SCHUR POSITIVITY AND SCHUR LOG-CONCAVITY

THOMAS LAM, ALEXANDER POSTNIKOV, PAVLO PYLYAVSKYY

Abstract. We prove Okounkov's conjecture, a conjecture of Fomin-Fulton-Li-Poon, and a special case of Lascoux-Leclerc-Thibon's conjecture on Schur positivity and give several more general statements using a recent result of Rhoades and Skandera. An alternative proof of this result is provided. We also give an intriguing log-concavity property of Schur functions.

1. Schur positivity conjectures

The ring of symmetric functions has a linear basis of *Schur functions* s_λ labelled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$, see [Mac]. These functions appear in representation theory as characters of irreducible representations of GL_n and in geometry as representatives of Schubert classes for Grassmannians. A symmetric function is called *Schur nonnegative* if it is a linear combination with nonnegative coefficients of the Schur functions, or, equivalently, if it is the character of a certain representation of GL_n. In particular, *skew Schur functions* $s_{\lambda/\mu}$ are Schur nonnegative. Recently, a lot of work has gone into studying whether certain expressions of the form $s_\lambda s_\mu - s_\nu s_\rho$ were Schur nonnegative. Schur positivity of an expression of this form is equivalent to some inequalities between Littlewood-Richardson coefficients. In a sense, characterizing such inequalities is a "higher analogue" of the Klyachko problem on nonzero Littlewood-Richardson coefficients. Let us mention several Schur positivity conjectures due to Okounkov, Fomin-Fulton-Li-Poon, and Lascoux-Leclerc-Thibon of the above form.

Okounkov [Oko] studied branching rules for classical Lie groups and proved that the multiplicities were "monomial log-concave" in some sense. An essential combinatorial ingredient in his construction was the theorem about monomial nonnegativity of some symmetric functions. He conjectured that these functions are Schur nonnegative, as well. For a partition λ with all even parts, let $\frac{\lambda}{2}$ denote the partition $(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \ldots)$. For two symmetric functions f and g, the notation $f \geq_s g$ means that $f - g$ is Schur nonnegative.

Conjecture 1. Okounkov [Oko] *For two skew shapes λ/μ and ν/ρ such that $\lambda+\nu$ and $\mu+\rho$ both have all even parts, we have $(s_{\frac{(\lambda+\nu)}{2}/\frac{(\mu+\rho)}{2}})^2 \geq_s s_{\lambda/\mu} s_{\nu/\rho}$.*

Fomin, Fulton, Li, and Poon [FFLP] studied the eigenvalues and singular values of sums of Hermitian and of complex matrices. Their study led to two combinatorial conjectures concerning differences of products of Schur functions. Let us

Date: February 18, 2005, updated September 26, 2005.

Key words and phrases. Schur functions, Schur positivity, Schur log-concavity, immanants, Kazhdan-Lusztig polynomials, Temperley-Lieb algebra, minors.

A.P. was supported in part by NSF grant DMS-0201494. P.P. would like to thank Institut Mittag-Leffler for their hospitality.

formulate one of these conjectures, which was also studied recently by Bergeron and McNamara [BM]. For two partitions λ and μ, let $\lambda \cup \mu = (\nu_1, \nu_2, \nu_3, \dots)$ be the partition obtained by rearranging all parts of λ and μ in the weakly decreasing order. Let $\mathrm{sort}_1(\lambda, \mu) := (\nu_1, \nu_3, \nu_5, \dots)$ and $\mathrm{sort}_2(\lambda, \mu) := (\nu_2, \nu_4, \nu_6, \dots)$.

Conjecture 2. Fomin-Fulton-Li-Poon [FFLP, Conjecture 2.7] *For two partitions λ and μ, we have $s_{\mathrm{sort}_1(\lambda,\mu)} s_{\mathrm{sort}_2(\lambda,\mu)} \geq_s s_\lambda s_\mu$.*

Lascoux, Leclerc, and Thibon [LLT] studied a family of symmetric functions $\mathcal{G}^{(n)}_\lambda(q, x)$ arising combinatorially from ribbon tableaux and algebraically from the Fock space representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. They conjectured that $\mathcal{G}^{(n)}_{n\lambda}(q, x) \geq_s \mathcal{G}^{(m)}_{m\lambda}(q, x)$ for $m \leq n$. For the case $q = 1$, their conjecture can be reformulated, as follows. For a partition λ and $1 \leq i \leq n$, let $\lambda^{[i,n]} := (\lambda_i, \lambda_{i+n}, \lambda_{i+2n}, \dots)$. In particular, $\mathrm{sort}_i(\lambda, \mu) = (\lambda \cup \mu)^{[i,2]}$, for $i = 1, 2$.

Conjecture 3. Lascoux-Leclerc-Thibon [LLT, Conjecture 6.4] *For integers $1 \leq m \leq n$ and a partition λ, we have $\prod_{i=1}^n s_{\lambda^{[i,n]}} \geq_s \prod_{i=1}^m s_{\lambda^{[i,m]}}$.*

Theorem 4. *Conjectures 1, 2 and 3 are true.*

In Section 4, we present and prove more general versions of these conjectures. Our approach is based on the following result. For two partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$, let us define partitions $\lambda \vee \mu := (\max(\lambda_1, \mu_1), \max(\lambda_2, \mu_2), \dots)$ and $\lambda \wedge \mu := (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \dots)$. The Young diagram of $\lambda \vee \mu$ is the set-theoretical union of the Young diagrams of λ and μ. Similarly, the Young diagram of $\lambda \wedge \mu$ is the set-theoretical intersection of the Young diagrams of λ and μ. For two skew shapes, define $(\lambda/\mu) \vee (\nu/\rho) := \lambda \vee \nu / \mu \vee \rho$ and $(\lambda/\mu) \wedge (\nu/\rho) := \lambda \wedge \nu / \mu \wedge \rho$.

Theorem 5. *Let λ/μ and ν/ρ be any two skew shapes. Then we have*

$$s_{(\lambda/\mu)\vee(\nu/\rho)}\, s_{(\lambda/\mu)\wedge(\nu/\rho)} \geq_s s_{\lambda/\mu}\, s_{\nu/\rho}.$$

This theorem was originally conjectured by Lam and Pylyavskyy in [LP].

2. Background

In this section we give an overview of some results of Haiman [Hai] and Rhoades-Skandera [RS2, RS1]. We include an alternative proof Rhoades-Skandera's result.

2.1. **Haiman's Schur positivity result.** Let $H_n(q)$ be the *Hecke algebra* associated with the symmetric group S_n. The Hecke algebra has the standard basis $\{T_w \mid w \in S_n\}$ and the *Kazhdan-Lusztig basis* $\{C'_w(q) \mid w \in S_n\}$ related by

$$q^{l(v)/2} C'_v(q) = \sum_{w \leq v} P_{w,v}(q)\, T_w \quad \text{and} \quad T_w = \sum_{v \leq w} (-1)^{l(vw)} Q_{v,w}(q)\, q^{l(v)/2} C'_v(q),$$

where $P_{w,v}(q)$ are the *Kazhdan-Lusztig polynomials* and $Q_{v,w}(q) = P_{w_\circ w, w_\circ v}(q)$, for the longest permutation $w_\circ \in S_n$, see [Hum] for more details.

For $w \in S_n$ and a $n \times n$ matrix $X = (x_{ij})$, the *Kazhdan-Lusztig immanant* was defined in [RS2] as

$$\mathrm{Imm}_w(X) := \sum_{v \in S_n} (-1)^{l(vw)} Q_{w,v}(1)\, x_{1,v(1)} \cdots x_{n,v(n)},$$

Let $h_k = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}$ be the k-th homogeneous symmetric function, where $h_0 = 1$ and $h_k = 0$ for $k < 0$. A *generalized Jacobi-Trudi matrix* is a $n \times n$

matrix of the form $\left(h_{\mu_i-\nu_j}\right)_{i,j=1}^{n}$, for partitions $\mu = (\mu_1 \geq \mu_2 \cdots \geq \mu_n \geq 0)$ and $\nu = (\nu_1 \geq \nu_2 \cdots \geq \nu_n \geq 0)$. Haiman's result can be reformulated as follows, see [RS2].

Theorem 6. Haiman [Hai, Theorem 1.5] *The immanants* Imm_w *of a generalized Jacobi-Trudi matrix are Schur non-negative.*

Haiman's proof of this result is based on the Kazhdan-Lusztig conjecture proven by Beilinson-Bernstein and Brylinski-Kashiwara. This conjecture expresses the characters of Verma modules as sums of the characters of some irreducible highest weight representations of $\mathfrak{sl}_n$ with multiplicities equal to $P_{w,v}(1)$. One can derive from this conjecture that the coefficients of Schur functions in Imm_w are certain tensor product multiplicities of irreducible representations.

2.2. **Temperley-Lieb algebra.** The *Temperley-Lieb algebra* $TL_n(\xi)$ is the $\mathbb{C}[\xi]$-algebra generated by $t_1, \ldots, t_{n-1}$ subject to the relations $t_i^2 = \xi\, t_i$, $t_i t_j t_i = t_i$ if $|i-j| = 1$, $t_i t_j = t_j t_i$ if $|i-j| \geq 2$. The dimension of $TL_n(\xi)$ equals the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. A *321-avoiding permutation* is a permutation $w \in S_n$ that has no reduced decomposition of the form $w = \cdots s_i s_j s_i \cdots$ with $|i-j| = 1$. (These permutations are also called *fully-commutative.*) A natural basis of the Temperley-Lieb algebra is $\{t_w \mid w \text{ is a 321-avoiding permutation in } S_n\}$, where $t_w := t_{i_1} \cdots t_{i_l}$, for a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$.

The map $\theta : T_{s_i} \mapsto t_i - 1$ determines a homomorphism $\theta : H_n(1) = \mathbb{C}[S_n] \to TL_n(2)$. Indeed, the elements $t_i - 1$ in $TL_n(2)$ satisfy the Coxeter relations.

Theorem 7. Fan-Green [FG] *The homomorphism θ acts on the Kazhdan-Lusztig basis $\{C'_w(1)\}$ of $H_n(1)$ as follows:*

$$\theta(C'_w(1)) = \begin{cases} t_w & \textit{if } w \textit{ is } 321\textit{-avoiding,} \\ 0 & \textit{otherwise.} \end{cases}$$

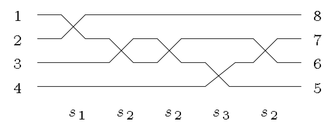
For any permutation $v \in S_n$ and a 321-avoiding permutation $w \in S_n$, let $f_w(v)$ be the coefficient of the basis element $t_w \in TL_n(2)$ in the basis expansion of $\theta(T_v) = (t_{i_1} - 1) \cdots (t_{i_l} - 1) \in TL_n(2)$, for a reduced decomposition $v = s_{i_1} \cdots s_{i_l}$. Rhoades and Skandera [RS1] defined the *Temperley-Lieb immanant* $\mathrm{Imm}_w^{\mathrm{TL}}(x)$ of an $n \times n$ matrix $X = (x_{ij})$ by

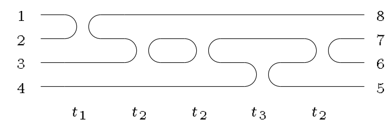
$$\mathrm{Imm}_w^{\mathrm{TL}}(X) := \sum_{v \in S_n} f_w(v)\, x_{1,v(1)} \cdots x_{n,v(n)}.$$

Theorem 8. Rhoades-Skandera [RS1] *For a 321-avoiding permutation $w \in S_n$, we have* $\mathrm{Imm}_w^{\mathrm{TL}}(X) = \mathrm{Imm}_w(X)$.

Proof. Applying the map θ to $T_v = \sum_{w \leq v} (-1)^{l(vw)} Q_{w,v}(1)\, C'_w(1)$ and using Theorem 7 we obtain $\theta(T_v) = \sum (-1)^{l(vw)} Q_{w,v}(1)\, t_w$, where the sum is over 321-avoiding permutations w. Thus $f_w(v) = (-1)^{l(vw)} Q_{w,v}(1)$ and $\mathrm{Imm}_w^{\mathrm{TL}} = \mathrm{Imm}_w$. $\square$

A product of generators (decomposition) $t_{i_1} \cdots t_{i_l}$ in the Temperley-Lieb algebra TL_n can be graphically presented by a *Temperley-Lieb diagram* with n non-crossing strands connecting the vertices $1, \ldots, 2n$ and, possibly, with some internal loops. This diagram is obtained from the wiring diagram of the decomposition $w = s_{i_1} \cdots s_{i_l} \in S_n$ by replacing each crossing "$\times$" with a *vertical uncrossing* "$)($". For example, the following figure shows the wiring diagram for $s_1 s_2 s_2 s_3 s_2 \in S_4$ and the Temperley-Lieb diagram for $t_1 t_2 t_2 t_3 t_2 \in TL_4$.





Pairs of vertices connected by strands of a wiring diagram are $(2n+1-i, w(i))$, for $i = 1, \ldots, n$. Pairs of vertices connected by strands in a Temperley-Lieb diagram form a *non-crossing matching*, i.e., a graph on the vertices $1, \ldots, 2n$ with n disjoint edges that contains no pair of edges (a, c) and (b, d) with $a < b < c < d$. If two Temperley-Lieb diagrams give the same matching and have the same number of internal loops, then the corresponding products of generators of TL_n are equal to each other. If the diagram of a is obtained from the diagram of b by removing k internal loops, then $b = \xi^k a$ in TL_n.

The map that sends t_w to the non-crossing matching given by its Temperley-Lieb diagram is a bijection between basis elements t_w of TL_n, where w is 321-avoiding, and non-crossing matchings on the vertex set $[2n]$. For example, the basis element $t_1 t_3 t_2$ of TL_4 corresponds to the non-crosssing matching with the edges $(1, 2), (3, 4), (5, 8), (6, 7)$.

2.3. **An identity for products of minors.** For a subset $S \subset [2n]$, let us say that a Temperley-Lieb diagram (or the associated element in TL_n) is *S-compatible* if each strand of the diagram has one end-point in S and the other end-point in its complement $[2n] \setminus S$. Coloring vertices in S black and the remaining vertices white, a basis element t_w is S-compatible if and only if each edge in the associated matching has two vertices of different colors. Let $\Theta(S)$ denote the set of all 321-avoiding permutation $w \in S_n$ such that t_w is S-compatible.

For two subsets $I, J \subset [n]$ of the same cardinality let $\Delta_{I,J}(X)$ denote the *minor* of an $n \times n$ matrix X in the row set I and the column set J. Let $\bar{I} := [n] \setminus I$ and let $I^\wedge := \{2n + 1 - i \mid i \in I\}$.

Theorem 9. Rhoades-Skandera [RS1, Proposition 4.3], cf. Skandera [Ska] *For two subsets $I, J \subset [n]$ of the same cardinality and $S = J \cup (\bar{I})^\wedge$, we have*

$$\Delta_{I,J}(X) \cdot \Delta_{\bar{I},\bar{J}}(X) = \sum_{w \in \Theta(S)} \mathrm{Imm}_w^{\mathrm{TL}}(X).$$

The proof given in [RS1] employs planar networks. We give a more direct proof that uses the involution principle.

Proof. Let us fix a permutation $v \in S_n$ with a reduced decomposition $v = s_{i_1} \cdots s_{i_l}$. The coefficient of the monomial $x_{1,v(1)} \cdots x_{n,v(n)}$ in the expansion of the product of two minors $\Delta_{I,J}(X) \cdot \Delta_{\bar{I},\bar{J}}(X)$ equals

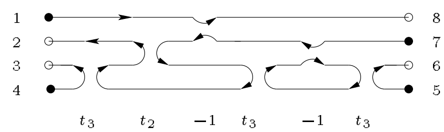
$$\begin{cases} (-1)^{\mathrm{inv}(I) + \mathrm{inv}(\bar{I})} & \text{if } v(I) = J, \\ 0 & \text{if } v(I) \neq J, \end{cases}$$

where $\mathrm{inv}(I)$ is the number of inversions $i < j$, $v(i) > v(j)$ such that $i, j \in I$.

On the other hand, by the definition of $\mathrm{Imm}_w^{\mathrm{TL}}$, the coefficient of $x_{1,v(1)} \cdots x_{n,v(n)}$ in the right-hand side of the identity equals the sum $\sum (-1)^r \, 2^s$ over all diagrams obtained from the wiring diagram of the reduced decomposition $s_{i_1} \cdots s_{i_l}$ by replacing each crossing " $\times$ " with either a *vertical uncrossing* ") (" or a *horizontal uncrossing* " $\asymp$ " so that the resulting diagram is S-compatible, where r is the number of horizontal uncrossings " $\asymp$ " and s is the number of internal loops in the resulting diagram. Indeed, the choice of ") (" corresponds to the choice of "t_{i_k}" and the choice of " " corresponds to the choice of "-1" in the k-th term of the product $(t_{i_1}-1)\cdots(t_{i_l}-1) \in TL_n(2)$, for $k = 1,\ldots,l$.

Let us pick directions of all strands and loops in such diagrams so that the initial vertex in each strand belongs to S (and, thus, the end-point is not in S). There are 2^s ways to pick directions of s internal loops. Thus the above sum can be written as the sum $\sum(-1)^r$ over such *directed Temperley-Lieb* diagrams.

Here is an example of a directed diagram for $v = s_3s_2s_1s_3s_2s_3$ and $S = \{1,4,5,7\}$ corresponding to the term $t_3t_2(-1)t_3(-1)t_3$ in the expansion of the product $(t_3-1)(t_2-1)(t_1-1)(t_3-1)(t_2-1)(t_3-1)$. This diagram comes with the sign $(-1)^2$.



Let us construct a sign reversing partial involution ι on the set of such directed Temperley-Lieb diagrams. If a diagram has a *misaligned uncrossing*, i.e., an uncrossing of the form " ", " ", " ", or " ", then ι switches the leftmost such uncrossing according to the rules ι : $\leftrightarrow$ and ι : $\leftrightarrow$. Otherwise, when the diagram involves only *aligned uncrossings* " ", " ", " ", " ", the involution ι is not defined.

For example, in the above diagram, the involution ι switches the second uncrossing, which has the form " ", to " ". The resulting diagram corresponds to the term $t_3(-1)(-1)t_3(-1)t_3$.

Since the involution ι reverses signs, this shows that the total contribution of all diagrams with at least one misaligned uncrossing is zero. Let us show that there is at most one S-compatible directed Temperley-Lieb diagram with all aligned uncrossings. If we have a such diagram, then we can direct the strands of the wiring diagram for $v = s_{i_1}\ldots s_{i_l}$ so that each segment of the wiring diagram has the same direction as in the Temperley-Lieb diagram. In particular, the end-points of strands in the wiring diagram should have different colors. Thus each strand starting at an element of J should finish at an element of $I^\wedge$, or, equivalently, $v(I) = J$. The directed Temperley-Lieb diagram can be uniquely recovered from this directed wiring diagram by replacing the crossings with uncrossings, as follows: $\to$, $\to$, $\to$, $\to$. Thus the coefficient of $x_{1,v(1)}\cdots x_{n,v(n)}$ in the right-hand side of the needed identity is zero, if $v(I) \neq J$, and is $(-1)^r$, if $v(I) = J$, where r is the number of crossings of the form " " or " " in the wiring diagram. In other words, r equals the number of crossings such that the right end-points of the pair of crossing strands have the same color. This is exactly the same as the expression for the coefficient in the left-hand side of the needed identity. $\square$

3. Proof of Theorem 5

For two subsets $I, J \subseteq [n]$ of the same cardinality, let $\Delta_{I,J}(H)$ denote the minor of the Jacobi-Trudi matrix $H = (h_{j-i})_{1\le i,j\le n}$ with row set I and column set J, where h_i is the i-th homogeneous symmetric function, as before. According to the Jacobi-Trudi formula, see [Mac], the minors $\Delta_{I,J}(H)$ are precisely the skew Schur functions

$$\Delta_{I,J}(H) = s_{\lambda/\mu},$$

where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, $\mu = (\mu_1 \geq \cdots \geq \mu_k \geq 0)$ and the associated subsets are $I = \{\mu_k+1 < \mu_{k-1}+2 < \cdots < \mu_1+k\}$, $J = \{\lambda_k+1 < \lambda_{k-1}+2 < \cdots < \lambda_1+k\}$.

For two sets $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$, let us define $I \vee J := \{\max(i_1, j_1) < \cdots < \max(i_k, j_k)\}$ and $I \wedge J := \{\min(i_1, j_1) < \cdots < \min(i_k, j_k)\}$.

Theorem 5 can be reformulated in terms of minors, as follows. Without loss of generality we can assume that all partitions λ, μ, ν, ρ in Theorem 5 have the same number k of parts, some of which might be zero. Note that generalized Jacobi-Trudi matrices are obtained from H by skipping or duplicating rows and columns.

Theorem 10. *Let I, J, I', J' be k element subsets in $[n]$. Then we have*

$$\Delta_{I\vee I',\, J\vee J'}(X) \cdot \Delta_{I\wedge I',\, J\wedge J'}(X) \geq_s \Delta_{I,J}(X) \cdot \Delta_{I',J'}(X),$$

for a generalized Jacobi-Trudi matrix X.

Proof. Let us denote $\bar{I} := [n] \setminus I$ and $\check{S} := [2n] \setminus S$. By skipping or duplicating rows and columns of the matrix X, we may assume that $I' = \bar{I}$ and $J' = \bar{J}$. Then $I \vee I' = \overline{I \wedge I'}$ and $J \vee J' = \overline{J \wedge J'}$. Let $S := J \cup (\bar{I})^\wedge$ and $T := (J \vee J') \cup (\overline{I \vee I'})^\wedge$. Then we have $T = S \vee \check{S}$ and $\check{T} = S \wedge \check{S}$.

Let us show that $\Theta(S) \subseteq \Theta(T)$, i.e., every S-compatible non-crossing matching on $[2n]$ is also T-compatible. Let $S = \{s_1 < \cdots < s_n\}$ and $\check{S} = \{\check{s}_1 < \cdots < \check{s}_n\}$. Then $T = \{\max(s_1, \check{s}_1), \ldots, \max(s_n, \check{s}_n)\}$ and $\check{T} = \{\min(s_1, \check{s}_1), \ldots, \min(s_n, \check{s}_n)\}$. Let M be an S-compatible non-crossing matching on $[2n]$ and let $(a < b)$ be an edge of M. Without loss of generality we may assume that $a = s_i \in S$ and $b = \check{s}_j \in \check{S}$. We must show that either ($a \in T$ and $b \in \check{T}$) or ($a \in \check{T}$ and $b \in T$). Since no edge of M can cross (a, b), the elements of S in the interval $[a+1, b-1]$ are matched with the elements of $\check{S}$ in this interval. Let $k = \#(S \cap [a+1, b-1]) = \#(\check{S} \cap [a+1, b-1])$. Suppose that $a, b \in T$, or, equivalently, $\check{s}_i < s_i$ and $s_j < \check{s}_j$. Since there are at least k elements of $\check{S}$ in the interval $[\check{s}_i + 1, \check{s}_j - 1]$, we have $i + k + 1 \leq j$. On the other hand, since there are at most $k - 1$ elements of S in the interval $[s_i + 1, s_j - 1]$, we have $i + k \geq j$. We obtain a contradiction. The case $a, b \in \check{T}$ is analogous.

Now Theorem 9 implies that the difference $\Delta_{I\vee I',\, J\vee J'} \cdot \Delta_{I\wedge I',\, J\wedge J'} - \Delta_{I,J} \cdot \Delta_{I',J'}$ is a nonnegative combination of Temperley-Lieb immanants. Theorems 6 and 8 imply its Schur nonnegativity. □

4. Proof of conjectures and generalizations

In this section we prove generalized versions of Conjectures 1-3, which were conjectured by Kirillov [Kir, Section 6.8]. Corollary 12 was also conjectured by Bergeron-McNamara [BM, Conjecture 5.2] who showed that it implies Theorem 13.

Let $\lfloor x \rfloor$ be the maximal integer $\leq x$ and $\lceil x \rceil$ be the minimal integer $\geq x$. For vectors v and w and a positive integer n, we assume that the operations $v + w$, $\frac{v}{n}$, $\lfloor v \rfloor$, $\lceil v \rceil$ are performed coordinate-wise. In particular, we have well-defined operations $\lfloor \frac{\lambda+\nu}{2} \rfloor$ and $\lceil \frac{\lambda+\nu}{2} \rceil$ on pairs of partitions.

The next claim extends Okounkov's conjecture (Conjecture 1).

Theorem 11. *Let λ/μ and ν/ρ be any two skew shapes. Then we have*

$$s_{\lfloor \frac{\lambda+\nu}{2} \rfloor / \lfloor \frac{\mu+\rho}{2} \rfloor}\, s_{\lceil \frac{\lambda+\nu}{2} \rceil / \lceil \frac{\mu+\rho}{2} \rceil} \geq_s s_{\lambda/\mu}\, s_{\nu/\rho}.$$

Proof. We will assume that all partitions have the same fixed number k of parts, some of which might be zero. For a skew shape $\lambda/\mu = (\lambda_1, \ldots, \lambda_k)/(\mu_1, \ldots, \mu_k)$, define

$$\overrightarrow{\lambda/\mu} := (\lambda_1+1,\ldots,\lambda_k+1)/(\mu_1+1,\ldots,\mu_k+1),$$

that is, $\overrightarrow{\lambda/\mu}$ is the skew shape obtained by shifting the shape λ/μ one step to the right. Similarly, define the left shift of λ/μ by

$$\overleftarrow{\lambda/\mu} := (\lambda_1-1,\ldots,\lambda_k-1)/(\mu_1-1,\ldots,\mu_k-1),$$

assuming that the result is a legitimate skew shape. Note that $s_{\lambda/\mu} = s_{\overleftarrow{\lambda/\mu}} = s_{\overrightarrow{\lambda/\mu}}$.

Let θ be the operation on pairs of skew shapes given by

$$\theta : (\lambda/\mu, \nu/\rho) \longmapsto ((\lambda/\mu)\vee(\nu/\rho), (\lambda/\mu)\wedge(\nu/\rho)).$$

According to Theorem 5, the product of the two skew Schur functions corresponding to the shapes in $\theta(\lambda/\mu,\nu/\rho)$ is $\geq_s s_{\lambda/\mu}\, s_{\nu/\rho}$. Let us show that we can repeatedly apply the operation θ together with the left and right shifts of shapes and the flips $(\lambda/\mu,\nu/\rho)\mapsto(\nu/\rho,\lambda/\mu)$ in order to obtain the pair of skew shapes $(\lfloor\frac{\lambda+\nu}{2}\rfloor/\lfloor\frac{\mu+\rho}{2}\rfloor, \lceil\frac{\lambda+\nu}{2}\rceil/\lceil\frac{\mu+\rho}{2}\rceil)$ from $(\lambda/\mu,\nu/\rho)$.

Let us define two operations ϕ and ψ on ordered pairs of skew shapes by conjugating θ with the right and left shifts and the flips, as follows:

$$\phi : (\lambda/\mu,\nu/\rho) \longmapsto ((\lambda/\mu)\wedge(\overrightarrow{\nu/\rho}), \overleftarrow{(\lambda/\mu)\vee(\overrightarrow{\nu/\rho})}),$$

$$\psi : (\lambda/\mu,\nu/\rho) \longmapsto (\overleftarrow{(\overrightarrow{\lambda/\mu})\vee(\nu/\rho)}, (\overrightarrow{\lambda/\mu})\wedge(\nu/\rho)).$$

In this definition the application of the left shift "$\leftarrow$" always makes sense. Indeed, in both cases, before the application of "$\leftarrow$", we apply "$\rightarrow$" and then "$\vee$". As we noted above, both products of skew Schur functions for shapes in $\phi(\lambda/\mu,\nu/\rho)$ and in $\psi(\lambda/\mu,\nu/\rho)$ are $\geq_s s_{\lambda/\mu}\, s_{\nu/\rho}$.

It is convenient to write the operations ϕ and ψ in the coordinates $\lambda_i,\mu_i,\nu_i,\rho_i$, for $i=1,\ldots,k$. These operations independently act on the pairs (λ_i,ν_i) by

$$\begin{aligned}\phi &: (\lambda_i,\nu_i)\mapsto(\min(\lambda_i,\nu_i+1),\max(\lambda_i,\nu_i+1)-1),\\ \psi &: (\lambda_i,\nu_i)\mapsto(\max(\lambda_i+1,\nu_i)-1,\min(\lambda_i+1,\nu_i)),\end{aligned}$$

and independently act on the pairs (μ_i,ρ_i) by exactly the same formulas. Note that both operations ϕ and ψ preserve the sums $\lambda_i+\nu_i$ and $\mu_i+\rho_i$.

The operations ϕ and ψ transform the differences $\lambda_i-\nu_i$ and $\mu_i-\rho_i$ according to the following piecewise-linear maps:

$$\bar\phi(x)=\begin{cases} x & \text{if } x\leq 1,\\ 2-x & \text{if } x\geq 1,\end{cases} \qquad\text{and}\qquad \bar\psi(x)=\begin{cases} x & \text{if } x\geq -1,\\ -2-x & \text{if } x\leq -1.\end{cases}$$

Whenever we apply the composition $\phi\circ\psi$ of these operations, all absolute values $|\lambda_i-\nu_i|$ and $|\mu_i-\rho_i|$ strictly decrease, if these absolute values are ≥ 2. It follows that, for a sufficiently large integer N, we have $(\phi\circ\psi)^N(\lambda/\mu,\nu/\rho) = (\tilde\lambda/\tilde\mu,\tilde\nu/\tilde\rho)$ with $\tilde\lambda_i+\tilde\nu_i=\lambda_i+\nu_i$, $\tilde\mu_i+\tilde\rho_i=\mu_i+\rho_i$, and $|\tilde\lambda_i-\tilde\nu_i|\leq 1$, $|\tilde\mu_i-\tilde\rho_i|\leq 1$, for all i. Finally, applying the operation θ, we obtain $\theta(\tilde\lambda/\tilde\mu,\tilde\nu/\tilde\rho) = (\lceil\frac{\lambda+\nu}{2}\rceil/\lceil\frac{\mu+\rho}{2}\rceil, \lfloor\frac{\lambda+\nu}{2}\rfloor/\lfloor\frac{\mu+\rho}{2}\rfloor)$, as needed. $\square$

The following conjugate version of Theorem 11 extends Fomin-Fulton-Li-Poon's conjecture (Conjecture 2) to skew shapes.

Corollary 12. *Let λ/μ and ν/ρ be two skew shapes. Then we have*

$$s_{\mathrm{sort}_1(\lambda,\nu)/\mathrm{sort}_1(\mu,\rho)}\, s_{\mathrm{sort}_2(\lambda,\nu)/\mathrm{sort}_2(\mu,\rho)} \geq_s s_{\lambda/\mu}\, s_{\nu/\rho}.$$

Proof. This statement is obtained from Theorem 11 by conjugating the shapes. Indeed, $\lceil \frac{\lambda+\mu}{2} \rceil' = \mathrm{sort}_1(\lambda', \mu')$ and $\lfloor \frac{\lambda+\mu}{2} \rfloor' = \mathrm{sort}_2(\lambda', \mu')$. Here λ' denote the partition conjugate to λ. $\square$

Theorem 13. *Let $\lambda^{(1)}/\mu^{(1)}, \dots, \lambda^{(n)}/\mu^{(n)}$ be n skew shapes, let $\lambda = \bigcup \lambda^{(i)}$ be the partition obtained by the decreasing rearrangement of the parts in all $\lambda^{(i)}$, and, similarly, let $\mu = \bigcup \mu^{(i)}$. Then we have $\prod_{i=1}^{n} s_{\lambda^{[i,n]}/\mu^{[i,n]}} \geq_s \prod_{i=1}^{n} s_{\lambda^{(i)}/\mu^{(i)}}$.*

This theorem extends Corollary 12 and Conjecture 2. Also note that Lascoux-Leclerc-Thibon's conjecture (Conjecture 3) is a special case of Theorem 13 for the n-tuple of partitions $(\lambda^{[1,m]}, \dots, \lambda^{[m,m]}, \emptyset, \dots, \emptyset)$.

Proof. Let us derive the statement by applying Corollary 12 repeatedly. For a sequence $v = (v_1, v_2, \dots, v_l)$ of integers, the *anti-inversion number* is $\mathrm{ainv}(v) := \#\{(i,j) \mid i < j,\ v_i < v_j\}$. Let $L = (\lambda^{(1)}/\mu^{(1)}, \dots, \lambda^{(n)}/\mu^{(n)})$ be a sequence of skew shapes. Define its anti-inversion number as

$$\begin{aligned} \mathrm{ainv}(L) = \mathrm{ainv}(&\lambda_1^{(1)}, \lambda_1^{(2)}, \dots, \lambda_1^{(n)}, \lambda_2^{(1)}, \dots, \lambda_2^{(n)}, \lambda_3^{(1)}, \dots, \lambda_3^{(n)}, \dots) \\ &+ \mathrm{ainv}(\mu_1^{(1)}, \mu_1^{(2)}, \dots, \mu_1^{(n)}, \mu_2^{(1)}, \dots, \mu_2^{(n)}, \mu_3^{(1)}, \dots, \mu_3^{(n)}, \dots). \end{aligned}$$

If $\mathrm{ainv}(L) \neq 0$ then there is a pair $k < l$ such that $\mathrm{ainv}(\lambda^{(k)}/\mu^{(k)}, \lambda^{(l)}/\mu^{(l)}) \neq 0$. Let $\tilde{L}$ be the sequence of skew shapes obtained from L by replacing the two terms $\lambda^{(k)}/\mu^{(k)}$ and $\lambda^{(l)}/\mu^{(l)}$ with the terms

$$\mathrm{sort}_1(\lambda^{(k)}, \lambda^{(l)})/\mathrm{sort}_1(\mu^{(k)}, \mu^{(l)}) \quad \text{and} \quad \mathrm{sort}_2(\lambda^{(k)}, \lambda^{(l)})/\mathrm{sort}_2(\mu^{(k)}, \mu^{(l)}),$$

correspondingly. Then $\mathrm{ainv}(\tilde{L}) < \mathrm{ainv}(L)$. Indeed, if we rearrange a subsequence in a sequence in the decreasing order, the total number of anti-inversions decreases. According to Corollary 12, we have $s_{\tilde{L}} \geq_s s_L$, where $s_L := \prod_{i=1}^{n} s_{\lambda^{(i)}/\mu^{(i)}}$. Note that the operation $L \mapsto \tilde{L}$ does not change the unions of partitions $\bigcup \lambda^{(i)}$ and $\bigcup \mu^{(i)}$. Let us apply the operations $L \mapsto \tilde{L}$ for various pairs (k, l) until we obtain a sequence of skew shapes $\hat{L} = (\hat{\lambda}^{(1)}/\hat{\mu}^{(1)}, \dots, \hat{\lambda}^{(n)}/\hat{\mu}^{(n)})$ with $\mathrm{ainv}(\hat{L}) = 0$, i.e., the parts of all partitions must be sorted as $\hat{\lambda}_1^{(1)} \geq \cdots \geq \hat{\lambda}_1^{(n)} \geq \hat{\lambda}_2^{(1)} \geq \cdots \geq \hat{\lambda}_2^{(n)} \geq \hat{\lambda}_3^{(1)} \geq \cdots \geq \hat{\lambda}_3^{(n)} \geq \cdots$, and the same inequalities hold for the $\hat{\mu}_j^{(i)}$. This means that $\hat{\lambda}^{(i)}/\hat{\mu}^{(i)} = \lambda^{[i,n]}/\mu^{[i,n]}$, for $i = 1, \dots, n$. Thus $s_{\hat{L}} = \prod s_{\lambda^{[i,n]}/\mu^{[i,n]}} \geq_s s_L$, as needed. $\square$

Let us define $\lambda^{\{i,n\}} := ((\lambda')^{[i,n]})'$, for $i = 1, \dots, n$. Here λ' again denotes the partition conjugate to λ. The partitions $\lambda^{\{i,n\}}$ are uniquely defined by the conditions $\lceil \frac{\lambda}{n} \rceil \supseteq \lambda^{\{1,n\}} \supseteq \cdots \supseteq \lambda^{\{n,n\}} \supseteq \lfloor \frac{\lambda}{n} \rfloor$ and $\sum_{i=1}^{n} \lambda^{\{i,n\}} = \lambda$. In particular, $\lambda^{\{1,2\}} = \lceil \frac{\lambda}{2} \rceil$ and $\lambda^{\{2,2\}} = \lfloor \frac{\lambda}{2} \rfloor$. If $\frac{\lambda}{n}$ is a partition, i.e., all parts of λ are divisible by n, then $\lambda^{\{i,n\}} = \frac{\lambda}{n}$ for each $1 \leq i \leq n$.

Corollary 14. *Let $\lambda^{(1)}/\mu^{(1)}, \dots, \lambda^{(n)}/\mu^{(n)}$ be n skew shapes, let $\lambda = \lambda^{(1)} + \cdots + \lambda^{(n)}$ and $\mu = \mu^{(1)} + \cdots + \mu^{(n)}$. Then we have $\prod_{i=1}^{n} s_{\lambda^{\{i,n\}}/\mu^{\{i,n\}}} \geq_s \prod_{i=1}^{n} s_{\lambda^{(i)}/\mu^{(i)}}$.*

Proof. This claim is obtained from Theorem 13 by conjugating the shapes. Indeed, $\left(\bigcup \lambda^{(i)}\right)' = \sum (\lambda^{(i)})'$. $\square$

For a skew shape λ/μ and a positive integer n, define $s^{(n)}_{\frac{\lambda}{n}/\frac{\mu}{n}} := \prod_{i=1}^{n} s_{\lambda^{\{i,n\}}/\mu^{\{i,n\}}}$. In particular, if $\frac{\lambda}{n}$ and $\frac{\mu}{n}$ are partitions, then $s^{(n)}_{\frac{\lambda}{n}/\frac{\mu}{n}} = \left(s_{\frac{\lambda}{n}/\frac{\mu}{n}}\right)^n$.

Corollary 15. *Let c and d be positive integers and $n = c + d$. Let λ/μ and ν/ρ be two skew shapes. Then* $s^{\langle n \rangle}_{\frac{c\lambda+d\nu}{n} / \frac{c\mu+d\rho}{n}} \geq_s s^c_{\lambda/\mu} s^d_{\nu/\rho}$.

Theorem 11 is a special case of Corollary 15 for $c = d = 1$.

Proof. This claim follows from Corollary 14 for the sequence of skew shapes that consists of λ/μ repeated c times and ν/ρ repeated d times. $\square$

Corollary 15 implies that the map $S : \lambda \mapsto s_\lambda$ from the set of partitions to symmetric functions satisfies the following "Schur log-concavity" property.

Corollary 16. *For positive integers c, d and partitions λ, μ such that $\frac{c\lambda+d\mu}{c+d}$ is a partition, we have* $\left(S\left(\frac{c\lambda+d\mu}{c+d}\right)\right)^{c+d} \geq_s S(\lambda)^c S(\mu)^d$.

This notion of Schur log-concavity is inspired by Okounkov's notion of log-concavity; see [Oko].

Acknowledgements: We thank Richard Stanley for useful conversations. We are grateful to Sergey Fomin for helpful comments and suggestions and to Mark Skandera for help with the references.

References

[BM] F. Bergeron and P. McNamara: Some positive differences of products of Schur functions, arXiv: `math.CO/0412289`.

[FFLP] S. Fomin, W. Fulton, C.-K. Li and Y.-T. Poon: Eigenvalues, singular values, and Littlewood-Richardson coefficients, *American Journal of Mathematics*, **127** (2005), 101-127.

[FG] K. Fan and R.M. Green: Monomials and Temperley-Lieb algebras, *Journal of Algebra*, **190** (1997), 498-517.

[Hai] M. Haiman: Hecke algebra characters and immanant conjectures, *J. Amer. Math. Soc.*, **6** (1993), 569-595.

[Hum] J. Humphreys: *Reflection Groups and Coxeter Groups,* Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1992.

[Kir] A. Kirillov: An invitation to the generalized saturation conjecture, *Publications of RIMS Kyoto University* **40** (2004), 1147-1239.

[LP] T. Lam and P. Pylyavskyy: Cell transfer and monomial positivity, arXiv: `math.CO/ 0505273`.

[LLT] A. Lascoux, B. Leclerc and J.-Y. Thibon: Ribbon tableaux, Hall-Littlewood symmetric functions, quantum affine algebras, and unipotent varieties, *Journal of Mathematical Physics*, **38**(3) (1997), 1041-1068.

[Mac] I. G. Macdonald: *Symmetric Functions and Hall Polynomials,* Oxford, 1970.

[Oko] A. Okounkov: Log-concavity of multiplicities with applications to characters of $U(\infty)$, *Advances in Mathematics*, **127** no. 2 (1997), 258-282.

[RS1] B. Rhoades and M. Skandera: Temperley-Lieb immanants, to appear in *Annals of Combinatorics*; `http://www.math.dartmouth.edu/~skan/papers.htm`.

[RS2] B. Rhoades and M. Skandera: Kazhdan-Lusztig immanants and products of matrix minors, preprint, November 19, 2004; `http://www.math.dartmouth.edu/~skan/papers.htm`.

[Ska] M. Skandera: Inequalities in products of minors of totally nonnegative matrices, *Journal of Algebraic Combinatorics* **20** (2004), no. 2, 195–211.

Department of Mathematics, M.I.T., Cambridge, MA 02139
E-mail address: `thomasl (at) math (dot) mit (dot) edu`
E-mail address: `apost (at) math (dot) mit (dot) edu`
E-mail address: `pasha (at) math (dot) mit (dot) edu`