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QUANTUM BRUHAT GRAPH AND SCHUBERT POLYNOMIALS

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ABSTRACT. The quantum Bruhat graph, which is an extension of the graph formed by covering relations in the Bruhat order, is naturally related to the quantum cohomology ring of G/B. We enhance a result of Fulton and Woodward by showing that the minimal monomial in the quantum parameters that occurs in the quantum product of two Schubert classes has a simple interpretation in terms of directed paths in this graph.

We define path Schubert polynomials, which are quantum cohomology analogs of skew Schubert polynomials recently introduced by Lenart and Sottile. They are given by sums over paths in the quantum Bruhat graph of type A. The 3-point Gromov-Witten invariants for the flag manifold are expressed in terms of these polynomials. This construction gives a combinatorial description for the set of all monomials in the quantum parameters that occur in the quantum product of two Schubert classes.

1. INTRODUCTION

The famous Littlewood-Richardson coefficients are the structure constants of the cohomology ring of the Grassmannian in the basis of Schubert classes. It is an open question to give a combinatorial interpretation of the generalized Littlewood-Richardson coefficients for the flag manifold and, more generally, for the homogeneous space G/P. This article discusses the 3-point *Gromov-Witten invariants*, which are the structure constants of the small *quantum cohomology ring*. They extend the Littlewood-Richardson coefficients in an orthogonal "quantum" direction.

Fulton and Woodward [F-W] described the set of minimal monomials q^d in the quantum parameters that occur in the quantum product $\sigma_u * \sigma_v$ of two Schubert classes in $\mathrm{QH}^*(G/P)$. We demonstrate that the problem of multiplying the Schubert classes in the quantum cohomology ring is naturally related to the study of paths in the *quantum Bruhat graph* from [BFP]. This is the directed graph obtained by adding some "quantum" edges to the graph formed by covering relations in the Bruhat order. The quantum Bruhat graph encodes the terms that appear in the quantum Chevalley-Monk formula. We show, for G/B, that there is a *unique* minimal monomial q^d that occurs in the quantum product $\sigma_u * \sigma_v$. This fact does not immediately follow from [F-W]. The minimal monomial q^d is equal to the weight of any *shortest* directed path from u to $w_\circ v$ in the quantum Bruhat graph.

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Lenart and Sottile [L-S] recently defined *skew Schubert polynomials.* The expansion coefficients of these polynomials in the basis of usual Schubert polynomials are equal to the generalized Littlewood-Richardson coefficients for the flag manifold. On the other hand, in [Po3], we introduced toric Schur polynomials, whose expansion coefficients in the basis of usual Schur polynomials are the Gromov-Witten invariants for the Grassmannian.

In this article, we put these two approaches together. We define the *path Schubert polynomials*, whose expansion coefficients in the basis of usual Schubert polynomials give the Gromov-Witten invariants for the flag manifold. The path Schubert polynomials can be defined as sums over directed paths in the quantum Bruhat graph on the symmetric group. The proof of the result on their Schubert expansion readily follows from a combination of the Cauchy identity, the elementary quantization rule from [FGP], and the quantum Pieri formula.

This construction implies a combinatorial description for the set of *all* monomials q^d in the quantum parameters that occur, with nonzero coefficients, in the quantum product $\sigma_u * \sigma_v$ of two Schubert classes in the quantum cohomology of the flag manifold. A monomial q^d occurs in $\sigma_u * \sigma_v$ if and only if there is a directed path of weight q^d from u to $w_\circ v$ in the quantum Bruhat graph that satisfies a certain additional condition. This solves a problem posed by Fulton and Woodward [F-W] for the flag manifold.

The article consists of two parts: one related to the generalized flag manifold G/B, and the other concerned with the classical flag manifold $\mathrm{Fl}_n = \mathrm{SL}(n,\mathbb{C})/B$. In Section 2, we discuss quantum cohomology of G/B in a uniform Lie-theoretic setup. We enhance a result of Fulton and Woodward using the quantum Bruhat graph. In Section 3, we review some notions and results peculiar to the classical flag manifold. In Section 4, we use these combinatorial techniques to obtain stronger results for the quantum cohomology of Fl_n.

2. Quantum cohomology of G/B

In this section we discuss the quantum cohomology ring of the generalized flag manifold G/B. First, we briefly recall some general Lie-theoretic notation; see [Hum] for more details. Then we recall a few facts related to Schubert classes and quantum cohomology, and formulate the quantum Chevalley-Monk formula; see [F-W]. This formula leads to the quantum Bruhat graph that was introduced in [BFP]. We show that the minimal monomial q^d that occurs in the quantum product of two Schubert classes has a simple interpretation in terms of directed paths in the quantum Bruhat graph.

Let G be a simply connected complex semi-simple Lie group together with a Borel subgroup B. The homogeneous space G/B is a compact complex manifold.

Let $\mathfrak{h}$ be the *Cartan subalgebra* in the Lie algebra $\mathfrak{g}$ of G, and let $\Phi \subset \mathfrak{h}^*$ be the *root system* associated with G. For a root $\alpha \in \Phi$, let $h_\alpha \in \mathfrak{h}$ denote the corresponding *coroot* in the dual root system $\Phi^\vee$. The *Weyl group* W is generated by the reflections $s_\alpha \in \mathrm{End}(\mathfrak{h}^*)$, $\alpha \in \Phi$, given by $s_\alpha : \lambda \mapsto \lambda - \lambda(h_\alpha)\,\alpha$. If $\langle \alpha \mid \beta \rangle$ is a W-invariant inner product on $\mathfrak{h}^*$, then $\mathfrak{h}$ can be identified with $\mathfrak{h}^*$ via this inner product and $h_\alpha = 2\alpha/\langle \alpha \mid \alpha \rangle$. The choice of Borel subgroup B determines the set of *positive roots* $\Phi_+ \subset \Phi$, the basis of *simple roots* $\alpha_1, \ldots, \alpha_r$ in the root system Φ, and the basis of simple coroots $h_{\alpha_1}, \ldots, h_{\alpha_r}$ in the dual root system $\Phi^\vee$. The *fundamental weights* $\lambda_1, \ldots, \lambda_r \in \mathfrak{h}^*$ are defined as elements of the basis dual to $\{h_{\alpha_1}, \ldots, h_{\alpha_r}\}$, i.e., $\lambda_i(h_{\alpha_j}) = \delta_{ij}$. The Weyl group W is generated by the set of *Coxeter generators* $s_i = s_{\alpha_i}$. The *length* $\ell(w)$ of an element $w \in W$ is the minimal number of generators s_i in a decomposition for w.

The *cohomology ring* $\mathrm{H}^*(G/B)$ is a free $\mathbb{Z}$-module linearly generated by the *Schubert classes* σ_w labeled by the elements $w \in W$ of the Weyl group. The *Poincaré duality* preserves the basis of Schubert classes:

$$\int \sigma_u \cdot \sigma_v = \delta_{u, w_\circ v} \qquad \text{(Kronecker's delta)}.$$

Here and everywhere below $w_\circ$ is the unique Weyl group element of maximal possible length, called the *longest element*.

The Schubert classes σ_{s_i} generate the cohomology ring of G/B. According to *Borel's theorem*, the maps $\sigma_{s_i} \mapsto \lambda_i$ extend to the canonical isomorphism:

$$\mathrm{H}^*(G/B, \mathbb{Q}) \simeq \mathrm{Sym}(\mathfrak{h}^*_{\mathbb{Q}})/I_W, \tag{1}$$

where $\mathrm{Sym}(\mathfrak{h}^*_{\mathbb{Q}})$ is the symmetric algebra of the $\mathbb{Q}$-span $\mathfrak{h}^*_{\mathbb{Q}} \subset \mathfrak{h}^*$ of the fundamental weights λ_i and I_W is the ideal in this algebra generated by W-invariant elements without constant term.

Let $\mathbb{Z}[q] = \mathbb{Z}[q_1, \ldots, q_r]$. The small *quantum cohomology ring* $\mathrm{QH}^*(G/B)$ of G/B equals, as a $\mathbb{Z}[q]$-module, the tensor product $\mathrm{H}^*(G/B) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. Thus the Schubert classes σ_w, $w \in W$, form a $\mathbb{Z}[q]$-linear basis of $\mathrm{QH}^*(G/B)$. The multiplicative structure of the quantum cohomology is a deformation of the usual product in $\mathrm{H}^*(G/B)$. We will use the symbol "$*$" to denote the quantum product, i.e., the product in the quantum cohomology ring. The structure constants of the quantum cohomology are 3-point *Gromov-Witten invariants*:

$$\sigma_u * \sigma_v = \sum_{w,\, d} \langle \sigma_u, \sigma_v, \sigma_{w_\circ w} \rangle_d \, q^d \, \sigma_w$$

where the sum is over $w \in W$ and $d = (d_1, \ldots, d_r) \in \mathbb{Z}^r_{\geq 0}$, and $q^d = q_1^{d_1} \cdots q_r^{d_r}$. The Gromov-Witten invariants $\langle \sigma_u, \sigma_v, \sigma_{w_\circ w} \rangle_d$ are nonnegative integers that count the numbers of certain rational curves in G/B. Their geometric definition implies that the quantum product is a commutative and associative operation.

For a root $\alpha \in \Phi_+$ such that $h_\alpha = d_1 \, h_{\alpha_1} + \cdots + d_r \, h_{\alpha_r}$, let $q^{h_\alpha} = q_1^{d_1} \cdots q_r^{d_r}$. Let us assume that the variables q_i are of degree 2. Thus $\deg(q^{h_\alpha}) = 2\,|h_\alpha|$, where $|h_\alpha| = (\lambda_1 + \cdots + \lambda_r)(h_\alpha)$ is the *height* of the coroot h_α.

Let us define, for a positive root $\alpha \in \Phi_+$, the operator $\mathcal{T}_\alpha$ acting $\mathbb{Z}[q]$-linearly on the quantum cohomology ring $\mathrm{QH}^*(G/B)$ as

$$\mathcal{T}_\alpha : \sigma_w \longmapsto \begin{cases} \sigma_{w s_\alpha} & \text{if } \ell(w s_\alpha) = \ell(w) + 1, \\ q^{h_\alpha} \, \sigma_{w s_\alpha} & \text{if } \ell(w s_\alpha) = \ell(w) + 1 - 2\,|h_\alpha|, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The space $\mathrm{QH}^*(G/B)$ has the $\mathbb{Z}$-linear degree function $\deg(q^d \, \sigma_w) = 2\,|d| + \ell(w)$. Then the $\mathcal{T}_\alpha$ are homogeneous operators on $\mathrm{QH}^*(G/B)$ of degree 1.

The quantum cohomology $\mathrm{QH}^*(G/B)$ is generated, as an algebra over $\mathbb{Z}[q]$, by the Schubert classes σ_{s_i}. Thus the quantum product is uniquely determined by the following *quantum Chevalley-Monk formula*:

$$\sigma_{s_i} * \sigma_w = \sum_{\alpha \in \Phi_+} \lambda_i(h_\alpha) \, \mathcal{T}_\alpha(\sigma_w)\,. \tag{3}$$

For root systems of type A, this formula was established in [FGP]. For the case of an arbitrary type it was found by Peterson [Pet] (unpublished). A more general version of the quantum Chevalley-Monk formula, for $\mathrm{QH}^*(G/P)$, was given and rigorously proved by Fulton and Woodward [F-W].

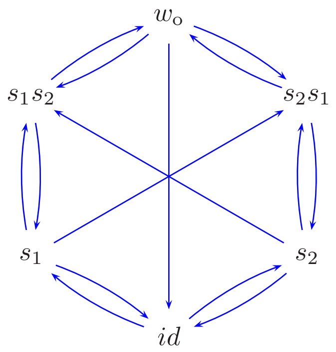

FIGURE 1. Quantum Bruhat graph Γ_Φ of type A_2

Motivated by the quantum Chevalley-Monk formula (3), let us define the *quantum Bruhat graph* Γ_Φ as the following directed graph on the Weyl group elements $w \in W$ with weighted edges. Two elements $u, v \in W$ are connected by a directed edge $e : u \to v$ if and only if $v = u s_\alpha$ and one of the following two conditions is satisfied:

$$\ell(v) = \ell(u) + 1 \quad \text{or} \quad \ell(v) = \ell(u) + 1 - 2\,|h_\alpha|.$$

If $\ell(v) = \ell(u)+1$, then the weight of edge e equals 1, and if $\ell(v) = \ell(u)+1-2\,|h_\alpha|$, then the weight of edge e equals q^{h_α}. The weight of a directed path in the graph Γ_Φ is the product of the weights of its edges.

The graph Γ_Φ was investigated in [BFP] and, for type A, in [Po2]. The upward edges in Γ_Φ, i.e., the edges that increase the length by 1, are exactly the covering relations in the Bruhat order on the Weyl group W. The downward edges correspond to additional "quantum" terms that appear in the quantum Chevalley-Monk formula (3). This is why we call Γ_Φ the quantum Bruhat graph.

Let us say that a directed path in Γ_Φ from u to v is *shortest* if it has the minimal possible length among all directed paths from u to v.

Lemma 1. *Let $u, v \in W$ be any two Weyl group elements.* (1) *There exists a directed path from u to v in the graph Γ_Φ.* (2) *All shortest paths from u to v have the same weight $q^{d_{\min}}$.* (3) *The weight of any path from u to v is divisible by $q^{d_{\min}}$.*

This lemma says that, for any $u, v \in W$, there is a well-defined *minimal degree* $d_{\min} = d_{\min}(u, v) \in \mathbb{Z}^r_{\geq 0}$ such that $q^{d_{\min}}$ is the minimal possible weight of a directed path from u to v. The lemma easily follows from results of [BFP].

Proof. Suppose that a covers b in the *weak* Bruhat order, i.e., $a, b \in W$, $a = b\, s_i$, and $\ell(a) = \ell(b) + 1$. Then both directed edges $a \to b$ and $b \to a$ are present in the quantum Bruhat graph Γ_Φ. Thus any two Weyl group elements u and v can be connected by a directed path in Γ_Φ.

According to [BFP, Lemma 6.7], for a path $a \to b \to c$ of length 2 in the graph Γ_Φ with $a \neq c$, there exists a unique path $a \to b' \to c$ such that $b' \neq b$. The proof of Theorem 6.4 in [BFP] implies that any two shortest paths from u to v can be obtained from each other by applying several switches of this type to pairs of consecutive edges in a path. Moreover, any path from u to v can be reduced to a shortest path by applying a sequence of such switches and cancellations of pairs of opposite edges $a \to b \to a$; see [BFP, Section 6].

These switches of pairs of consecutive edges in a path preserve its weight. Indeed, it is enough to verify the statement only for 3 types A_2, B_2, and G_2 of rank 2 root systems (cf. proof of [BFP, Lemma 6.7]), which can be easily done by a direct observation. Thus all shortest paths from u to v have the same weight $q^{d_{\min}}$, and the weight of any path from u to v is divisible by $q^{d_{\min}}$. □

Let $\ell(u, v)$ be the length of a shortest path from u to v in the graph Γ_Φ.

Theorem 2. *Fix two Weyl group elements $u, v \in W$. For any $w \in W$, the coefficient of σ_v in $\sigma_u * \sigma_w$ is divisible by $q^{d_{\min}(u,v)}$. There exists $w \in W$ such that the coefficient of σ_v in $\sigma_u * \sigma_w$ equals $q^{d_{\min}(u,v)}$ times a nonzero integer. Moreover, for any such w, we have $\ell(w) = \ell(u, v)$.*

Proof. Let $S = \sigma_{s_1} + \ldots + \sigma_{s_r}$. The expression $S^{*l} = S * \cdots * S$ expands as a *positive* integer combination of the terms $q^d \sigma_w$. For any $w \in W$ of length $\ell(w) = l$, the expansion of S^{*l} contains the class σ_w with a strictly positive integer coefficient (without any quantum parameters q_i). According to the quantum Chevalley-Monk formula (3), the coefficient of σ_v in $S^{*l} * \sigma_u$ is given by the sum over directed paths from u to v, and each path comes with its weight times some positive integer. By Lemma 1, all these terms, including the contribution of $\sigma_w * \sigma_u$, should be divisible by $q^{d_{\min}(u,v)}$. This proves the first claim of the theorem.

On the other hand, suppose that $l = \ell(u, v)$. Then there exists at least one path from u to v of length l. By Lemma 1 and the quantum Chevalley-Monk formula (3), the coefficient of σ_v in $S^{*l} * \sigma_u$ is equal to $q^{d_{\min}(u,v)}$ times a strictly positive integer. The expansion of S^{*l} in the basis of Schubert classes may also contain terms $q^d \sigma_{w'}$ with nonzero d. For all such terms, we have $\ell(w') = l' < l$ because $l = \deg(S^{*l}) = \deg(q^d \sigma_{w'}) = 2\,|d| + l'$. Thus $\sigma_{w'}$ appears with a strictly positive coefficient in the expansion of $S^{*l'}$. But, the coefficient of σ_v in $S^{*l'} * \sigma_u$ is 0 because there are no paths from u to v of length $l' < l$. Thus the terms $q^d \sigma_{w'}$ in S^{*l} cannot make any contribution to the (nonzero) coefficient of σ_v in $S^{*l} * \sigma_u$.

This means that there exists at least one term Const $\cdot\, \sigma_w$ in the expansion of S^{*l} with $\ell(w) = \ell(u, v)$ that *does make* a contribution to the coefficient of σ_v in $S^{*l} * \sigma_u$. In other words, the coefficient of σ_v in $\sigma_w * \sigma_u$ equals $q^{d_{\min}(u,v)}$ times a nonzero integer, which proves the second claim of the theorem. □

The next claim strengthens a result due to Fulton and Woodward [F-W, Theorem 9.1]. Let us say that a monomial q^d *occurs* in the quantum product $\sigma_u * \sigma_v$ of two Schubert classes if there exists $w \in W$ such that the Gromov-Witten invariant $\langle \sigma_u, \sigma_v, \sigma_w \rangle_d$ is nonzero. Let us say that q^d is a *minimal monomial* that occurs in $\sigma_u * \sigma_v$ if (1) q^d occurs in $\sigma_u * \sigma_v$, and (2) any monomial strictly dividing q^d does not occur in $\sigma_u * \sigma_v$. Theorem 2 can be equivalently reformulated, as follows.

Corollary 3. *For any pair $u, v \in W$, the monomial $q^{d_{\min}(u, w_\circ v)}$ is the unique minimal monomial that occurs in the quantum product $\sigma_u * \sigma_v$.*

Note that Fulton and Woodward [F-W] described the set of minimal monomials that occur in $\sigma_u * \sigma_v$, but their construction does not immediately imply that this set consists of a single element.

It would be interesting to describe *all* monomials q^d that occur in a quantum product $\sigma_u * \sigma_v$. Theorem 2 suggests that such monomials q^d should be weights of paths from u to $w_\circ v$ in the graph Γ_Φ that satisfy certain additional conditions. In the following sections we describe such a class of paths for type A root systems.

3. Flag manifold and Schubert polynomials

There are some notions and results related to (quantum) cohomology, which are peculiar for type A. In this section we discuss this type A theory, which includes the Cauchy identity for Schubert polynomials (see [Mac]), the elementary quantization rule, and the quantum Pieri operators (see [FGP, Po1] for more details).

For type A_{n-1}, the homogeneous space G/B is the complex *flag manifold* $\mathrm{Fl}_n = \mathrm{SL}(n, \mathbb{C})/B$. The corresponding Weyl group is the symmetric group $W = S_n$ of order n permutations. Thus the Schubert classes σ_w, which form a $\mathbb{Z}$-basis of the cohomology $\mathrm{H}^*(\mathrm{Fl}_n)$, are labeled by permutations $w \in S_n$.

In this case, Borel's theorem (1) implies that the cohomology ring is isomorphic to the following quotient of the polynomial ring:

$$\mathrm{H}^*(\mathrm{Fl}_n) \simeq \mathbb{Z}[x_1, \dots, x_n]/\langle e_1, \dots, e_n \rangle, \tag{4}$$

where $e_i = e_i(x_1, \dots, x_n)$ are the elementary symmetric polynomials.

Lascoux and Schützenberger [LSc], using constructions of Bernstein-Gelfand-Gelfand and Demazure, defined the *Schubert polynomials* $\mathfrak{S}_w(x) = \mathfrak{S}_w(x_1, \dots, x_n)$ in the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$, which are particularly nice polynomial representatives of the Schubert classes σ_w. These polynomials have nonnegative integer coefficients and are stable under the standard embedding $S_n \hookrightarrow S_{n+1}$. Another important property of the Schubert polynomials is the following *Cauchy identity* (see [Mac]):

$$\prod_{i+j \le n} (x_i + y_j) = \sum_{w \in S_n} \mathfrak{S}_w(x_1, \dots, x_n) \cdot \mathfrak{S}_{w w_\circ}(y_1, \dots, y_n). \tag{5}$$

For type A_{n-1}, the longest permutation in S_n is given by $w_\circ = n \dots 2\,1$.

The left-hand side of (5) is the double Schubert polynomial $\mathfrak{S}_{w_\circ}(x, -y)$. It can be expanded in terms of elementary symmetric polynomials as

$$\prod_{i+j \le n} (x_i + y_j) = \prod_{k=1}^{n-1} \sum_{i=0}^{k} y_{n-k}^{k-i}\, e_i(x_1, \dots, x_k). \tag{6}$$

As a linear space, the quantum cohomology ring $\mathrm{QH}^*(\mathrm{Fl}_n)$ of the flag manifold equals $\mathrm{H}^*(\mathrm{Fl}_n) \otimes \mathbb{Z}[q]$, where $\mathbb{Z}[q] = \mathbb{Z}[q_1, \dots, q_{n-1}]$.

For type A_{n-1}, the definition (2) of the operators $\mathcal{T}_\alpha$ can be written as follows. For $1 \le i < j \le n$, let $\mathcal{T}_{ij}$ be the operator that acts $\mathbb{Z}[q]$-linearly on the quantum cohomology $\mathrm{QH}^*(\mathrm{Fl}_n)$ by the formula

$$\mathcal{T}_{ij} : \sigma_w \longmapsto \begin{cases} \sigma_{w s_{ij}} & \text{if } \ell(w s_{ij}) = \ell(w) + 1, \\ q_{ij}\, \sigma_{w s_{ij}} & \text{if } \ell(w s_{ij}) = \ell(w) + 1 - 2(j-i), \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where $\ell(w)$ denote the length of the permutation w, $s_{ij} \in S_n$ is the transposition of i and j, and $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$. The Coxeter generators of S_n are the adjacent transpositions $s_k = s_{k\,k+1}$, $k = 1, \ldots, n-1$. The quantum Chevalley-Monk formula (3) specializes to the following formula proved in [FGP]:

$$\sigma_{s_k} * \sigma_w = \sum_{i \le k < j} \mathcal{T}_{ij}(\sigma_w), \tag{8}$$

for any $w \in S_n$ and $k = 1, \ldots, n-1$.

Let us define an involution ω on the quantum cohomology ring $\mathrm{QH}^*(\mathrm{Fl}_n)$ by setting $\omega : f(q_1, \ldots, q_{n-1}) \mapsto f(q_{n-1}, \ldots, q_1)$ and

$$\omega : q^d \sigma_w \mapsto \omega(q^d)\, \sigma_{w_\circ w w_\circ}\,,$$

and extending it by linearity. An easy observation shows that the quantum Chevalley-Monk formula (8) is invariant under ω. Thus ω is an automorphism of $\mathrm{QH}^*(\mathrm{Fl}_n)$.

Let $e_i^{(k)} = \sigma_{(k-i+1,\ldots,k,k+1)}$ and $h_i^{(k)} = \sigma_{(k+i,\ldots,k+1,k)}$ be the Schubert classes represented in (4) by the elementary and complete homogeneous symmetric functions in the first k variables: $e_i(x_1, \ldots, x_k) = \mathfrak{S}_{(k-i+1,\ldots,k+1)}$ and $h_i(x_1, \ldots, x_k) = \mathfrak{S}_{(k+i,\ldots,k)}$. Here we use cycle notation for permutations: $(k-i+1, \ldots, k+1) = s_{k-i+1} \cdots s_{k-1} s_k$ and $(k+i, \ldots, k) = s_{k+i-1} \cdots s_{k+1} s_k$. The involution ω on $\mathrm{H}^*(\mathrm{Fl}_n)$ switches these two families of Schubert classes $e_i^{(k)} \overset{\omega}{\longleftrightarrow} h_i^{(n-k)}$.

Theorem 1.1 from [FGP] is equivalent to saying that the quantum product of the Schubert classes $e_{i_1}^{(1)}, e_{i_2}^{(2)}, \ldots, e_{i_{n-1}}^{(n-1)}$ in the ring $\mathrm{QH}^*(\mathrm{Fl}_n)$ is exactly the same as the classical product of these Schubert classes in $\mathrm{H}^*(\mathrm{Fl}_n)$:

$$e_{i_1}^{(1)} * e_{i_2}^{(2)} * \cdots * e_{i_{n-1}}^{(n-1)} = e_{i_1}^{(1)} \cdot e_{i_2}^{(2)} \cdot \cdots \cdot e_{i_{n-1}}^{(n-1)}. \tag{9}$$

We call this identity the *elementary quantization rule*. Applying the involution ω to both sides, we get a similar statement for the $h_i^{(k)}$:

$$h_{i_1}^{(1)} * h_{i_2}^{(2)} * \cdots * h_{i_{n-1}}^{(n-1)} = h_{i_1}^{(1)} \cdot h_{i_2}^{(2)} \cdot \cdots \cdot h_{i_{n-1}}^{(n-1)}.$$

Let us define the *quantum Pieri operators* $\mathcal{E}_i^{(k)}$ and $\mathcal{H}_i^{(k)}$, acting on the quantum cohomology $\mathrm{QH}^*(\mathrm{Fl}_n)$, as

$$\begin{aligned} \mathcal{E}_i^{(k)} &= \sum \mathcal{T}_{a_1 b_1} \cdots \mathcal{T}_{a_i b_i}, \\ \mathcal{H}_i^{(k)} &= \sum \mathcal{T}_{c_1 d_1} \cdots \mathcal{T}_{c_i d_i}, \end{aligned} \tag{10}$$

where the first sum is over $a_1, \ldots, a_i, b_1, \ldots, b_i$ such that

$$a_1, \ldots, a_i \le k < b_1 \le \cdots \le b_i \text{ and } a_1, \ldots, a_i \text{ are distinct;}$$

and the second sum is over $c_1, \ldots, c_i, d_1, \ldots, d_i$ such that

$$c_1 \le \cdots \le c_i \le k < d_1, \ldots, d_i \text{ and } d_1, \ldots, d_i \text{ are distinct.}$$

In [Po1] we showed how to deduce from the quantum Chevalley-Monk formula (8) the following *quantum Pieri formulas* for the quantum product of $e_i^{(k)}$ and $h_i^{(k)}$ with any Schubert class:

$$\begin{aligned} e_i^{(k)} * \sigma_w &= \mathcal{E}_i^{(k)}(\sigma_w), \\ h_i^{(k)} * \sigma_w &= \mathcal{H}_i^{(k)}(\sigma_w). \end{aligned} \tag{11}$$

In a different form, the quantum Pieri formulas were earlier given by Ciocan-Fontanine in [C-F]. In the proof of these formulas given in [Po1] we only used the quadratic relations for the operators $\mathcal{T}_{ij}$ found by Fomin and Kirillov [F-K].

4. Path Schubert polynomials

In this section we define path Schubert polynomials and establish their relation with the Gromov-Witten invariants for the flag manifold. These polynomials can be expressed in terms of paths in the quantum Bruhat graph. As a corollary, we obtain a combinatorial rule for all monomials q^d that occur in the quantum product $\sigma_u * \sigma_v$ in terms of paths.

Let us define the operator $\mathcal{H}(y)$, acting on the space $\mathrm{QH}^*(\mathrm{Fl}_n) \otimes \mathbb{Z}[y_1, \ldots, y_n]$, as the following combination of quantum Pieri operators:

$$\mathcal{H}(y) = \sum_{\beta \leq \delta} y^{\delta-\beta}\, \mathcal{H}^{(1)}_{\beta_1} \cdots \mathcal{H}^{(n-1)}_{\beta_{n-1}}$$

where the sum is over nonnegative sequences $\beta = (\beta_1, \ldots, \beta_{n-1})$ such that $\beta \leq \delta$ termwise, $\delta = (n-1, n-2, \ldots, 1)$, and $y^\gamma = y_1^{\gamma_1} y_2^{\gamma_2} \cdots y_{n-1}^{\gamma_{n-1}}$, for the termwise difference $\gamma = \beta - \delta$.

Let us define, for all pairs $u, v \in S_n$, the *path Schubert polynomials* $\mathfrak{S}_{u,v} = \mathfrak{S}_{u,v}(y, q)$ in the polynomial ring $\mathbb{Z}[y_1, \ldots, y_n, q_1, \ldots, q_{n-1}]$ as the matrix elements of the operator $\mathcal{H}(y)$ in the basis of Schubert classes:

$$\mathcal{H}(y) : \sigma_u \longmapsto \sum_{v \in S_n} \mathfrak{S}_{u,v} \cdot \sigma_v .$$

It follows from the definitions that the $\mathfrak{S}_{u,v}$ are polynomials with nonnegative integer coefficients.

The specialization of the $\mathfrak{S}_{u,v}$ for $q_1 = \cdots = q_{n-1} = 0$ are exactly the *skew Schubert polynomials* $\mathfrak{S}_{v/u} \in \mathbb{Z}[x_1, \ldots, x_n]$ recently introduced by Lenart and Sottile [L-S]. We remark that the quantum Schubert polynomials $\mathfrak{S}^q_w$ from [FGP] *cannot* be obtained by a specialization of the path Schubert polynomials $\mathfrak{S}_{u,v}$. Indeed, all coefficients of $\mathfrak{S}_{u,v}$ are nonnegative, whereas $\mathfrak{S}^q_w$ may have negative coefficients. Nevertheless, as we will see, the polynomials $\mathfrak{S}_{u,v}$ are intimately related to the quantum cohomology $\mathrm{QH}^*(\mathrm{Fl}_n)$.

The definition of path Schubert polynomials $\mathfrak{S}_{u,v}$ can be reformulated in terms of paths in the quantum Bruhat graph. Let $\Gamma_n = \Gamma_\Phi$ be the quantum Bruhat graph for a type A_{n-1} root system Φ. The graph Γ_n is a certain directed graph on the set of permutations in S_n. Each edge e in Γ_n is assigned a certain weight and a certain label. Two permutations u and v are connected by the directed edge $e : u \xrightarrow{i\,j} v$ labeled (i, j), $1 \leq i < j \leq n$, if and only if $v = us_{ij}$ and one of the following two conditions is satisfied:

$$\ell(v) = \ell(u) + 1 \quad \text{or} \quad \ell(v) = \ell(u) + 1 - 2(j-i).$$

If $\ell(v) = \ell(u)+1$, then the weight of edge e equals 1, and if $\ell(v) = \ell(u)+1-2(j-i)$, then the weight of edge e equals $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$. The weight of a directed path in the graph Γ_n is the product of the weights of its edges.

Let us say that a directed path $u_0 \xrightarrow{a_1\, b_1} u_1 \xrightarrow{a_2\, b_2} \cdots \xrightarrow{a_i\, b_i} u_i$ in the graph Γ_n is y^i_k*-admissible* if $a_1 \leq \cdots \leq a_i \leq k < b_1, \ldots, b_i$ and $b_1, \ldots, b_i$ are distinct; cf. the definition (10) of quantum Pieri operators $\mathcal{H}^{(k)}_i$. More generally, let us say that a path P in the graph Γ_n is y^β*-admissible*, for a monomial $y^\beta = y_1^{\beta_1} \dots y_{n-1}^{\beta_{n-1}}$, if the path P is a concatenation of $n-1$ paths $P = P_1 \circ \dots \circ P_{n-1}$ such that P_k is $y_k^{\beta_k}$-admissible, for $k = 1, \dots, n-1$.

By definition, the path Schubert polynomial $\mathfrak{S}_{u,v}$ is equal to

$$\mathfrak{S}_{u,v} = \sum_\beta y^{\delta-\beta} \sum_P \mathrm{weight}(P)$$

where the second summation is over all y^β-admissible paths P in the graph Γ_n with initial vertex u and terminal vertex v.

Every polynomial in the y_i with $\deg_{y_i} \le n-i$ can be expressed as a linear combination of the usual Schubert polynomials $\mathfrak{S}_w(y) = \mathfrak{S}_w(y_1, \dots, y_n)$. The next theorem claims that the coefficients in such an expansion for path Schubert polynomials are exactly the Gromov-Witten invariants.

Theorem 4. *For any $u, v \in S_n$, we have*

$$\mathfrak{S}_{u,w_\circ v}(y,q) = \sum_{w,\,d} \langle \sigma_u, \sigma_v, \sigma_{w_\circ w}\rangle_d \, q^d \, \mathfrak{S}_w(y),$$

where the sum is over $w \in S_n$ and $d \in \mathbb{Z}_{\ge 0}^{n-1}$. In other words, the coefficient of $\mathfrak{S}_w$ in the Schubert expansion of $\mathfrak{S}_{u,w_\circ v}$ is equal to the coefficient of the Schubert class σ_w in the quantum product $\sigma_u * \sigma_v$.

This statement is a generalization to the quantum cohomology of a recent result by Lenart and Sottile [L-S, Theorem 1]. On the other hand, it is a flag manifold analog of [Po3, Theorem 6.3].

Theorem 4 implies, in particular, that

$$\mathfrak{S}_{w,w_\circ}(y,q) = \mathfrak{S}_{1,w_\circ w}(y,q) = \mathfrak{S}_w(y)$$

is the usual Schubert polynomial. Thus all y^β-admissible paths in Γ_n with the initial vertex 1 or with the terminal vertex $w_\circ$ have weight 1, i.e., they are increasing saturated chains in the Bruhat order.

Following [L-S, Remark 6], note that both sides of the identity in Theorem 4 can be interpreted in terms of paths in the quantum Bruhat graph. It would be interesting to present a bijection between these two collections of paths. Such a bijection would provide a generalization of *jeu de taquin* to paths in the quantum Bruhat graph. Actually, such a bijection would imply a combinatorial interpretation for the Gromov-Witten invariants $\langle \sigma_u, \sigma_v, \sigma_{w_\circ w}\rangle_d$ similar to the jeu de taquin version of the classical Littlewood-Richardson rule.

Theorem 4 easily follows from the Cauchy identity (5), the elementary quantization rule (9), and the quantum Pieri formula (11).

Proof of Theorem 4. Taking the image of the Cauchy identity (5), with the expanded left-hand side (6), in the ring $\mathrm{H}^*(\mathrm{Fl}_n) \otimes \mathbb{Z}[y_1, \dots, y_n]$, we get

$$\sum_\beta y^{\delta-\beta} \, e_{\beta_1}^{(n-1)} \cdots e_{\beta_{n-1}}^{(1)} = \sum_{w \in S_n} \mathfrak{S}_w(y) \cdot \sigma_{w w_\circ}.$$

According to the elementary quantization rule (9), we can write the following identity in the ring $\mathrm{QH}^*(\mathrm{Fl}_n) \otimes \mathbb{Z}[y_1, \dots, y_n]$:

$$\sum_\beta y^{\delta-\beta} \, e_{\beta_1}^{(n-1)} * \cdots * e_{\beta_{n-1}}^{(1)} = \sum_{w \in S_n} \mathfrak{S}_w(y) \cdot \sigma_{w w_\circ}.$$

Applying the involution $\omega : \mathrm{QH}^*(\mathrm{Fl}_n) \to \mathrm{QH}^*(\mathrm{Fl}_n)$, we get

$$\sum_\beta y^{\delta-\beta}\, h^{(1)}_{\beta_1} * \cdots * h^{(n-1)}_{\beta_{n-1}} = \sum_{w\in S_n} \mathfrak{S}_w(y)\cdot \sigma_{w_\circ w}\,.$$

Since the operators of quantum multiplications by the $h_i^{(k)}$ are given by the quantum Pieri operators $\mathcal{H}_i^{(k)}$, the previous identity implies that

$$\mathcal{H}(y) : \sigma_u \mapsto \sum_{w\in S_n} \mathfrak{S}_w(y)\cdot(\sigma_{w_\circ w} * \sigma_u).$$

The last formula is equivalent to the claim of the theorem. □

Observe that the Cauchy identity that appears in the proof of Theorem 4 was an essential ingredient in the approach of Kirillov and Maeno [K-M] to quantum Schubert calculus.

Theorem 4 implies a combinatorial description for the set of monomials q^d in the quantum parameters that occur in the quantum product of two Schubert classes. Let us say that a directed path $u_0 \xrightarrow{a_1 b_1} u_1 \xrightarrow{a_2 b_2} \cdots \xrightarrow{a_l b_l} u_l$ in the graph Γ_n is *admissible* if there exists a sequence $k_1 \le \cdots \le k_l$ such that $a_i \le k_i < b_i$, for $i = 1, \ldots, l$, and all pairs $(k_1, b_1), \ldots, (k_l, b_l)$ are distinct.

Corollary 5. *For any $u, v \in S_n$ and $d \in \mathbb{Z}^{n-1}_{\ge 0}$, the monomial q^d occurs in $\sigma_u * \sigma_v$ if and only if there exists an admissible path of weight q^d from u to $w_\circ v$ in the graph Γ_n.*

Proof. Let $\mathfrak{S}_{u,v,d} \in \mathbb{Z}[y_1, \ldots, y_n]$ be the coefficient of q^d in the path Schubert polynomial $\mathfrak{S}_{u,v}$. By Theorem 4, a monomial q^d occurs in $\sigma_u * \sigma_v$ if and only if $\mathfrak{S}_{u,w_\circ v,d}$ is nonzero. On the other hand, by the definition of path Schubert polynomials, $\mathfrak{S}_{u,w_\circ v,d}$ is nonzero if and only if there exists a path from u to $w_\circ v$ of weight q^d, which is y^β-admissible, for some β. Every y^β-admissible path is admissible and every admissible path reduces to a y^β-admissible path. □

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