $\pi(i) = i$ and $col(i) = -1$. (That is we regard white fixed points as anti-exceedances.) Let $I(\pi^{:}) \subset [n]$ be the set of all anti-exceedances in $\pi^{:}$. For $r \in [n]$, let us also define the *shifted anti-exceedance set* $I_r(\pi^{:})$ as the set of indices $i \in [n]$ such that $i <_r \pi^{-1}(i)$ or $(\pi(i) = i$ and $col(i) = -1)$, where $<_r$ is the cyclical shift of the usual linear order on $[n]$: $r <_r (r+1) <_r \cdots <_r n <_r < 1 <_r \cdots <_r (r-1)$, i.e., $I_r(\pi^{:})$ is the anti-exceedance set with respect to the linear order $<_r$. In particular, $I_1(\pi^{:}) = I(\pi^{:})$. Let $\mathcal{I}(\pi^{:}) = (I_1, \dots, I_n)$, where $I_r = I_r(\pi^{:})$, for $r \in [n]$.

Lemma 16.2. *The maps $\pi^{:} \mapsto \mathcal{I}(\pi^{:})$ and $\mathcal{I} \mapsto \pi^{:}(\mathcal{I})$ are inverse to each other. They give a bijection between decorated permutations $\pi^{:}$ of size n and Grassmann necklaces $\mathcal{I}$ of size n.*

The proof of this lemma is quite straightforward (an exercise for the reader).

We can graphically present decorated permutation $\pi^{:}$ by arranging the vertices $b_1, \dots, b_n$ clockwise on a circle, drawing straight directed chords $[b_i, b_{\pi(i)}]$, whenever $\pi(i) \neq i$; drawing a counterclockwise loop at b_j for each black fixed point $col(j) = 1$; and drawing a clockwise loop at b_l for each white fixed point $col(l) = -1$.

For example, Figure 16.1 shows the decorated permutation $\pi^{:}$ with $\pi = 3\,1\,5\,\underline{4}\,2\,\underline{6}$ with two fixed points 4 and 6 colored $col(4) = 1$ (black) and $col(6) = -1$ (white). This decorated permutation has the following shifted anti-exceedance sets $I_1 = \{1, 2, 6\}$, $I_2 = \{2, 3, 6\}$, $I_3 = \{3, 6, 1\}$, $I_4 = \{5, 6, 1\}$, $I_5 = \{5, 6, 1\}$, $I_6 = \{6, 1, 2\}$.

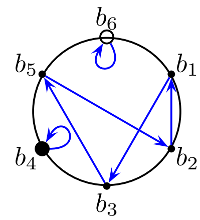


FIGURE 16.1. A decorated permutation $\pi^{:}$

For a matroid $\mathcal{M} \subseteq \binom{[n]}{k}$ of rank k on the set $[n]$, let $\mathcal{I}_\mathcal{M} = (I_1, \dots, I_n)$ be the sequence of subsets in $[n]$ such that, for $i \in [n]$, I_i is the lexicographically minimal base of $\mathcal{M}$ with respect to the shifted linear order $<_i$ on $[n]$.

Lemma 16.3. *For a matroid $\mathcal{M}$, the sequence $\mathcal{I}_\mathcal{M}$ is a Grassmann necklace.*

Proof. Let $\mathcal{I}(\mathcal{M}) = (I_1, \dots, I_n)$. By the cyclic symmetry, it is enough to check that $I_2 = (I_1 \setminus \{1\}) \cup \{j\}$ for some j, or $I_2 = I_1$. The subset $I_1 = \{i_1 < \cdots < i_k\}$ is the lex minimal base of $\mathcal{M}$ (with respect to the usual order on $[n]$). If $i_1 \neq 1$ then I_1 is also the lex minimal base with respect to the order $<_2$, and thus $I_2 = I_1$. Assume that $i_1 = 1$ and $I_2 = \{j_1 < \cdots < j_k\} \neq I_1$. Let r be the index such that $j_s = i_{s+1}$ for all $s < r$ and $j_r \notin I_1$. Then $j_r \in [i_r + 1, i_{r+1} - 1]$ (or $j_r \in [i_r + 1, n]$ if $r = k$) and $\mathcal{M}$ has a circuit (minimal dependence) involving i_1, j_r and some of the $j_1, \dots, j_{r-1}$. That implies that, for any $(k-r)$-element subset $S \subset [j_r + 1, n]$, $\{i_1, \dots, i_r\} \cup S$ is a base of $\mathcal{M}$ if and only if $\{j_1, \dots, j_r\} \cup S$ is a base. Because of the lex minimality of I_1 and I_2, we have $i_t = j_t$ for all $t > r$. Thus $I_2 = (I_1 \setminus \{1\}) \cup \{j_r\}$, as needed. $\square$

Recall that, for a plabic graph G, the image of the boundary measurement map $\tilde{Meas}_G$ belongs to a totally nonnegative Grassmann cell $S_\mathcal{M}^{\mathrm{tnn}}$ where $\mathcal{M} = \mathcal{M}_G$; see Proposition 11.7.

Proposition 16.4. *Let G be a reduced plabic graph, and let $\mathcal{M} = \mathcal{M}_G$ be the associated matroid. Then the Grassmann necklace $\mathcal{I}_\mathcal{M}$ of the matroid $\mathcal{M}$ equals the necklace $\mathcal{I}(\pi^{:}_G)$ corresponding to the decorated trip permutation of G.*

Proof. Black boundary leaves of G correspond to isolated boundary sinks in directed network, which correspond to zeros of the matroid $\mathcal{M}$, i.e., the elements $i \in [n]$ which do not appear in any base of $\mathcal{M}$. These elements never appear in the necklace $\mathcal{I}_\mathcal{M}$. They give black fixed points of the decorated permutation $\pi^{:}(\mathcal{I}_\mathcal{M})$, as needed. Similarly, white boundary leaves of G give cozeros of $\mathcal{M}$, i.e., the element $i \in [n]$ that appear in all bases of $\mathcal{M}$. They give white fixed points in $\pi^{:}(\mathcal{I}_\mathcal{M})$.

Thus we can remove all boundary leaves and assume that the reduced plabic graph G has no boundary leaves. Then the trip permutation π_G has no fixed points.

According to the cyclic symmetry (Remark 3.3) it is enough to show that the anti-exceedance set $I(\pi_G)$ equals the lexicographically minimal base I of $\mathcal{M}$. Let G' be the plabic graph with the same trip permutation obtained from a monotone triple diagram as shown in Section 14. Then by Theorem 13.4 the graphs G and G' can be obtained from each other by moves (M1)–(M3). These moves never change the trip permutation π_G and never change the matroid $\mathcal{M}_G$ (because they do not change the image $\mathrm{Image}(\tilde{M}_G) \subseteq S_\mathcal{M}^{\mathrm{tnn}}$). Thus the needed claim follows from Lemma 14.9. $\square$

Now we can finally prove Theorems 12.1 and 12.7.

Proof of Theorems 12.1 and 12.7. Let N and N' be two perfectly orientable plabic networks with graphs G and G' such that $\tilde{Meas}(N) = \tilde{Meas}(N')$. According to Lemma 13.6, we can transform these networks by the moves and reductions into networks with reduced graphs and maybe some singleton components. If there are singleton components then the graph(s) are not perfectly orientable. Thus we may assume that the plabic graphs G and G' are reduced.

The boundary measurement map sends N and N' into into the same cell $S^{\mathrm{tnn}}_{\mathcal{M}}$. Thus, by Proposition 16.4, the graphs G and G' have the same necklaces $\mathcal{I}(\pi^{:}_G) = \mathcal{I}(\pi^{:}_G) = \mathcal{I}_{\mathcal{M}}$. Thus by Lemma 16.2, the decorated trip permutations $\pi^{:}_G$ and $\pi^{:}_G$ are the same. According to Theorem 13.4, the graphs G and G' are move-equivalent.

We know that for any cell $S^{\mathrm{tnn}}_{\mathcal{M}}$ there is a plabic graph G'' such that $\tilde{Meas}_{G''}$ is a subtraction-free parametrization of $S^{\mathrm{tnn}}_{\mathcal{M}}$. Indeed, we can take the Ⅎ-diagram associated with $S^{\mathrm{tnn}}_{\mathcal{M}}$ (see Theorem 6.5) and transform it into a plabic graph. The graph G'' must be reduced. (Otherwise, we can kill some parameter and the map $\tilde{Meas}_{G''}$ would not be a parametrization; see Remark 12.8.) Thus again G'' has the same decorated trip permutation $\pi^{:}_{G''} = \pi^{:}_G$ and is move-equivalent to G and G'.

Note that every time when we perform moves (M1)–(M3), the face variables y_f are transformed by invertible subtraction-free rational maps. Thus for the graph G (and any other graph obtained from G'' by the moves), the map $\tilde{Meas}_G$ is obtained from $\tilde{Meas}_{G''}$ by a sequence of these reparametrization maps, and thus $\tilde{Meas}_G$ is also a subtraction-free rational parametrization of $S^{\mathrm{tnn}}_{\mathcal{M}}$.

So any two networks with the same graph G that maps into the same point in the Grassmannian must be equal to each other. That means that if we transform the network N' by the moves into a network with the graph G we will get the network N. Thus the networks N and N' are move-equivalent. □

Corollary 16.5. *For any perfectly orientable plabic graph G (not necessarily reduced) that corresponds to the cell $S^{\mathrm{tnn}}_{\mathcal{M}}$, we have*

$$\tilde{Meas}_G(\mathbb{R}^{F(G)-1}_{>0}) = S^{\mathrm{tnn}}_{\mathcal{M}},$$

that is the image of $\tilde{Meas}_G$ is the whole cell $S^{\mathrm{tnn}}_{\mathcal{M}}$.

Proof. For a reduced graph G this follow from Theorem 12.7. Other graphs can be transformed into reduced ones by the moves and reductions (Lemma 13.6), but the moves and reductions do not change the image of the map $\tilde{Meas}_G$. □

17. Circular Bruhat order

In this section we show that each nonnegative Grassmann cell $S^{\mathrm{tnn}}_{\mathcal{M}}$ is an intersection of n Schubert cells. Then we combinatorially describe the partial order on the cells $S^{\mathrm{tnn}}_{\mathcal{M}}$ by containment of their closures.

Let us use notation of Section 16. Let us say that a decorated permutation $\pi^{:}$ has *type* (k, n) if $\pi^{:}$ has size n and it has k anti-exceedances. Also say that a Grassmann necklace has *type* (k, n) if it consists of k-element subsets in $[n]$. Clearly, the types of corresponding decorated permutations and Grassmann necklaces are the same.

Theorem 17.1. *The map $S^{\mathrm{tnn}}_{\mathcal{M}} \mapsto \pi^{:}(\mathcal{I}_{\mathcal{M}})$ is a bijection between nonnegative Grassmann cells $S^{\mathrm{tnn}}_{\mathcal{M}} \subset Gr^{\mathrm{tnn}}_{kn}$ and decorated permutations of type (k, n). Equivalently,* *the map* $S^{\rm tnn}_{\mathcal{M}} \mapsto \mathcal{I}_{\mathcal{M}}$ *is a bijection between nonnegative Grassmann cells in* $Gr^{\rm tnn}_{kn}$ *and Grassmann necklaces of type* (k,n).

Proof. By Theorems 12.7 and 13.4, all reduced plabic graphs with the same decorated trip permutation $\pi^{:}$ correspond to the same nonnegative Grassmann cell $S^{\rm tnn}_{\mathcal{M}}$. By Propositions 11.7 and 16.4, we have $\pi^{:} = \pi^{:}(\mathcal{I}_{\mathcal{M}})$. Thus, two reduced plabic graphs with different trip permutations correspond to different Grassmann cell. $\square$

Recall that $\Omega^w_\lambda := w(\Omega_\lambda)$, for $w \in S_n$, are the permuted Schubert cells; see Section 2. Let us also use the subset notation for these Schubert cells

$$\Omega^w_I := \{V \in Gr_{kn} \mid I \text{ is the lex minimal base of } \mathcal{M}_V \text{ with respect to } <_w\},$$

where $<_w$ is the linear order on $[n]$ given $w(1) < w(2) < \cdots < w(n)$, cf. Section 2.3. The cells Ω^w_I are exactly the cells Ω^w_λ labelled by subsets rather than partitions. Each matroid strata $S_{\mathcal{M}}$ is an intersection of several permuted Schubert cells; see Remark 2.1. For a nonnegative Grassmann cell $S^{\rm tnn}_{\mathcal{M}}$ only n Schubert cells are needed. Let $c = (1,\dots,n) \in S_n$ be the long cycle.

Theorem 17.2. *Let* $S^{\rm tnn}_{\mathcal{M}} \subset Gr^{\rm tnn}_{kn}$ *be a nonnegative Grassmann cell, and let* $\mathcal{I}_{\mathcal{M}} = (I_1,\dots,I_n)$ *be the Grassmann necklace corresponding to* $\mathcal{M}$. *Then*

$$S^{\rm tnn}_{\mathcal{M}} = \bigcap_{i=1}^{n} \Omega^{c^{i-1}}_{I_i} \cap Gr^{\rm tnn}_{kn}.$$

Moreover, for any arbitrary collection of k*-subsets* $I_1,\dots,I_n \subset [n]$, *the intersection in the right-hand-side is nonempty if and only if* $(I_1,\dots,I_n)$ *is a Grassmann necklace.*

Proof. Note that $<_{c^{i-1}}$ is exactly the shifted order $<_i$ on $[n]$. By the definition of $\mathcal{I}_{\mathcal{M}}$, the cell $S^{\rm tnn}_{\mathcal{M}}$ belongs to the intersection of Schubert cells in the right-hand-side. Suppose that this intersection contains an element of another cell $S^{\rm tnn}_{\mathcal{M}'}$. Then $\mathcal{I}_{\mathcal{M}'} = \mathcal{I}_{\mathcal{M}}$, so the cell $S^{\rm tnn}_{\mathcal{M}'}$ corresponds to the same decorated trip permutation $\pi^{:}(\mathcal{I}_{\mathcal{M}}) = \pi^{:}(\mathcal{I}_{\mathcal{M}'})$, which is impossible by Theorem 17.1. The second claim follows from Lemma 16.3. $\square$

Let $\overline{S^{\rm tnn}_{\mathcal{M}}} \subseteq Gr^{\rm tnn}_{kn}$ denotes the closure of the cell $S^{\rm tnn}_{\mathcal{M}} \subset Gr^{\rm tnn}_{kn}$. Define the partial order $\leq$ on nonnegative Grassmann cells such that $S^{\rm tnn}_{\mathcal{M}} \leq S^{\rm tnn}_{\mathcal{M}'}$ if and only if $\overline{S^{\rm tnn}_{\mathcal{M}}} \subseteq \overline{S^{\rm tnn}_{\mathcal{M}'}}$. Recall that the cell $S^{\rm tnn}_{\mathcal{M}}$ is given in the Plücker coordinates by the conditions $\Delta_I > 0$ for $I \in \mathcal{M}$, and $\Delta_J = 0$ for $J \notin \mathcal{M}$; see Section 3. Thus, for two nonempty cells, we have $S^{\rm tnn}_{\mathcal{M}} \leq S^{\rm tnn}_{\mathcal{M}'}$ if and only if $\mathcal{M} \subseteq \mathcal{M}'$.

Recall that the partial order on the Schubert cells by the containment has a simple combinatorial description $\overline{\Omega}_\lambda \subseteq \overline{\Omega}_\mu$ if and only if $\lambda \subseteq \mu$ (meaning the inclusion of Young diagrams); see [Fult].

Recall the standard bijection $\lambda \mapsto I(\lambda)$ between partitions $\lambda \subseteq (n-k)^k$ and k-subsets $I \subset [n]$; see Section 2. Let $I \mapsto \lambda(I)$ be the inverse bijection.

Lemma 17.3. *Let* $S^{\rm tnn}_{\mathcal{M}}$ *be a nonnegative Grassmann cell. Let* I *be the lexicographically minimal base of* $\mathcal{M}$. *Then, for any other base* $J \in \mathcal{M}$, *we have* $\lambda(J) \subseteq \lambda(I)$.

Proof. According to Theorem 17.1, Lemma 14.6, and the second part of Lemma 14.9, any base J of $\mathcal{M}$ is obtained from I by switching some entries $i_1,\dots,i_s \in I$ with some $j_1,\dots,j_s$ such that $j_1 > i_1, \dots, j_s > i_s$. But this exactly means that for the corresponding partitions we have $\lambda(J) \subseteq \lambda(I)$. $\square$

Let $\mathcal{I}_\mathcal{M} = (I_1, \dots, I_n)$ be the Grassmann necklace corresponding to a matroid $\mathcal{M} \subseteq \binom{[n]}{k}$. Let us also denote $\Lambda_\mathcal{M} = (\lambda_{(1)}, \dots, \lambda_{(n)})$ the sequence of partitions $\lambda_{(i)} = \lambda(c^{-i+1}(I_i))$, for $i \in [n]$. In other words, the partitions $\lambda_{(i)}$ are chosen so that $\Omega_{I_i}^{c^{i-1}} = \Omega_{\lambda_{(i)}}^{c^{i-1}}$.

Proposition 17.4. *Let $S_\mathcal{M}^{\text{tnn}}, S_{\mathcal{M}'}^{\text{tnn}}$ be two cells in Gr_{kn}^{tnn}, $\Lambda_\mathcal{M} = (\lambda_{(1)}, \dots, \lambda_{(n)})$ and $\Lambda_{\mathcal{M}'} = (\lambda'_{(1)}, \dots, \lambda'_{(n)})$. Then $S_\mathcal{M}^{\text{tnn}} \leq S_{\mathcal{M}'}^{\text{tnn}}$ if and only if $\lambda_{(i)} \subseteq \lambda'_{(i)}$, for all $i \in [n]$.*

Proof. If $\lambda_{(i)} \subseteq \lambda'_{(i)}$ for all $i \in [n]$, then $S_\mathcal{M}^{\text{tnn}} \leq S_{\mathcal{M}'}^{\text{tnn}}$, by Theorem 17.2. On the other hand, suppose that $S_\mathcal{M}^{\text{tnn}} \leq S_{\mathcal{M}'}^{\text{tnn}}$. Then $\mathcal{M} \subseteq \mathcal{M}'$. In particular, the lexicographically minimal base I of of $\mathcal{M}$ is also a base of $\mathcal{M}'$. By Lemma 17.3, this implies that $\lambda_{(1)} \subseteq \lambda'_{(1)}$. Taking cyclic shifts and using the cyclic symmetry of the construction, we get $\lambda_{(i)} \subseteq \lambda'_{(i)}$, for any $i \in [n]$. □

Let us now describe the partial order on the cells $S_\mathcal{M}^{\text{tnn}}$ in terms of decorated permutations.

Definition 17.5. The *circular Bruhat order CB_{kn}* is the partial order $\leq$ on the set of decorated permutations of type (k, n) such that, for two decorated permutations $\pi^:$ and $\sigma^:$ corresponding to the cells $S_\mathcal{M}^{\text{tnn}}$ and $S_{\mathcal{M}'}^{\text{tnn}}$ in Gr_{kn}^{tnn}, we have $\pi^: \leq \sigma^:$ if and only if $S_\mathcal{M}^{\text{tnn}} \leq S_{\mathcal{M}'}^{\text{tnn}}$.

Lemma 17.6. *The circular Bruhat order CB_{kn} has a unique top element given by the decorated permutation $\pi_{top} : i \mapsto i + k \pmod n$, for $i \in [n]$ (for $k = 0$, all fixed points of π_{top} are colored black, and for $k = n$ all fixed points of π_{top} are colored white). The circular Bruhat order CB_{kn} has $\binom{n}{k}$ minimal elements corresponding to the identity permutation permutation with some k fixed points colored in white and remaining $(n - k)$ fixed points colored in black.*

Proof. The top element of CB_{kn} corresponds to the top cell $S_\mathcal{M}^{\text{tnn}} \subset Gr_{kn}^{\text{tnn}}$, where $\mathcal{M} = \binom{[n]}{k}$ is the complete matroid of rank k on $[n]$. By Lemmas 16.2 and 16.3, this matroid corresponds to the Grassmann necklace $\mathcal{I}_\mathcal{M} = (I_1, \dots, I_n)$ with $I_i = \{i, i+1, \dots, i+k\}$, for $i \in [n]$ (elements are taken modulo n); and this necklace corresponds to decorated permutation $\pi_{top} : i \mapsto i + k$.

On the other hand, minimal elements of CB_{kn} correspond to 0-dimensional cells $S_\mathcal{M}^{\text{tnn}}$. These cells are fixed points of the torus action on the Grassmannian Gr_{kn}. In other words, they correspond to matroids with a single base $\mathcal{M} = \{I\}$. Under the correspondence $\mathcal{M} \mapsto \pi^:$, the k elements of I give k white fixed points of $\pi^:$ and $n - k$ elements of $[n] \setminus I$ give $n - k$ black fixed points of $\pi^:$. □

For $a, b \in [n]$, define the *cyclic interval* $[a, b]^{cyc}$ as $\{a, a+1, \dots, b\}$ if $a \leq b$, and as $\{a, a+1, \dots, n, 1, \dots, b-1\}$ if $a > b$. In other words, a cyclic interval is a sequence of consecutive numbers arranged on a circle in the clockwise order. For a decorated permutation $\pi^:$ and a pair $a, b \in [n]$, let us define the number $r_{ab}(\pi^:)$ as the number of shifted anti-exceedances i of $\pi^:$ with respect to the shifted order $<_a$ such that $i \in [a, b]^{cyc}$. In other words, if $\mathcal{I}(\pi^:) = (I_1, \dots, I_n)$, then $r_{ab}(\pi^:) = |I_a \cap \{a, a+1, \dots, b\}|$. In particular, for a decorated permutation of type (k, n), we have $r_{a,a-1}(\pi^:) = k$, for any $a \in [n]$. (Here we take indices a, b modulo n.)

Corollary 17.7. *For two decorated permutations $\pi^:$ and $\sigma^:$ of the same type, we have $\pi^: \leq \sigma^:$ if and only if $r_{ab}(\pi^:) \leq r_{ab}(\sigma^:)$ for all $a, b \in [n]$.*

Proof. Follows from Proposition 17.4 and the fact that, for two partitions $\lambda, \mu \subseteq (n-k)^k$, we have $\lambda \subseteq \mu$ if and only if $|I(\lambda) \cap [b]| \leq |I(\mu) \cap [b]|$ for any $b \in [n]$. ☐

Recall, that we presented each decorated permutation $\pi^:$ by a chord diagram; see Figure 16.1. Also recall *crossings, alignments,* and *misalignments* from Section 5. This notions can be adapted for decorated permutations, as follows.

Let $\pi^:$ be a decorated permutation, and let $(b_i, b_{\pi(i)})$ and $(b_j, b_{\pi(j)})$, $i \neq j$, be a pair of chords (or loops). We say that this pair is a *crossing* if $\pi(j) \in [i, \pi(i)]^{cyc}$ and $j \in [\pi(i), i]^{cyc}$. We say that this pair is an *alignment* if $\pi(i) \in [i, \pi(j)]^{cyc}$ and $j \in [\pi(j), i]^{cyc}$; if $\pi(i) = i$ then i must be colored $col(i) = 1$ (counterclockwise loop), and if $\pi(j) = j$ then j must be colored $col(j) = -1$ (clockwise loop); see Figure 17.1. Note that in a crossing the vertex b_i is allowed to coincide with $b_{\pi(j)}$, and the vertex b_j is allowed to coincide with $b_{\pi(i)}$. But a loop can never participate in a crossing. On the other hand, two chords in an alignment never have common vertices, but the vertex b_i can coincide with $b_{\pi(i)}$ and form a counterclockwise loop, and similarly $(b_j, b_{\pi(j)}$ can form a clockwise loop. In particular, any counterclockwise loop forms an alignment with any clockwise loop.

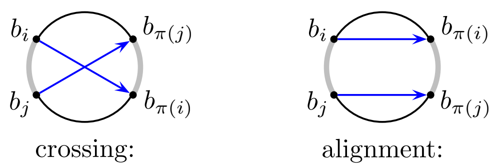


FIGURE 17.1. A crossing and an alignment

Let us say that a crossing as above is a *simple crossing* if there are no any other chords $(l, \pi(l))$ such that $l \in [j, i]^{cyc}$ and $\pi(l) \in [\pi(j), \pi(i)]^{cyc}$. Similarly, a *simple alignment* is an alignment such that there are not other chords $(l, \pi(l))$ such that $l \in [j, i]^{cyc}$ and $\pi(l) \in [\pi(i), \pi(j)]^{cyc}$. In other words, simple crossings and alignments should have no other chords that start on the left between b_i and b_j and end on the right between $b_{\pi(i)}$ and $b_{\pi(j)}$. Notice if $\pi^:$ has a crossing/alignment then it should also have a simple crossing/alignment. (Just pick the one where b_i and b_j are closest to each other.)

For two decorated permutations $\pi^:, \sigma :$ of the same type, let $\pi^: \lessdot \sigma^:$ denotes the covering relation in the circular Bruhat order.

Theorem 17.8. *In the circular Bruhat order CB_{kn} we have $\pi^: \lessdot \sigma^:$ if and only if the chord diagram $\pi^:$ is obtained from the chord diagram of $\sigma^:$ by replacing a simple crossing with the corresponding simple alignment as shown on Figure 17.2.*

Note again that we allow $i = \sigma(j)$ and/or $j = \sigma(i)$. In this case $\pi^:$ should have a counterclockwise loop at $b_i = b_{\pi(i)}$ and/or a clockwise loop at $b_j = b_{\pi(j)}$. In the case when $i = \sigma(j)$ and $j = \sigma(i)$, we can also switch i and j and get the decorated permutation clockwise loop at b_i and counterclockwise loop at b_j, which is also covered by $\sigma^:$.

Proof. One directly verifies that for any pair $\pi^:$ and $\sigma^:$ related by "undoing a crossing" as above, we have $r_{ab}(\pi^:) \leq r_{ab}(\sigma^:)$ for any $a, b \in [n]$. Thus we have $\pi^: < \sigma^:$. One the other hand, let $\mathcal{I}(\pi^:) = (I_1, \ldots, I_n)$ and $\mathcal{I}(\sigma^:) = (I'_1, \ldots, I'_n)$. Then the relation $\pi^: \leq \sigma^:$ mean that I_i is obtained from I'_i by moving some elements "to the

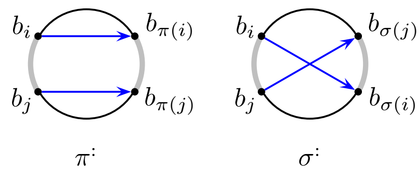


FIGURE 17.2. A covering relation $\pi^{:} \lessdot \sigma^{:}$ in the circular Bruhat order

right" (with respect to the linear order $<_i$ on $[n]$). We may assume that $I_1 \neq I_1'$. (Otherwise cyclic shift all elements using the cyclic symmetry of the construction.) Suppose that I_1 is obtained from I_1' by switching the elements $i_1, \dots, i_s \in I_1'$ with $j_1, \dots, j_s$, where $j_1 > i_1, \dots, j_s > i_s$. Let $a = \pi^{-1}(i_1)$ and $b = \pi^{-1}(j_1)$. Then $a \leq i_1$ and $b \geq j_1$ because j_1 belongs to the anti-exceedance set of $\pi^{:}$ and i_1 does not belong to this set. Note that for $i \in [b+1, a]^{cyc}$, the set I_i contains j_1 and does not contain i_1. Let $(I_1'', \dots, I_n'')$ be the Grassmann necklace such that $I_i'' = (I_i \setminus \{j_1\}) \cup \{i_1\}$, for $i \in [b+1, a]^{cyc}$, and $I_i'' = I_i$ otherwise. Let $\rho^{:}$ be the decorated permutation corresponding to $(I_1'', \dots, I_n'')$. Then we have $\pi^{:} < \rho^{:} \leq \sigma^{:}$ and $\pi^{:}$ is obtained from $\rho^{:}$ by undoing a crossing. $\square$

In Section 18 we will explicitly describe how the cells are glued to each other, which will explain where these "undoings of crossings" are coming from.

Let us define the *alignment number* $A(\pi^{:})$ of a decorated permutation as the total number of pairs of chords (or loops) in $\pi^{:}$ forming an alignment. The following claim is obtained by an easy verification.

Lemma 17.9. *Suppose that $\pi^{:}$ is obtained from $\sigma^{:}$ by undoing a simple crossing as shown on Figure 17.2. Then $A(\pi^{:}) = A(\sigma^{:}) + 1$.*

Notice that the maximal element π_{top} in CB_{kn} has no alignments. On the other hand, all minimal elements $\pi^{:} \in CB_{kn}$ have $k(n-k)$ alignments; see Lemma 17.6. Also notice that the dimension of the top cell in Gr_{kn}^{tnn} is $k(n-k)$. As we go down from the top element π_{top} to a minimal element in CB_{kn} by undoing simple crossings, the alignment number $A(\pi^{:})$ increases by 1 at each step. On the other hand, dimensions of the corresponding cells should drop by at least 1 at each step. Since the dimension of the top cell is the same as the number of steps, we obtain the following claim.

Proposition 17.10. *For the decorated permutation $\pi^{:} \in CB_{kn}$ associated with a cell $S_{\mathcal{M}}^{\mathrm{tnn}}$, we have $\dim S_{\mathcal{M}}^{\mathrm{tnn}} = k(n-k) - A(\pi^{:})$. The circular Bruhat order is a ranked poset with the rank function $\mathrm{rank}(\pi^{:}) = k(n-k) - A(\pi^{:})$.*

Let us now explain the reason why we call the partial order CB_{kn} the circular Bruhat order. Actually, one can embed the usual (strong) Bruhat order on the symmetric group S_k (and also the product of two copies of the Bruhat order) into $CB_{k,2k}$ as a certain interval.

Let $S_{\mathcal{M}}^{\mathrm{tnn}}$ be a cell in $Gr_{k,2k}^{\mathrm{tnn}}$ such that $[k], [k+1, 2k] \in \mathcal{M}$. Such cells are exactly the *double Bruhat cells* of Fomin-Zelevinsky, see Remark 3.11. Then the corresponding Bruhat necklace $\mathcal{I}_{\mathcal{M}}$ has the entries $I_1 = [k]$ and $I_{k+1} = [k+1, 2k]$. This means that the corresponding permutation $\pi = \pi(\mathcal{I}_{\mathcal{M}})$ satisfies the following condition: $\pi : [k] \to [k+1, 2k]$ and $\pi : [k+1, 2k] \to [k]$. (Such permutations have no fixed points so there is no need to decorate them.) In other words, π can be subdivided into two permutations from S_k, as follows. For two permutations $u, v \in S_k$, let $\pi = \pi(u, v)$ be the permutation in S_{2k}, given by $\pi(i) = \overline{u(i)}$ and $\pi(\bar{i}) = v(i)$ for $i = 1, \ldots, k$, where $\bar{i} = 2k + 1 - i$.

Proposition 17.11. *For two permutations $\pi(u, v)$ and $\pi(u', v')$, where $u, v, u', v' \in S_k$ we have $\pi(u, v) \le \pi(u', v')$ in the circular Bruhat order $CB_{k,2k}$ if and only if $u \le u'$ and $v \le v'$ in usual Bruhat order on S_k. The interval $[\pi(1, 1), \pi_{top}]$ in $CB_{k,2k}$ is isomorphic to the direct product of two copies of the usual Bruhat order on S_k.*

Proof. For permutations of the form $\pi(u, v)$, the description of the circular Bruhat order from Corollary 17.7 is equivalent to the well known description of the usual Bruhat order on S_k: $w \le w'$ if and only if $|\{i \in [a] \mid w(i) \in [b]\}| \ge |\{i \in [a] \mid w'(i) \in [b]\}|$ for any $a, b \in [k]$. The second claim follows from the fact that any element element π which is greater than $\pi(1, 1)$ in $CB_{k,2k}$ should have the form $\pi = \pi(u, v)$. □

18. Gluing of cells

In this section, we will explicitly describe how the nonnegative Grassmann cells $S_{\mathcal{M}}^{\mathrm{tnn}}$ are glued to each other using the network parametrization $Meas_G$ of the cells.

Let G be a plabic (undirected) graph and let $\mathcal{O}$ be a perfect orientation of its edges. Denote by $G' = (G, \mathcal{O})$ the corresponding directed graph. According to Theorem 12.7 (or Theorem 4.11), for each cell $S_{\mathcal{M}}^{\mathrm{tnn}} \subset Gr_{kn}^{\mathrm{tnn}}$ there is a a directed graph G' as above such that the boundary measurement map $Meas_{G'}$ maps $\mathbb{R}_{>0}^{E(G')}/\{\text{gauge transformations}\}$ onto $S_{\mathcal{M}}^{\mathrm{tnn}}$. By a slight abuse of notation, we will also denote by $Meas_{G'}$ the map $\mathbb{R}_{>0}^{E(G')} \to S_{\mathcal{M}}^{\mathrm{tnn}}$.

Lemma 18.1. *The map $Meas_{G'} : \mathbb{R}_{>0}^{E(G')} \to Gr_{kn}^{\mathrm{tnn}}$ uniquely extends to the continuous map $\overline{Meas}_{G'} : \mathbb{R}_{\ge 0}^{E(G')} \to Gr_{kn}^{\mathrm{tnn}}$.*

This claim is trivial for an acyclic graph G'. But we allow G' to have cycles.

Proof. The uniqueness is clear because $\mathbb{R}_{>0}^{E(G')}$ is a dense subset in $\mathbb{R}_{\ge 0}^{E(G')}$. We need to check that $Meas_{G'}$ does not have have poles as some of the edge variables x_e approach 0. This follows from Lemma 4.4, which says that the boundary measurements M_{ij} are well-defined functions on $\mathbb{R}_{\ge 0}^{E(G')}$. □

Clearly, the image of the map $\overline{Meas}_{G'}$ belongs to the closure $\overline{S_{\mathcal{M}}^{\mathrm{tnn}}}$. Moreover, this image consists of the union of some cells $S_{\mathcal{M}''}^{\mathrm{tnn}}$. Indeed, these are the cells that correspond (as in Theorem 4.11) to all directed graphs H' obtained from G' by removing some edges.

Let us show that the opposite is true. For any fixed (perfectly orientable) plabic graph G and any point p in the closure $\overline{S_{\mathcal{M}}^{\mathrm{tnn}}}$ of the corresponding cell, there exists a directed graph G' obtained by a perfect orientation of edges of G such that $p \in \mathrm{Image}(\overline{Meas}_{G'})$.

Let $p \in \overline{S_{\mathcal{M}}^{\mathrm{tnn}}}$. Let us pick the graph $G' = (G, \mathcal{O})$ for some perfect orientation $\mathcal{O}$ of G. Then we can find nonnegative functions $x_e(t) :]0, 1] \to \mathbb{R}_{>0}$ on edges e of G' such that $\lim_{t\to 0} Meas_{G'}(\{x_e(t)\}) = p$ and each $x_e(t)$ is of the form $x_e(t) = t^{m_e} f_e(t)$, where $m_e \in \mathbb{R}$ and $f_e(t)$ is real-valued analytic function such that $f_e(0) > 0$.

Lemma 18.2. *For any collection of edge functions $x_e(t)$ as above one can apply gauge transformations at vertices and switch edge directions along some paths and/or closed cycles as in Section 10 to transform the $x_e(t)$ into functions $x'_e(t)$ that have no poles at $t = 0$, for all edges e.*

Proof. Let $M = \min(m_e)$. We may assume that we have already transformed the edge functions by gauge transformations and switching edge directions so that M is a big as possible and the number of edges e such that $m_e = M$ is as small as possible (for this M.) If $M \geq 0$ then the $x_e(t)$ have no poles. Assume that $M < 0$. For any directed edge $e = (u, v)$ such that $m_e = M$, where v is an internal vertex, there should be an outgoing edge e' from the vertex v such that $m_{e'} = M$. Otherwise, $m_{e'} > M$ for all edges e' outgoing from v, and we could apply a gauge transformation $t_v = t^{-\epsilon}$ at this vertex so that m_e increases by ϵ and the $m_{e'}$ decrease by $-\epsilon$. For a sufficiently small $\epsilon > 0$ this would make the number of edges with $m_e = M$ smaller, which is impossible by our assumption. Similarly, there should be an edge e'' incoming to the vertex u such that $m_{e''} = M$. This means that we can always find a directed path P in G' joining two boundary vertices (or a closed cycle C) such that for all edges e in this path/cycle we have $m_e = M$. Let us switch directions of edges in P (or C) and invert the edge functions $x_e(t)$ for these edges; see Section 10. This switch would transform the $m_e = M$ into $-M$ for all edges in the path/cycle, so again this would make the number of edges with $m_e = M$ smaller, which is impossible by our assumption. Thus we should have $M \geq 0$, as needed. □

According to Lemma 18.2, the graph G' and the edge functions $x_e(t)$ can be chosen so that the $x_e(t)$ have no poles at $t = 0$. That means that the point $p = \lim_{t\to 0} Meas_{G'}(\{x_e(t)\}) = \overline{Meas}_{G'}\{x_e(0)\}$ belongs to the image of $\overline{Meas}_{G'}$.

We have proved the following result.

Theorem 18.3. *For a cell $S^{\mathrm{tnn}}_{\mathcal{M}}$, pick any (perfectly orientable) plabic graph G such that $\tilde{Meas}_G$ maps onto $S^{\mathrm{tnn}}_{\mathcal{M}}$. Then the closure of this cell equals*

$$\overline{S^{\mathrm{tnn}}_{\mathcal{M}}} = \bigcup_{G'=(G,\mathcal{O})} \overline{Meas}_{G'}(\mathbb{R}^{E(G')}_{\geq 0}) \subseteq Gr^{\mathrm{tnn}}_{kn},$$

where $\mathcal{O}$ ranges over all perfect orientations of G.

Assume that G is a contracted plabic graph. (That is G is without unicolored edges, non-boundary leaves, and vertices of degree 2). For each $G' = (G, \mathcal{O})$, we have

$$\overline{Meas}_{G'}(\mathbb{R}^{E(G')}_{\geq 0}) = \bigcup_{H'} Meas_{H'}(\mathbb{R}^{E(G')}_{>0}),$$

where the union is over directed graphs H' obtained from G' by removing some edges (but keeping all vertices). If we remove a directed edge $e = (u, v)$ where u is a black vertex and v is white, then u will be an internal sink and v will be an internal source in the obtained graph. Instead of removing such edge e, we could remove all incoming edges to the vertex u (and get a graph with the same image of $Meas_{H'}$). Thus in the above union it is enough to take only graphs H' obtained from G' by removing some white-to-black directed edges $e = (u, v)$, i.e., u is white and v is black. Notice that such graphs H' will be perfectly oriented graphs with the same coloring of vertices as in the graph G.

For a contracted plabic graph G, let us say that a plabic graph H is a *subgraph* of G, and write $H \subseteq G$, if H is obtained from G by removing some edges while keeping all vertices in G. When we remove a nonleaf edge $e = (b_i, v)$ attached to a boundary vertex, we need to create a new boundary leaf at b_i, whose color is opposite to v. When we remove an edge $e = (b_i, b_j)$ between two boundary vertices, we need to create two boundary leaves at b_i and b_j of different colors. (So there are two different ways to "remove" such edge.) We are not allowed to remove boundary leaves. Note that each perfect orientation of a plabic subgraph $H \subseteq G$ uniquely extends to a perfect orientation G. (Just direct all removed edges from white vertices to black vertices.)

Let us call a plabic graph *almost-reduced* if it consists of a reduced graph possibly together with some dipoles (isolated components with a single edge and a pair of vertices of different colors).

Theorem 18.3 and the above discussion implies that the closure $\overline{S^{\mathrm{tnn}}_{\mathcal{M}}}$ is the union of Image($\tilde{Meas}_H$) over all perfectly orientable plabic subgraphs $H \subseteq G$. Let us show that it is enough to take only almost-reduced plabic subgraphs H.

Lemma 18.4. *Let G be a perfectly orientable contracted plabic graph. Then G has an almost-reduced plabic subgraph $H \subseteq G$ such that boundary measurement maps $\tilde{Meas}_G$ and $\tilde{Meas}_H$ have the same images in Gr^{tnn}_{kn}.*

Note that if we remove the edge from a dipole, then we would create a graph with a singleton (isolated component with a single vertex), which is not perfectly orientable. This is why we need to consider almost-reduced subgraphs (and not just reduced ones).

Proof. Suppose that G is not almost-reduced. Let us pick its perfect orientation G' and transform this (directed) graph by the moves (M1)–(M3) into a (directed) graph $\tilde{G}$, where we can apply a reduction (R1) or (R2); see Section 12. Then we can find an edge $\tilde{e} = (u, v)$ in $\tilde{G}$ directed from a white vertex u to a black vertex v such that by removing $\tilde{e}$ from $\tilde{G}$ we will not change the image of $Meas_{\tilde{G}}$. (We need to check 3 possible directions of edges in (R1); for (R2) there is only one possible direction of edges; see Figures 12.4 and 12.5.) Let us transform $\tilde{G}$ back to G' by moves (M1)–(M3) and keep track of the "marked for removal" edge $\tilde{e}$. In all cases when we perform a move $G'' \mapsto G'''$ and G'' has a white-to-black marked edge e'', one can find a white-to-black edge e''' in G''' such that the maps $Meas_{G''\setminus\{e''\}}$ and $Meas_{G'''\setminus\{e'''\}}$ have the same image. The only case that needs a special attention is square move (M1). Note that transformations of edge weights on Figure 12.7 involve only weights of the four edges in the square. Thus, if e'' is not one of these four edges, then we can just keep this edge $e''' = e''$. If e'' is one of the four edges in the square, the we can pick e''' to be the edge opposite to e'' in the square.

This shows that in the (directed) graph G' we can always find a white-to-black edge e' whose removal does not change the image of the map $Meas_G$. Thus in the undirected graph G we can find an edge e whose removal preserves perfect orientability and does not change the image of $\tilde{Meas}_G$. If the graph $G\setminus\{e\}$ is still not almost-reduced then we can repeatedly remove its edges using this procedure until we get an almost-reduced graph with the same image of the boundary measurement map. □

We proved the following result.

Theorem 18.5. *For a cell $S_\mathcal{M}^{\mathrm{tnn}}$, and any reduced contracted plabic graph G associated with this cell, we have*

$$\overline{S_\mathcal{M}^{\mathrm{tnn}}} = \bigcup_{H\subseteq G} \tilde{Meas}_H(\mathbb{R}_{>0}^{F(H)-1}),$$

where H ranges over all almost-reduced plabic subgraphs $H \subseteq G$.

Note that each term in the right-hand side of this expression is a certain cell $S_{\mathcal{M}'}^{\mathrm{tnn}}$ inside the closure $\overline{S_\mathcal{M}^{\mathrm{tnn}}}$. Each map $\tilde{Meas}_H$ gives a parametrization of this cell.

Remark 18.6. Let us describe bases of the matroid $\mathcal{M}$ using Theorem 18.5. Any almost-reduced subgraph $H \subseteq G$ should contain at least one edge incident to each internal vertex v of G. (If we remove all edges incident to v then we would get a graph, which is not perfectly orientable.) Thus *minimal* almost-reduced subgraphs are the subgraphs containing exactly one edge at each internal vertex. These are partial matchings of G, discussed in the end of Section 11. Each minimal subgraph $H \subseteq G$ gives a cell $S_{\mathcal{M}'}^{\mathrm{tnn}}$ that consists of a single point and $\mathcal{M}'$ contains a single base $I = \{i \mid b_i \text{ connected to a white boundary leaf}\}$. For any base of $\mathcal{M}$, there is a minimal subgraph of this form. We obtain the description $\mathcal{M}$ equivalent to the matching matroid $\mathcal{M}_G^m$ from Lemma 11.10.

Remark 18.7. There is an analogy between the usual Bruhat order on a Weyl group W and the circular Bruhat order CB_{kn}, as follows. The cells $S_\mathcal{M}^{\mathrm{tnn}}$ are analogues of Weyl group elements $w \in W$; plabic graphs G are analogues of reduced decompositions $w = s_{i_1} \cdots s_{i_l}$; plabic subgraphs $H \subseteq G$ are analogues of reduced subwords in a reduced decomposition. Then Theorem 18.5 corresponds to the following well-known description of the Bruhat order on W. For $w = s_{i_1} \cdots s_{i_l}$, all elements u such that $u \leq w$ (in the Bruhat order) have reduced decompositions obtained by taking subwords in $s_{i_1} \cdots s_{i_l}$.

This analogy can be made more precise. Each type A reduced decomposition $w = s_{i_1} \dots s_{i_l} \in S_k$ is graphically represented by a wiring diagram. We can transform the wiring diagram into a reduced plabic graph G with $n = 2k$ boundary vertices by replacing each crossing with a pair of trivalent vertices as shown on Figure 18.1. The trip permutation of G is $\pi : i \to 2k+1-w(i)$, for $i \in [k]$ and $\pi : i \to 2k+1-i$, for $i \in [k+1, 2k]$. Subgraphs $H \subseteq G$ obtained by removing some vertical edges of G correspond to subwords in the reduced decomposition.

Remark 18.8. Note that two different subgraphs of $H \subset G$ might correspond to the same cell in the closure $\overline{S_\mathcal{M}^{\mathrm{tnn}}}$. (Similarly, to the situation when different two subwords in a reduced decomposition give the same Weyl group element.) That means that possibly there are some nontrivial identifications of components in the right-hand side of expression in Theorem 18.5. This why the description of the geometrical structure of the closure $\overline{S_\mathcal{M}^{\mathrm{tnn}}}$ is a nontrivial problem. Conjecturally, the closure $\overline{S_\mathcal{M}^{\mathrm{tnn}}}$ is homeomorphic to an open ball.

Let us now describe the covering relation $\lessdot$ in the circular Bruhat order using Theorem 18.5. Suppose that the cell $S_\mathcal{M}^{\mathrm{tnn}}$ covers $S_{\mathcal{M}'}^{\mathrm{tnn}}$. Let us pick a reduced contracted plabic graph G such that $S_\mathcal{M}^{\mathrm{tnn}} = \mathrm{Image}(\tilde{Meas}_G)$ and G has no leaves and vertices of degree 2. By Theorem 18.5, there is an almost-reduced subgraph $H \subset G$ such that $S_{\mathcal{M}'}^{\mathrm{tnn}} = \mathrm{Image}(\tilde{Meas}_H)$. Note that if H is obtained from G

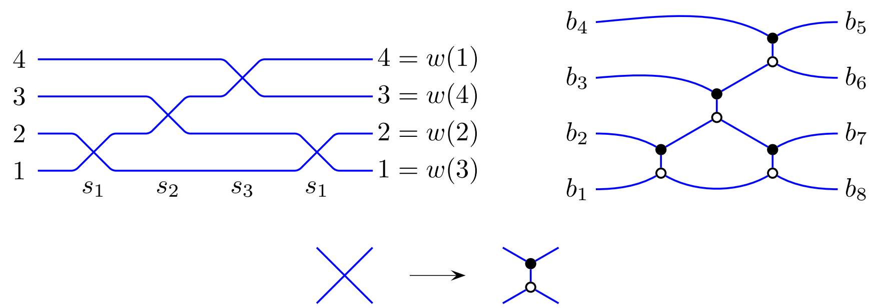


FIGURE 18.1. Transforming a wiring diagram of $w = s_1 s_2 s_3 s_1$ into a plabic graph G

by removing two or more edges, then it number of faces drops by at least 2. By Proposition 17.10, the circular Bruhat order CB_{kn} is a ranked poset with the rank function equal $\dim S^{\rm tnn}_{\mathcal{M}}$. Thus the codimension of $S^{\rm tnn}_{\mathcal{M}'}$ in $\overline{S^{\rm tnn}_{\mathcal{M}}}$ should be 1, that is H is obtained from G by removing a *single* edge. In this case we cannot create a dipole, so H should be reduced.

Let us call an edge e in a reduced contracted plabic graph G *removable* if $G \backslash \{e\} \subset G$ is a reduced plabic graph. By Theorem 18.5, removable edges in G are in one-to-one correspondence with the cells covered by $S^{\rm tnn}_{\mathcal{M}} = \mathrm{Image}(\widetilde{Meas}_G)$.

Let G be a reduced contracted plabic graph with the decorated trip permutation $\pi^{:} = \pi^{:}(G)$. For an edge e, let $T_1 : b_i \to b_{\pi(i)}$ and $T_2 : b_j \to b_{\pi(j)}$ be the two trips in G that contain e (and pass this edge in two different directions).

Lemma 18.9. *The edge e is removable if and only if the pair $(i, \pi(i))$ and $(j, \pi(j))$ is a simple crossing in the decorated trip permutation $\pi^{:}$. In this case the decorated trip permutation of $G \setminus \{e\}$ is obtained from $\pi^{:}$ by replacing this simple crossing with the corresponding alignment; see Figure 17.2.*

Since we have already described covering relations in Theorem 17.8, this lemma follows. We can also easily deduce it from the reducedness criterion in Theorem 13.2.

Proof. The trips of $G \setminus \{e\}$ are exactly the same as the trips of G, except that we need to switch tails of T_1 and T_2 at the their (essential) intersection point at e. According to Theorem 13.2, the graph $G \setminus \{e\}$ is reduced if and only if the trips T_1 and T_2 have only one essential intersection at e; and there is no any other trip T_3 that intersects the part of T_1 before e and then the part of T_2 after e; and vise versa. The means that the pair $(i, \pi(i))$ and $(j, \pi(j))$ is a simple crossing in $\pi^{:}$. On the other hand, for a simple crossing in $\pi^{:}$, the corresponding pair of trips should intersect only once. Otherwise, if they intersect ≥ 3 times, then there is another trip T_3 that passes through, say, the second intersection point of T_1 and T_2. This trip cannot intersect the tails of T_1 and T_2 (after all their intersection points). Thus T_3 should end at a boundary point between $b_{\pi(j)}$ and $b_{\pi(i)}$. (Here we assume that the vertices are arranged on the circle as in the crossing on Figure 17.1 and the word "between" means "between in the clockwise order".) Similarly, T_3 should start at a boundary point between b_j and b_i. So the crossing in $\pi^{:}$ would not be simple. Also if there is a trip T_3' that first intersects with the initial part of T_1 before e and then intersects with the part of T_2 after e, then again the trip T_3' would form an obstruction for a simple crossing. $\square$

Corollary 18.10. *For a cell $S_{\mathcal{M}}^{\mathrm{tnn}}$ and a reduced contracted plabic graph G such that $S_{\mathcal{M}}^{\mathrm{tnn}} = \mathrm{Image}(\tilde{Meas}_G)$, the cells $S_{\mathcal{M}'}^{\mathrm{tnn}}$ that are covered by $S_{\mathcal{M}}^{\mathrm{tnn}}$ are in one-to-one correspondence with removable edges e of G. They have the form $S_{\mathcal{M}'}^{\mathrm{tnn}} = \mathrm{Image}(\tilde{Meas}_{G\setminus\{e\}})$.*

Let us show how to glue such adjacent cells $S_{\mathcal{M}}^{\mathrm{tnn}}$ and $S_{\mathcal{M}'}^{\mathrm{tnn}}$ together. For a reduced plabic graph G and a removable edge $e \in G$, pick a perfect orientation $\mathcal{O}$ of $G \setminus \{e\}$. Let G' be the graph obtained by directing edges of G so that the edge e is directed from a white vertex to a black vertex and other edges are directed as in $\mathcal{O}$.

Let $E = E(G')$ and $V = V(G')$, be the edge and vertex sets of G'. Let $\mathbb{R}_{>0}^{E-e} \times \mathbb{R}_{\geq 0}$ be the space of edge weights $\{x_f\}_{f\in E(G')}$ such that $x_f > 0$, for $f \neq e$, and $x_e \geq 0$. Then

$$(\mathbb{R}_{>0}^{E-e} \times \mathbb{R}_{\geq 0})/\{\text{gauge transformations}\} \simeq \mathbb{R}_{>0}^{|E|-|V|-1} \times \mathbb{R}_{\geq 0}.$$

Corollary 18.11. *The map $\overline{Meas}_{G'} : \mathbb{R}_{\geq 0}^{E} \to \overline{S_{\mathcal{M}}^{\mathrm{tnn}}}$ induces the bijective map*

$$\mathbb{R}_{>0}^{|E|-|V|-1} \times \mathbb{R}_{\geq 0} \to \overline{S_{\mathcal{M}}^{\mathrm{tnn}}}.$$

The restriction of this map to the subset given $x_e > 0$ is a parametrization of $S_{\mathcal{M}}^{\mathrm{tnn}}$ and the restriction of this map to the subset $x_e = 0$ is a parametrization of $S_{\mathcal{M}'}^{\mathrm{tnn}}$.

19. ⅃-diagrams and Bruhat intervals

Rietsch constructed cellular decomposition of the totally nonnegative part of G/P; see [Rie1, Rie2, MR]. In this section we show that Rietsch's cells are in one-to-one correspondence with the cells $S_{\mathcal{M}}^{\mathrm{tnn}}$.

Let us fix the pair (k, n) as before. Recall that a permutation $w \in S_n$ is called *Grassmannian* if w is the minimal length representative of a left coset $(S_k \times S_{n-k})\backslash S_n$. In other words, w is a Grassmannian permutation if $w(1) < w(2) < \cdots < w(k)$ and $w(k+1) < w(k+1) < \cdots < w(n)$.

In case of the Grassmannian Gr_{kn}, Rietsch's cells are labeled by pairs of permutations (u, w) in $W = S_n$ such that $u \leq w$ in the Bruhat order on W and w is a Grassmannian permutation. Let us construct a bijection between such pairs and ⅃-diagrams.

Grassmannian permutations are in bijection with partitions $\lambda \subseteq (n-k)^k$. This bijection can be described as follows. Rotate the Young diagram of shape λ by 45° counterclockwise and draw the wiring diagram with k wires going along the rows of λ and $(n-k)$ wires going along the columns of λ. Label both ends of the wires by numbers $1, \dots, n$ starting from the bottom. This wiring diagram represents a permutation w_λ such that the wires connect the indices i on the left with the $w_\lambda(i)$ on the right; see Figure 19.1. Using our earlier notation, we can describe this permutation as $w_\lambda = (\tilde{i}_k, \tilde{i}_{k-1}, \dots, \tilde{i}_1, \tilde{j}_{n-k}, \tilde{j}_{n-k-1}, \dots, \tilde{j}_1)$, where $I(\lambda) = \{i_1 < \cdots < i_k\}$, $[n] \setminus I(\lambda) = \{j_1 < \cdots < j_{n-k}\}$, and $\tilde{i} := n+1-i$; see Section 2. It is clear that the length $\ell(w_\lambda)$ (the number of inversions) equals $|\lambda|$.

Let D be a ⅃-diagram of shape λ. Again rotate it by 45° counterclockwise.[2] For each box of λ filled with a 1 in D (shown by a dot), we replace the corresponding

[2]After rotation, it should be called >-diagram.

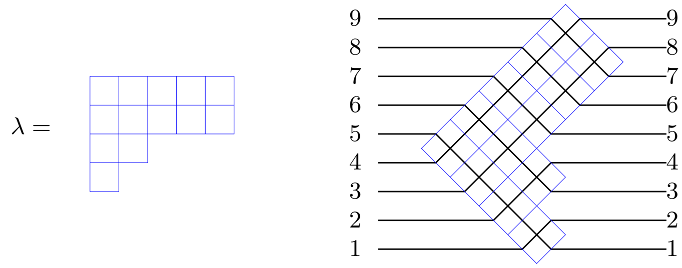


FIGURE 19.1. A Young diagram $\lambda = (5, 5, 2, 1)$ and the corresponding Grassmannian permutation $w_\lambda = 2\,4\,8\,9\,1\,3\,5\,6\,7$ for $(k, n) = (4, 9)$

crossing in the wiring diagram of w_λ by an uncrossing, as shown on Figure 19.2. The obtained "pipe dream" is a wiring diagram of a certain permutation in S_n, which we denote by $u_D \in S_n$.

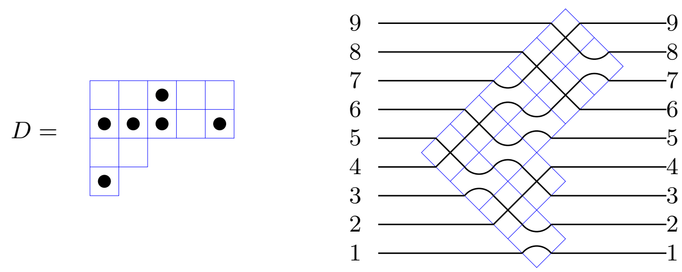


FIGURE 19.2. A ⅃-diagram D and the corresponding permutation $u_D = 1\,4\,2\,7\,3\,5\,9\,6\,8$

Theorem 19.1. *The map $D \mapsto u_D$ is a bijection between ⅃-diagrams of shape λ and permutations $u \in S_n$ such that $u \leq w_\lambda$ in the Bruhat order on S_n. The number of 1's in D equals $\ell(w_\lambda) - \ell(u_D)$.*

Lemma 19.2. Marsh-Rietsch [MR, Lemma 3.5] *Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced decomposition of a Weyl group element $w \in W$. Then, for any u such that $u \leq w$ in the Bruhat order, there is unique subset $J = \{j_1 < \cdots < j_s\} \subseteq [l]$ such that $u = s_{i_{j_1}} \cdots s_{i_{j_s}}$ and $(s'_{i_1} s'_{i_2} \cdots s'_{i_a})\, s_{i_{a+1}} > s'_{i_1} s'_{i_2} \cdots s'_{i_a}$, for all $a \in [l-1]$, where $s'_{i_j} = s_{i_j}$ for $j \in J$ and $s'_{i_j} = 1$ otherwise.*

Clearly, $u = s_{i_{j_1}} \cdots s_{i_{j_s}}$ should be a reduced decomposition.

It is well-known that for any $u \leq w$ there is a subword in a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$ which gives a reduced decomposition of u; see [Hum]. However, there are usually many such subwords. The above lemma describes one distinguished subword for each $u \leq w$. Remark that the subset J described in the Lemma 19.2 is exactly the *lexicographically minimal* subset in $[l]$ that produces a reduced decomposition of u.

In type A case, the condition of Lemma 19.2 have a simple combinatorial description. A reduced decomposition of w corresponds to a wiring diagram $\mathcal{W}$ where the wires are not allowed to cross more than once. Subwords in the reduced decomposition correspond to diagrams $\mathcal{W}'$ obtained from $\mathcal{W}$ by replacing some crossings of with uncrossings (like we did above for w_λ). The condition of Lemma 19.2 says that once two wires in $\mathcal{W}'$ intersect with each other they can never intersect or even *touch* each other again. Here "touch" means that the two wires participate in the same uncrossing of $\mathcal{W}'$.

For example, this condition fails for the wiring diagram shown on Figure 19.2. Indeed, the two wires whose left ends are labeled by 4 and 6 intersect each other and then arrive to the same uncrossing. However, if we take the mirror image of the condition (with respect to the vertical axis) then we will get exactly what we need.

Let $\mathrm{MR}(\lambda)$ be the set of pipe dreams obtained from the wiring diagram of w_λ by replacing some crossings with uncrossings so that the following conditions hold:

(1) Two wires can intersect at most once.
(2) If two wires do intersect at point P, then they cannot participate in the same uncrossing to the left of P.

The set $\mathrm{MR}(\lambda)$ corresponds to the subwords in a reduced decomposition of w_λ that satisfy the (mirror image) of Marsh-Rietsch's condition.

Lemma 19.3. *The set* $\mathrm{MR}(\lambda)$ *is exactly the set of pipe dreams coming from* ⅃*-diagrams, as described above.*

Proof. Recall that ⅃-diagrams D can be described as follows. For any box x filled with a 0 in D, either all boxes above x or all boxes below x are filled with 0's. Let us translate this ⅃-condition in the language of pipe dreams. It says that, for any crossing of two wires at point P, at least one of these wires does not participate in any uncrossing before P. So that one of these wires goes directly from the boundary of λ (from the North-West or from the South-West) to the intersection point P without making any turns. Clearly, such pipe dreams belong to the set $\mathrm{MR}(\lambda)$. Let us now show the opposite inclusion. Suppose that the ⅃-condition fails at some point P. That means that both wires intersecting at P diverge from the straight course before P. Let A and B be the points where the wires make the last turns before arriving to P; see Figure 19.3. Let us consider the rectangle

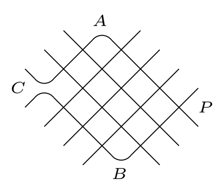


FIGURE 19.3. A failed ⅃-condition implies a failed MR-condition

R with the vertices A, B, P and another vertex C. Let us assume that our failed ⅃-condition was chosen so that the size of the rectangle R (say, its perimeter) is a small as possible. Then any other wire that intersects with the side AP or BP of R cannot make any turn in the rectangle R. Indeed, if it makes a turn then it gives another failed ⅃-condition with a smaller rectangle. Thus all other wires go straight through the rectangle R as shown on Figure 19.3. Thus means that the two wires that intersect at P should either intersect or touch each other at C. This means that the condition describing the set $\mathrm{MR}(\lambda)$ also fails. □

This proves Theorem 19.1 and shows that our cells are in bijection with Rietsch's cells. This implies Theorem 3.8 saying that these two cell decompositions coincide.

20. From Ⅎ-diagrams to decorated permutations (and back)

The cells $S_{\mathcal{M}}^{\mathrm{tnn}}$ are in bijection with Ⅎ-diagrams (Theorem 6.5) and also in bijection with decorated permutations (Theorem 17.1). In this section we discuss the induced bijection between Ⅎ-diagrams and decorated permutations.

Let us pick a Ⅎ-diagram D of shape $\lambda \subseteq (n-k)^k$, transform it into Γ-graph G_D^Γ, and then transform this graph into a plabic graph G_D^{plabic} (see Section 9). In other words, we need replace 4-valent vertices of G_D^Γ by pairs of trivalent vertices and color the vertices as shown on Figure 20.1. Theorems 6.5 and 17.1, and Proposition 16.4 imply that the corresponding decorated permutation is the decorated trip permutation of the obtained graph.

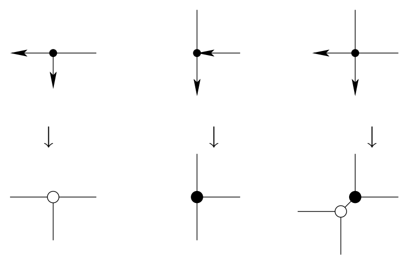


Figure 20.1. Transforming Γ-graphs into plabic graphs

Corollary 20.1. *The map $D \mapsto \pi^{:}(G_D^{plabic})$ is a bijection between Ⅎ-diagrams of shape $\lambda \subseteq (n-k)^k$ and decorated permutations with the anti-exceedance set $I(\lambda)$.*

The rules of the road for plabic networks (see Figure 13.1) translate into the rules of the road for Γ-graphs shown on Figure 20.2. So the decorated permutation $\pi^{:}$ corresponding to a Ⅎ-diagram D can be described as follows. The empty rows (columns) or D correspond to white (black) fixed points of $\pi^{:}$. For other entries we need to follow trips in the graph Γ-graph G_D^Γ. Let us trace *backwards* the trip that ends at a boundary vertex b_i located on a vertical segment of the boundary. We need to go from b_i all the way to the left until we hit a Γ-turn or a ⊢-fork (the first and the third segments shown on Figure 20.2), then turn down and go until the first junction, then turn right and go until the first junction, then turn down, then right, etc. We need to keep going in this zig-zag fashion until we hit the boundary at a boundary vertex b_j. Then we should have $\pi(j) = i$ in the corresponding permutation. The rule for trips that end on a horizontal segment of the boundary is the symmetric to the above rule (with respect to the axis $x + y = 0$).

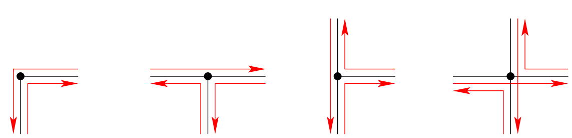


Figure 20.2. Rules of the road for Γ-graphs

The above reformulation of the bijection $D \mapsto \pi^{:}(G_D^{plabic})$ in terms of zig-zag paths was found by Steingrimsson-Williams [SW]. They used it to investigated nice properties of this bijection with respect to various statistics on permutations and ⅃-diagrams.

Let us give a simpler description of the bijection between ⅃-diagrams and decorated permutations using the bijection $D \mapsto u_D$ constructed in Section 19.

Let $w_\lambda = (w_1, \dots, w_n) = (\tilde{i}_k, \tilde{i}_{k-1}, \dots, \tilde{i}_1, \tilde{j}_{n-k}, \tilde{j}_{n-k-1}, \dots, \tilde{j}_1)$ be the Grassmannian permutation associated with $\lambda \subseteq (n-k)^k$, where $I(\lambda) = \{i_1 < \cdots < i_k\}$, $[n] \setminus I(\lambda) = \{j_1 < \cdots < j_{n-k}\}$, and $\tilde{i} := n + 1 - i$.

Lemma 20.2. *For a permutation $u \in S_n$, we have $u \leq w_\lambda$ in the Bruhat order if and only if $u_1 \leq w_1, \dots, u_k \leq w_k$ and $u_{k+1} \geq w_{k+1}, \dots, u_n \geq w_n$.*

Proof. According to the well-known description of the Bruhat order on S_n, we have $u \leq w_\lambda$ if and only if $\{i \in [a] \mid u(i) \in [b]\} \leq \{i \in [a] \mid w_\lambda(i) \in [b]\}$, for any $a, b \in [n]$. This translates into the needed inequalities. □

Let us now describe the map $u \mapsto \pi^{:} = \pi^{:}(u)$ from permutations $u \leq w_\lambda$ to decorated permutations $\pi^{:}$ with the anti-exceedance set $I(\lambda)$. It is given by $\pi^{-1}(i_1) = \tilde{u}_k$, …, $\pi^{-1}(i_k) = \tilde{u}_1$ (with fixed points colored in white), and $\pi^{-1}(j_1) = \tilde{u}_n$, …, $\pi^{-1}(j_{n-k}) = \tilde{u}_{k+1}$ (with fixed points colored in black).

Theorem 20.3. *The map $u \mapsto \pi^{:}(u)$ is a bijection between the Bruhat interval $\{u \mid u \leq w_\lambda\}$ and decorated permutations with anti-exceedance set $I(\lambda)$. The map $D \mapsto \pi^{:}(u_D)$ is the bijection between ⅃-diagrams and decorated permutations, which coincides with the above map $D \mapsto \pi^{:}(G_D^{plabic})$.*

Proof. The first claim follows directly from Lemma 20.2. The second claim is obtained by comparing the constructions of u_D and $\pi^{:}(G_D^{plabic})$. □

21. Cluster parametrization and chamber anzatz

22. Berenstein-Zelevinsky's string polytopes

23. Enumeration of nonnegative Grassmann cells

Let N_{kn} be the number of totally nonnegative cells in the Grassmannian Gr_{kn}. Recall that the *Eulerian number* A_{kn} is the number of permutations $w \in S_n$ such that w has $k-1$ descents: $\#\{1 \in [n-1] \mid w(i) > w(i+1)\} = k - 1$.

Proposition 23.1. *The numbers N_{kn} are related to the Eulerian numbers by $N_{kn} = \sum_{r=0}^{n} \binom{n}{r} A_{k,n-r}$. Their generating function is*

$$1 + \sum_{n \geq 1,\, 0 \leq k \leq n} x^k \frac{y^n}{n!} N_{kn} = e^{xy} \frac{x-1}{x - e^{y(x-1)}}.$$

Proof. According to Theorem 17.1, N_{kn} is the number of decorated permutations $\pi^{:}$ of size n with k anti-exceedances. If we remove black fixed points from $\pi^{:}$ we get a usual permutation with k anti-exceedances. It is well-know that the Eulerian number count such permutations. The second claim is obtained from the known generating function for the Eulerian numbers. □

The numbers N_{kn} appear in Sloan's On-Line Encyclopedia of Integer Sequences [Slo] with ID number A046802.

Let $N_n = \sum_{k=0}^{n} N_{kn}$ be the total number of decorated permutations of size n.

$n\backslash k$	0	1	2	3	4	5	6	$\cdots$
0	1							
1	1	1						
2	1	3	1					
3	1	7	7	1				
4	1	15	33	15	1			
5	1	31	131	131	31	1		
6	1	63	473	883	473	63	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

FIGURE 23.1. The numbers N_{kn} of nonnegative cells in Gr_{kn}^{tnn}

Proposition 23.2. *The numbers N_n satisfy the recurrence relation $N_n = n \cdot N_{n-1} + 1$, $N_0 = 1$. The exponential generating function for these numbers is $\sum_{n \geq 0} N_n \frac{x^n}{n!} = e^x/(1-x)$.*

The sequence N_n appears in [Slo] with ID number A000522.

Let $N_{kn}(q) = \sum q^{\dim S_{\mathcal{M}}^{\mathrm{tnn}}}$ be the generating function for the nonnegative cells $S_{\mathcal{M}}^{\mathrm{tnn}} \subset Gr_{kn}^{\mathrm{tnn}}$ counted according to their dimension. By Theorem 6.5, we have

$$N_{kn}(q) = \sum_D q^{|D|},$$

where the sum is over ⅃-diagrams whose shape fits inside the rectangle $(n-k)^k$, and $|D|$ denotes the number of 1's in the diagram.

Williams gave a formula for the polynomials $N_{kn}(q)$ by counting ⅃-diagrams.

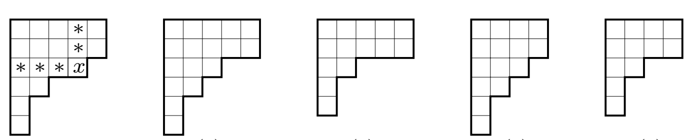
Theorem 23.3. Williams [W1, Theorem 4.1] *We have*

$$N_{kn}(q) = \sum_{i=1}^{k-1} \binom{n}{i} q^{-(k-i)^2} \left([i-k]_q^i \, [k-i+1]_q^{n-i} - [i-k+1]_q^i \, [k-i]_q^{n-i} \right),$$

where $[i]_q := \frac{1-q^i}{1-q}$.

For a partition λ, let $F_\lambda(q) = \sum_D q^{|D|}$, where the sum is over ⅃-diagrams D of shape λ.

The approach of [W1] is based on the following simple recurrence for the polynomials $F_\lambda(q)$. Let us pick a corner box x of the Young diagram of shape λ. Let $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$, and $\lambda^{(4)}$ be the Young diagrams obtained from λ by removing, respectively, the box x, the row containg x, the column containg x, the column and the row containing x; see Figure 23.2.



λ $\lambda^{(1)}$ $\lambda^{(2)}$ $\lambda^{(3)}$ $\lambda^{(4)}$

FIGURE 23.2. $F_\lambda = q\, F_{\lambda^{(1)}} + F_{\lambda^{(2)}} + F_{\lambda^{(3)}} - F_{\lambda^{(4)}}$

Lemma 23.4. [W1, Remark 4.2] $F_\lambda(q) = q\,F_{\lambda^{(1)}}(q) + F_{\lambda^{(2)}}(q) + F_{\lambda^{(3)}}(q) - F_{\lambda^{(4)}}(q)$.

([W1] formulated the case when x is in the last row of λ.)

Proof. The sum $\sum q^{|D|}$ over ⅃-diagrams of shape λ such that the box x is filled with a 1 equals $q\,F_{\lambda^{(1)}}(q)$. If D is a ⅃-diagram such that the box x is filled with a 0 then either all entries of D in the row of x are zeros, or all entries of D in the column of x are zeros. By the inclusion-exclusion principle, the sum over such ⅃-diagrams equals $F_{\lambda^{(2)}}(q) + F_{\lambda^{(3)}}(q) - F_{\lambda^{(4)}}(q)$. □

Steingrimsson-Williams [SW] studied various statistics on ⅃-diagrams. Corteel-Williams [CW] investigated the relationship between ⅃-diagrams and the asymmetric exclusion process.

24. Misce⅃⅃aneous

Recall that a *rook placement* on some board is a way to place rooks so that no rook attacks another rook.

Define a *teuton* as a chesspiece that can move only downwards and to the right. In order to attack a piece, two teutons need to simultaneously charge from two different directions in a wedge-shaped formation.[3] A *teuton placement* is a placement of several teutons on a board such that no pair of teutons can attack another teuton. In other words, a teuton placement is a subset S (possibly empty) of boxes on some board (say, a Young diagram) such that for any three boxes a, b, c in a ⅃-shaped pattern (as on Figure 6.1), if $a, c \in S$ then $b \notin S$.

For $\lambda \subseteq (n-k)^k$, let $\{i_1 < \cdots < i_k\} = I(\lambda)$, $\{j_1 < \cdots < j_{n-k}\} = [n] \setminus I(\lambda)$. Define the skew shape $\kappa_\lambda := (n^{n-k}, \tilde{i}_1, \dots, \tilde{i}_k)/(\tilde{j}_1 - 1, \dots, \tilde{j}_{n-k} - 1)$, where $\tilde{i} := n + 1 - i$.

For a permutation $w \in S_n$, the *inversion graph* G_w is the graph on the vertex set $[n]$ with edges (i, j) for any inversion $i < j$, $w(i) > w(j)$. Let A_w be the hyperplane arrangement in $\mathbb{R}^n$ that consists of the hyperplanes $x_i - x_j = 0$, for all inversions $i < j$, $w(i) > w(j)$.

Remind that $w_\lambda \in S_n$ is the Grassmannian permutation associated with a partition $\lambda \subseteq (n-k)^k$; see Section 19.

Theorem 24.1. *For a partition $\lambda \subseteq (n-k)^k$, the following numbers are equals:*

1. *The number of nonnegative cells $S_{\mathcal{M}}^{\mathrm{tnn}}$ inside the Schubert cell Ω_λ.*
2. *The number of ⅃-diagrams of shape λ.*
3. *The number of decorated permutations of size n with anti-exceedance set $I(\lambda)$.*
4. *The number of Grassmann necklaces $(I_1, \dots, I_n)$ with $I_1 = I(\lambda)$.*
5. *The number of permutations in the Bruhat interval $\{u \in S_n \mid u \leq w_\lambda\}$.*
6. *The number of teuton placements on the Young diagram of shape λ*
7. *The number of rook placements with n rooks on the skew Young diagram of shape κ_λ.*
8. *The number of acyclic orientations of the graph G_{w_λ}.*
9. *The number of regions of the hyperplane arrangement A_{w_λ}.*

[3] According to Tacitus, the ancient Teutons arranged their forces in the form of a wedge (*Germania,* 6). Later this wedge-shape phalanx was known as *svínfylking.*

The equality of (1), (2), (3), (4), and (5) is already proved bijectively in Theorems 6.5, 17.1, and 20.3. Teuton placements on the Young diagram of shape λ are in a simple bijection with Ⅎ-diagrams. Indeed, for a Ⅎ-diagram D place a teuton in each box x filled with a 1 such that all boxes above x or all boxes to the left of x are filled with 0's. This proves that the numbers (2) and (6) are equal. According to Lemma 20.2, permutations u in the Bruhat interval $\{u \mid u \leq w_\lambda\}$ are described by certain inequalities. These inequalities mean that the corresponding rook placement $\{(n+1-i, u(i)) \mid i \in [n]\}$ belongs to the skew shape κ_λ. This proves that the numbers (5) and (7) are equal. On the other hand, it is well-know (and trivial) that the number of regions of a graphical arrangement equals the number of acyclic orientations of the corresponding graph, which proves the equality (8) and (9). It remains to show that, say, the number (2) of Ⅎ-diagrams is equal to the number (8) of acyclic orientations of the graph G_{w_λ}.

For a partition $\lambda \subseteq (n-k)^k$, the graph $G_\lambda := G_{w_\lambda} \subseteq K_{k,n-k}$ is the bipartite graph on the vertices $1, \dots, k$ and $1', \dots, (n-k)'$ with edges (i, j') corresponding to boxes (i, j) of the Young diagram of shape λ.

Let $\chi_\lambda(t)$ be the *chromatic polynomial* of the graph G_λ. By the definition, the value of the polynomial $\chi_\lambda(t)$ at a positive integer t equals the number of ways to color vertices of G_λ with colors $\{1, \dots, t\}$ so that any two vertices connected by an edge have different colors. These are called *t-colorings.*

Lemma 24.2. *Pick a corner box x of the Young diagram of shape λ, and let $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$, $\lambda^{(4)}$ be the partitions as in Lemma 23.4. Then we have*

$$\chi_\lambda(t) = \chi_{\lambda^{(1)}}(t) - t^{-1}(\chi_{\lambda^{(2)}}(t) + \chi_{\lambda^{(3)}}(t) - \chi_{\lambda^{(4)}}(t)).$$

Proof. It is enough to prove the identity for a positive integer t. Suppose that $x = (i, j)$, i.e., the corner box x lies in the ith row and the jth column of λ. Then the graph G_λ contains the edge (i, j'). The vertex i is incident to the edges $(i, 1), (i, 2), \dots, (i, j')$ and the vertex the vertex j' is incident to the edges $(1, j'), (2, j'), \dots, (i, j')$. The graph $G_{\lambda^{(1)}}$ is obtained from G_λ by removing the edge (i, j'). Thus $\chi_{\lambda^{(1)}}(t) - \chi_\lambda(t)$ equals the number of t-coloring of $G_{\lambda^{(1)}}$ such that the vertices i and j' have the same color. Let $\tilde{G}_{\lambda^{(2)}}$, $\tilde{G}_{\lambda^{(3)}}$, $\tilde{G}_{\lambda^{(4)}}$ be the graphs obtained from G_λ by removing, respectively, all edges incident to the vertex i, all edges incident to the vertex j', all edges incident to i or j'. Then, for $r = 2, 3, 4$, the graph $\tilde{G}_{\lambda^{(r)}}$ is isomorphic to $G_{\lambda^{(r)}}$. Pick any t-coloring C of the graph $\tilde{G}_{\lambda^{(4)}}$ minus the vertices i and j'. Suppose that the vertices $1, \dots, i-1$ have c_1 different colors and the vertices $1', \dots, (j-1)'$ have c_2 different colors in this coloring C. Note that the graph $\tilde{G}_{\lambda^{(4)}}$ contains the complete bipartite graph $K_{i-1,j-1}$ on the vertices $1, \dots, i-1$ and $1', \dots, (j-1)'$. Thus all colors of the vertices $1, \dots, i-1$ must be different from the colors of the vertices $1', \dots, (j-1)'$. It follows that the vertices $1, \dots, i-1, 1', \dots, (j-1)'$ have c_1+c_2 different colors. Clearly, the number of ways to extend C to a t-coloring of $\tilde{G}_{\lambda^{(4)}}$ is t^2, because there are t^2 ways to pick colors of isolated vertices i and j'. The number of ways to extend C to a t-coloring of the graph $\tilde{G}_{\lambda^{(2)}}$ equals $t(t-c_1)$. Indeed, there are t ways to color the isolated vertex i and $t-c_1$ ways to color the vertex j' in a color which is different from all colors of adjacent vertices $1, \dots, i-1$. Similarly, there are $t(t-c_2)$ ways to extend C to a t-coloring of $\tilde{G}_{\lambda^{(3)}}$. Also there are $t - c_1 - c_2$ ways to extend C to a coloring of $G_{\lambda^{(1)}}$ where the vertices i and j' have the same color. Indeed, there are $t - c_1 - c_2$ ways to pick a color of these vertices, which should be different from colors of all adjacent vertices

$1,\dots,i-1,1',\dots,(j-1)'$. Now the identity, $t(t-c_1-c_2)=t(t-c_1)+t(t-c_2)-t^2$ implies that $t(\chi_{\lambda^{(1)}}(t)-\chi_\lambda(t))=\chi_{\lambda^{(2)}}(t)+\chi_{\lambda^{(3)}}(t)-\chi_{\lambda^{(4)}}(t)$, as needed. □

We can now finish the proof of Theorem 24.1. According to the well-known Stanley's result [Sta], the value $(-1)^n\chi_\lambda(-1)$ equals the number ao_λ of acyclic orientations of the graph G_λ. Specializing Lemma 24.2 at $t=-1$, we obtain $\mathrm{ao}_\lambda=\mathrm{ao}_{\lambda^{(1)}}+\mathrm{ao}_{\lambda^{(2)}}+\mathrm{ao}_{\lambda^{(3)}}-\mathrm{ao}_{\lambda^{(4)}}$. This is exactly the same as the $q=1$ specialization of the recurrence relation for $F_\lambda(q)$ given in Lemma 23.4. The initial condition $F_\emptyset(1)=\mathrm{ao}_\emptyset=1$ is trivial. Thus $F_\lambda(1)=\mathrm{ao}_\lambda$, for any λ, as needed.

Remark 24.3. The similarity of the recurrence relations in Lemmas 23.4 and 24.2 suggests that the polynomial $F_\lambda(q)$ and the chromatic polynomial $\chi_\lambda(t)$ might have a common generalization. It would be interesting to investigate the relation between these polynomials. A first guess is that the polynomial $F_\lambda(q)$ a certain specialization of the *Tutte dichromatic polynomial* of the graph G_λ. However, we did not find an exact relationship with the Tutte polynomial.

The polynomials $F_\lambda(q)$ and $\chi_\lambda(t)$ have natural generalizations for any permutation $w\in S_n$.

Let $F_w(q):=\sum_{u\leq w}q^{\ell(w)-\ell(u)}$, where the sum is over permutations u less than or equal to w in the Bruhat order. Then $F_\lambda(q)=F_{w_\lambda}(q)$; see Theorem 19.1. Also let $\chi_w(t)$ be the chromatic polynomial of the inversion graph G_w. The polynomial $q^{\ell(w)}F_w(q^{-1})$ is the Poincaré polynomial of the complex *Schubert variety* $X_w:=\overline{BwB/B}$ in $SL(n)/B$. On the other hand, the polynomial $(-t)^n\chi_w(-t^{-1})$ is the Poincaré polynomial of the complement to the complexified hyperplane arrangement A_w, or equivalently, the Hilbert polynomial of the *Orlik-Solomon algebra* of this arrangement. For example, for the longest permutation $w_\circ\in S_n$, we have $F_{w_\circ}(q)=(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$ and $\chi_{w_\circ}(t)=t(1-t)(2-t)\cdots(n-1-t)$.

Let us say that a permutation $w\in S_n$ is *chromobruhatic* if $F_w(1)=(-1)^n\chi_w(-1)$, that is the number elements in interval $[1,w]$ in the Bruhat order equals the number of acyclic orientations of G_w. According to Theorem 24.1, all Grassmannian permutations w_λ are chromobruhatic. It is also clear that the longest permutation $w_\circ\in S_n$ is chromobruhatic.

For two permutations $w\in S_n$ and $\sigma\in S_k$, we say that w *avoids the pattern* σ if there is no subset $\{i_1<\cdots<i_k\}\subseteq[n]$ such that $w(i_1),\dots,w(i_k)$ are in the same relative order as $\sigma(1),\dots,\sigma(k)$.

Conjecture 24.4. (1) *A permutation w is chromobruhatic if and only if it avoids the four patterns* 4231*,* 35142*,* 42513*,* 351624*.*

(2) *For any permutation $w\in S_n$, we have $(-1)^n\chi_w(-1)\leq F_w(1)$.*

We have verified this conjecture on a computer for $n\leq 8$. The numbers of chromobruhatic permutations in S_n for $n=1,\dots,8$ are $1,2,6,23,101,477,2343,11762$.

It would be interesting to find a common generalization of the polynomials $F_w(q)$ and $\chi_w(t)$ for chromobruhatic permutations w.

Remark 24.5. The conjecture implies that all 321-avoiding permutations are chromobruhatic, all 231-avoiding permutations are chromobruhatic, and all 312-avoiding permutations are chromobruhatic. Indeed, each of the four patterns in the conjecture contains the patterns 321, 231, and 312.

Remark 24.6. According to Lakshmibai-Sandhya's criterion [LS], a Schubert variety X_w is *smooth* if and only if the permutation w avoids the patterns 3412 and 4231. Also, according to [Car], X_w is smooth if and only if its Poincaré polynomial is palindromic: $F_w(q) = q^{\ell(w)} F_w(q^{-1})$. Conjecture 24.4 implies that all permutations w with smooth Schubert varieties X_w are chromobruhatic. Indeed, each of the four patterns in the conjecture contains one of the two patterns from Lakshmibai-Sandhya's criterion. It would be interesting to elucidate a relationship between chromobruhatic permutations w and the singular locus of the Schubert variety X_w.

Let us calculate the number of ⅃-diagrams of the rectangular shape $\lambda = (n-k)^k$. The *Stirling number of the second kind* $S(k,c)$ is the number of ways to subdivide the set $[k]$ into a disjoint union of c nonempty subsets.

Proposition 24.7. *The number of ⅃-diagrams of shape $(n-k)^k$ equals*

$$F_{(n-k)^k}(1) = \sum_{c=1}^{k} (-1)^{k-c} (c+1)^{n-k}\, c!\, S(k,c).$$

This number equals the number placements of n rooks on the $n \times n$ board with two triangular shapes of sizes $k-1$ and $n-k-1$ removed from the opposite corners of the board.

Proof. The second claim follows directly from parts (2) and (7) of Theorem 24.1. In this case, the graph G_{w_λ} is the complete bipartite graph $K_{k,n-k}$. Let us calculate its chromatic polynomial $\chi_{K_{k,n-k}}(t)$. The number of t-coloring of $K_{k,n-k}$ such that the k vertices in the first part have c different colors equals $\binom{t}{c}(t-c)^{n-k}\, c!\, S(k,c)$. Indeed, there are $\binom{t}{c}$ ways to pick a subset of c colors, there are $c!\, S(k,c)$ surjective maps from $[k]$ to the selected set of c colors, and there are $(t-c)^{n-k}$ ways to pick colors of vertices in the second part different from the selected c colors. Thus the chromatic polynomial $\chi_{K_{k,n-k}}(t)$ equals

$$\sum_{c=1}^{k} \binom{t}{c} (t-c)^{n-k}\, c!\, S(k,c) = \sum_{c=1}^{k} t(t-1)(t-2)\cdots(t-c+1)\,(t-c)^{n-k}\, S(k,n).$$

Plugging in $t=-1$ and using Theorem 24.1(8), we get the needed expression for $F_{(n-k)^k}(1) = (-1)^n \chi_{K_{k,n-k}}(-1)$. □

Theorem 24.8. *The number of ⅃-diagrams D of the triangular shape $\lambda = (n, n-1, \dots, 1)$ such that D contains no 1's in the n corner boxes equals $n!$. The bijection from this set of diagrams to permutations is constructed as follows. For $i \in [n]$, let $a_{i1} > a_{i2} > \cdots > a_{i,k_i}$ be the positions of 1's in the i-th column of D. The the permutation $w \in S_n$ corresponding to D is given by the following product of cycles:*

$$w = (n, a_{11}, a_{12}, \dots, a_{1,k_1})(n-1, a_{21}, a_{22}, \dots)(n-2, a_{31}, a_{32}, \dots)\cdots(1).$$

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