Symmetries of Gromov-Witten Invariants

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Abstract

The group $(\mathbb{Z}/n\mathbb{Z})^2$ is shown to act on the Gromov-Witten invariants of the complex flag manifold. We also deduce several corollaries of this result.

1 Introduction

The aim of this paper is to present certain symmetry properties of the Gromov-Witten invariants for type A complex flag manifolds.

Recall that the cohomology ring of the complex flag manifold Fl_n has an additive basis of Schubert classes σ_w, which are indexed by permutations w in the symmetric group S_n. For permutations $u, v, w \in S_n$, the Schubert number $c_{u,v,w}$ is the structure constant of the cohomology ring in the basis of Schubert classes:

$$\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c_{u,v,w}\, \sigma_{w_\circ w}\,,$$

where $w_\circ$ is the longest permutation in S_n. Equivalently,

$$c_{u,v,w} = \int \sigma_u \cdot \sigma_u \cdot \sigma_w$$

is the intersection number of Schubert varieties. Thus these numbers are nonnegative integers symmetric in u, v, and w. They generalize the famous Littlewood-Richardson coefficients. If $\ell(u)+\ell(v)+\ell(w) \neq \frac{n(n-1)}{2}$ then the Schubert number $c_{u,v,w}$ is zero for an obvious degree reason.

A long standing open problem is to find an algebraic or combinatorial construction for the coefficients $c_{u,v,w}$ that would imply their nonnegativity. A possible approach to this problem could be in its generalization to the quantum cohomology ring of the flag manifold Fl_n. The structure constants of this ring are certain polynomials whose coefficients are the Gromov-Witten invariants $\langle \sigma_u, \sigma_v, \sigma_w \rangle_{(d_1,\dots,d_{n-1})}$ The Schubert number $c_{u,v,w}$ is a special case of the Gromov-Witten invariants: $c_{u,v,w} = \langle \sigma_u, \sigma_v, \sigma_w \rangle_{(0,\dots,0)}$. These invariants are defined as numbers of certain rational curves in Fl_n. The geometric definition of the Gromov-Witten invariants implies their nonnegativity.

In this paper we establish cyclic symmetries of the Gromov-Witten invariants that could not be detected in their full generality on the "classical" level of the Schubert numbers $c_{u,v,w}$. Several related results for the $c_{u,v,w}$ when u is a Grassmannian permutation were, however, found by Bergeron and Sottile, see [2, Theorems 1.3.4, 1.3.4]. In case of the Gromov-Witten invariants we do not need to restrict the rule to Grassmannian permutations. Similar symmetries of the Gromov-Witten invariants for Grassmannian varieties were found in [1].

2 Gromov-Witten invariants

Let Fl_n denote the manifold of complete flags of subspaces in the complex n-dimensional linear space $\mathbb{C}^n$. One can also define the *flag manifold* as $Fl_n = GL_n(\mathbb{C})/B$, where B is the Borel subgroup of upper triangular matrices in the general linear group. The flag manifold is a compact smooth complex manifold. For a permutation $w \in S_n$, the *Schubert variety* X_w is the closure of the *Schubert cell* $B_- wB/B$ in Fl_n, where B_- is the subgroup of lower triangular matrices and w is viewed as a permutation matrix in GL_n. The *Schubert classes* $\sigma_w \in \mathrm{H}^*(Fl_n, \mathbb{Z})$, indexed by permutations $w \in S_n$, are defined as the Poincaré duals of the homology classes $[X_w]$ of Schubert manifolds. They form an additive $\mathbb{Z}$-basis of the cohomology ring $\mathrm{H}^*(Fl_n, \mathbb{Z})$. Moreover, $\sigma_w \in \mathrm{H}^{2l}(Fl_n, \mathbb{Z})$, where $l = \ell(w)$ is the *length* of permutation w, i.e., its number of inversions.

Recently, attention has been drawn to the (small) *quantum cohomology ring* $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ of the flag manifold. The definition of quantum cohomology can be found, for example, in [5]. Here we briefly outline several notions and results.

As a vector space, the quantum cohomology of Fl_n is the usual cohomology tensored with the polynomial ring in $n-1$ variables:

$$\mathrm{QH}^*(Fl_n, \mathbb{Z}) \cong \mathrm{H}^*(Fl_n, \mathbb{Z}) \otimes \mathbb{Z}[q_1, \ldots, q_{n-1}]. \tag{1}$$

The Schubert classes σ_w, thus, form a $\mathbb{Z}[q_1, \ldots, q_{n-1}]$-basis of the quantum cohomology ring.

The multiplication in $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ (quantum product) is a commutative $\mathbb{Z}[q_1, \ldots, q_{n-1}]$-linear operation. It is therefore sufficient to specify the quantum product of any two Schubert classes. To avoid confusion with the multiplication in the usual cohomology ring, we will use "$*$" to denote the quantum product. The quantum product $\sigma_u * \sigma_v$ of two Schubert classes can be expressed in the basis of the Schubert classes as

$$\sigma_u * \sigma_v = \sum_{w \in S_n} C_{u,v,w}\, \sigma_{w_\circ w}\,, \tag{2}$$

where $C_{u,v,w} \in \mathbb{Z}[q_1, \ldots, q_{n-1}]$ and $w_\circ = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$ is the longest permutation in S_n.

The coefficient of $q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ in the polynomial $C_{u,v,w}$ is the *Gromov-Witten invariant* $\langle \sigma_u, \sigma_v, \sigma_w \rangle_{(d_1,\ldots,d_{n-1})}$. The Gromov-Witten invariants are defined geometrically as numbers of certain rational curves in Fl_n. (See [5] or [3] for details.) Let us summarize the main properties of these invariants. It will be more convenient for us to work with the polynomials $C_{u,v,w}$.

1. (Nonnegativity) *All coefficients of the $C_{u,v,w}$ are nonnegative integers.*

2. (S_3-symmetry) *The polynomials $C_{u,v,w}$ are invariant with respect to permuting u, v, and w.*

3. (Degree condition) *The polynomial $C_{u,v,w}$ is a homogeneous polynomial of degree $\frac{1}{2}(\ell(u) + \ell(v) + \ell(w) - \frac{n(n-1)}{2})$.*

4. (Classical limit) *The Schubert number $c_{u,v,w}$ is the constant term of the polynomial $C_{u,v,w}$.*

5. (Associativity) *The operation "$*$" defined by* (2) *via the polynomials $C_{u,v,w}$ is associative.*

The first four properties are clear from geometric definitions. It was conjectured in [3] that nonnegativity, associativity, degree condition, and classical limit condition uniquely determine the Gromov-Witten invariants.

The conditions **3** and **4** immediately imply the following statement.

Proposition 1 *We have*

$$C_{u,v,w} = \begin{cases} 0 & \text{if } \ell(u) + \ell(v) + \ell(w) < \frac{n(n-1)}{2}, \\ 0 & \text{if } \ell(u) + \ell(v) + \ell(w) - \frac{n(n-1)}{2} \text{ is odd,} \\ c_{u,v,w} & \text{if } \ell(u) + \ell(v) + \ell(w) = \frac{n(n-1)}{2}, \\ ??? & \text{overwise.} \end{cases}$$

In [3] we gave a method for calculation of the Gromov-Witten invariants. Among several approaches presented in that paper, one is based on the quantum analogue of Monk's formula.

For $1 \le i < j \le n$, let s_{ij} be the transposition in S_n that permutes i and j. Then $s_i = s_{i\,i+1}$ is an adjacent transposition. Also, let q_{ij} be a shorthand for the product $q_i q_{i+1} \cdots q_{j-1}$.

Proposition 2 [3, Theorem 1.3] (quantum Monk's formula) *For $w \in S_n$ and $1 \le k < n$, the quantum product of the Schubert classes σ_{s_k} and σ_w is given by*

$$\sigma_{s_k} * \sigma_w = \sum \sigma_{w s_{ab}} + \sum q_{cd}\, \sigma_{w s_{cd}}\,, \tag{3}$$

where the first sum is over all transpositions s_{ab} such that $a \le k < b$ and $\ell(w s_{ab}) = \ell(w) + 1$, and the second sum is over all transpositions s_{cd} such that $c \le k < d$ and $\ell(w s_{cd}) = \ell(w) - \ell(s_{cd}) = \ell(w) - 2(d-c) + 1$.

Remark 3 The two-dimensional Schubert classes σ_{s_k} generate the quantum cohomology ring. Thus formula (3) uniquely determines the multiplicative structure of $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ and, therefore, the Gromov-Witten invariants.

3 Cyclic symmetry

Let $o = (1, 2, \ldots, n)$ be the cyclic permutation in S_n given by

$$o(i) = i + 1, \text{ for } i = 1, \ldots, n-1, \qquad o(n) = 1.$$

Recall that $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$ for $i < j$. We also define $q_{ij} = q_{ji}^{-1}$ for $i > j$ and $q_{ii} = 1$.

Theorem 4 *For any $u, v, w \in S_n$ we have*

$$C_{u,v,w} = q_{ij}\, C_{u,o^{-1}v,ow}\,, \tag{4}$$

where $i = v^{-1}(1)$ and $j = w^{-1}(n)$.

The S_3-invariance of the $C_{u,v,w}$ under permuting u, v, and w implies a more general statement.

For $w \in S_n$ and $1 \leq a \leq n$, define the following Laurent monomials in the q_i

$$Q_{w,a} = \prod_{i\,:\,w(i) \geq n-a+1} q_{1i}\,, \qquad Q_{w,-a} = \prod_{j\,:\,w(j) \leq a} (q_{1j})^{-1},$$

and let $Q_{w,0} = 1$.

Corollary 5 *For any $u, v, w \in S_n$ and $-n \leq a, b, c \leq n$ such that $a + b + c = 0$, we have*

$$C_{u,v,w} = Q_{u,a} Q_{v,b} Q_{w,c}\, C_{o^a u, o^b v, o^c w}\,. \tag{5}$$

In many cases Corollary 5 and Proposition 1 allow us to reduce the polynomials $C_{u,v,w}$ to the Schubert numbers $c_{u,v,w}$:

Corollary 6 *For $u, v, w \in S_n$, let us find a triple $-n \leq a, b, c \leq n$, $a+b+c = 0$, for which the expression*

$$\ell_{a,b,c} = \ell(o^a u) + \ell(o^b v) + \ell(o^c w)$$

is as small as possible. If $\ell_{a,b,c} < \frac{n(n-1)}{2}$ then $C_{u,v,w} = 0$. If $\ell_{a,b,c} = \frac{n(n-1)}{2}$ then $C_{u,v,w} = Q_{u,a} Q_{v,b} Q_{w,c}\, c_{o^a u, o^b v, o^c w}$.

Remark 7 (Reduction of Gromov-Witten invariants) The Gromov-Witten invariants have the following *stability property.* If $u, v, w \in S_n$ are three permutations such that $u(n) = v(n) = n$ and $w(n) = 1$ then $C_{u,v,w} = C_{u',v',w'}$, where

$u', v', w' \in S_{n-1}$ are permutations obtained from u, v, w by removing the last entry (and subtracting 1 from all entries of w).

For a triple of permutation $u, v, w \in S_n$ such that $u(n) + v(n) + w(n) \equiv 1 \pmod{n}$, we can use the relation (5) to transform the triple to the above case when we can use the stability property. This shows that $1/n$ of all Gromov-Witten invariants for Fl_n can be reduced to the Gromov-Witten invariants of Fl_{n-1}. Analogously, we can reduce the problem to a lower level for a triple of permutations $u, v, w \in S_n$ such that $u(1) + v(1) + w(1) \equiv 2 \pmod{n}$.

Remark 8 (New rules for multiplication of Schubert classes) Suppose that a rule is know for the quantum multiplication of an arbitrary Schubert class by certain Schubert class σ_u. Theorem 4 immediately produces a new rule for the quantum multiplication by $\sigma_{o^a u}$, where $a \in \mathbb{Z}$. For example, we get for free a rule for $\sigma_{o^a} * \sigma_w$. Quantum Monk's formula (3) can be extended to a rule for $\sigma_{o^a s_k} * \sigma_w$. More generally, quantum Pieri's formula [6, Corollary 4.3] extends to an explicit rule for $\sigma_{o^a u} * \sigma_w$, where u is a permutation of the form $u = s_k s_{k+1} \cdots s_{k+l}$ or $u = s_k s_{k-1} \cdots s_{k-l}$.

4 Twisted cyclic shift

Let T_{ij}, $1 \le i < j \le n$, be the $\mathbb{Z}[q_1, \ldots, q_{n-1}]$-linear operators that act on the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ by

$$T_{ij} \,:\, \sigma_w \longmapsto \begin{cases} \sigma_{ws_{ij}} & \text{if } \ell(ws_{ij}) = \ell(w) + 1, \\ q_{ij}\, \sigma_{ws_{ij}} & \text{if } \ell(ws_{ij}) = \ell(w) - 2(j-i) + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Then quantum Monk's formula (3) can be written as:

$$\sigma_{s_k} * \sigma_w = \sum_{i \le k < j} T_{ij}(\sigma_w). \tag{7}$$

The operators T_{ij} satisfy certain simple quadratic relations. The formal algebra defined by these relations was studied in [4] and [6].

Let us also define the *twisted cyclic shift operator* O that acts on the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$, linearly over $\mathbb{Z}[q_1, \ldots, q_{n-1}]$, by

$$O \,:\, \sigma_w \longmapsto q^{(w)}\, \sigma_{ow}\,,$$

where $q^{(w)} = q_{rn}$ with $r = w^{-1}(n)$.

Proposition 9 *For any $1 \le i < j \le n$, the operators T_{ij} and O commute:*

$$T_{ij}\, O = O\, T_{ij}\,.$$

The following lemma clarifies the conditions in the right-hand side of (6). Its proof is a straightforward observation.

Lemma 10 *Let $w \in S_n$ and $1 \le i < j \le n$. Then*

1. $\ell(w\, s_{ij}) = \ell(w) + 1$ *if and only if for all* $i \le k \le j$ *we have*
$$w(k) \ge w(j) \ge w(i) \quad \text{or} \quad w(j) \ge w(i) \ge w(k)\,;$$

2. $\ell(w\, s_{ij}) = \ell(w) - \ell(s_{ij}) = \ell(w) - 2(j-i) + 1$ *if and only if for all* $i \le k \le j$ *we have*
$$w(i) \ge w(k) \ge w(j)\,.$$

Proof of Proposition 9 — The crucial observation is that, for fixed $i \le k \le j$, the set of permutations w such that

$$w(k) \ge w(j) \ge w(i) \quad \text{or} \quad w(j) \ge w(i) \ge w(k) \quad \text{or} \quad w(i) \ge w(k) \ge w(j)$$

is invariant under the left multiplications of w by the cycle o. This fact, together with Lemma 10, implies that $(T_{ij}\, O)(\sigma_w)$ is nonzero if and only if $T_{ij}(\sigma_w)$ is nonzero. Assume that $T_{ij}(\sigma_w) \ne 0$ and consider three cases:

I. Neither $w(i)$ nor $w(j)$ is equal to n. Then either of the conditions in the right-hand side of (6) is satisfied for w if and only if the same condition is satisfied for ow. Also $q^{(w)} = q^{(ws_{ij})}$. Thus $(T_{ij}\, O)(\sigma_w) = (O\, T_{ij})(\sigma_w)$.

II. We have $w(j) = n$. Then $w(i) < w(j)$ and $ow(i) > ow(j)$. Thus $\ell(ws_{ij}) = \ell(w) + 1$ and $\ell(ows_{ij}) = \ell(ow) - \ell(s_{ij})$. Thus $T_{ij}(\sigma_w) = \sigma_{ws_{ij}}$ and $T_{ij}(\sigma_{ow}) = q_{ij}\sigma_{ows_{ij}}$. Also we have $q^{(w)} = q_{jn}$ and $q^{(ws_{ij})} = q_{in}$. Therefore, $(T_{ij}\, O)(\sigma_w) = q_{ij}q_{jn}\sigma_{ows_{ij}} = q_{in}\sigma_{ows_{ij}} = (O\, T_{ij})(\sigma_w)$.

III. We have $w(i) = n$. Then $w(i) > w(j)$ and $ow(i) < ow(j)$. Thus $\ell(ws_{ij}) = \ell(w) - \ell(s_{ij})$ and $\ell(ows_{ij}) = \ell(ow) + 1$. Thus $T_{ij}(\sigma_w) = q_{ij}\sigma_{ws_{ij}}$ and $T_{ij}(\sigma_{ow}) = \sigma_{ows_{ij}}$. Also we have $q^{(w)} = q_{in}$ and $q^{(ws_{ij})} = q_{jn}$. Therefore, $(T_{ij}\, O)(\sigma_w) = q_{in}\sigma_{ows_{ij}} = q_{ij}q_{jn}\sigma_{ows_{ij}} = (O\, T_{ij})(\sigma_w)$. □

Corollary 11 *For any $w \in S_n$, the operator of quantum multiplication by the Schubert class σ_w commutes with the operator O.*

Proof — Proposition 9 and quantum Monk's formula (7) imply that the operator of quantum multiplication by a two-dimensional Schubert class σ_{s_k} commutes with the twisted cyclic shift operator O. By Remark 3, for any $w \in S_n$, the operator of quantum multiplication by σ_w commutes with O. □

This also proves Theorem 4, because it is equivalent to Corollary 11.

5 Transition graph

The *Bruhat order* Br_n is the partial order on the set of all permutations in S_n given by the following covering relation: $u \to w$ if $w = u\, s_{ab}$ and $\ell(w) = \ell(u)+1$. In other words, $u \to w$ if σ_w appear in the expansion of $\sigma_{s_k} \cdot \sigma_u$ for some $1 \le k < n$ (the product in the usual cohomology ring).

The analogue of the Bruhat order for the quantum cohomology ring is the following transition graph. The *transition graph* Tr_n is the directed graph on the set of permutations in S_n. Two permutations are connected by an edge $u \to w$ in Tr_n if $w = u\, s_{ab}$ and either $\ell(w) = \ell(u) + 1$ or $\ell(w) = \ell(u) - \ell(s_{ab})$. We will label the edge $u \to u\, s_{ab}$ by the pair (a, b). Equivalently, two permutations are connected by the edge $u \to w$ in Tr_n whenever σ_w appear in the expansion of the quantum product $\sigma_{s_k} * \sigma_u$ for some $1 \le k < n$.

Proposition 9 implies the cyclic symmetry of the transition graph:

Corollary 12 *The transition graph Tr_n is invariant under the cyclic shift: $w \mapsto o\, w$, for $w \in S_n$.*

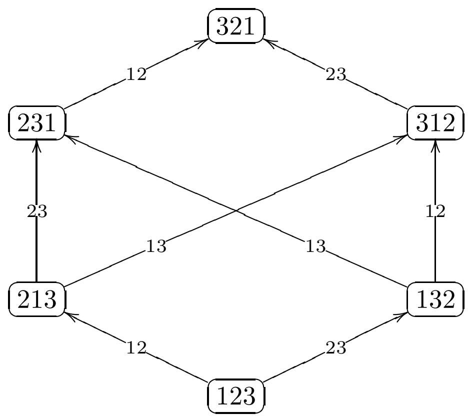

Figure 1: Bruhat order Br_3.

Figures 1 and 2 show the Bruhat order Br_3 and the transition graph Tr_3. The transition graph Tr_3 is obtained by adding several new edges to Br_3, which makes the picture symmetric with respect to the cyclic group $\mathbb{Z}/3\mathbb{Z}$. The generator o of the cyclic group rotates the graph Tr_3 by 180° clockwise.

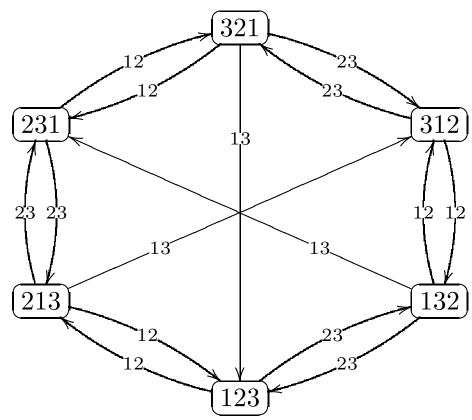

Figure 2: Transition graph Tr_3.

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