Noncommutative Lagrange Theorem and Inversion Polynomials

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This is a part of a handwritten manuscript of the same authors. To read it is enough to be familiar with basic definitions, properties and notations of the theory of quasideterminants of noncommutative matrices. (See the paper by I.M. Gelfand and V.S. Retakh, "A theory of noncommutative determinants and characteristic functions of graphs," Funct. Anal. Appl., 26 (1992), 1–20; Publ. LACIM. UQAM, Montreal. 14, 1–26; or
D. Krob and B. Leclerc, "Minor identities for quasideterminants and quantum determinants," preprint LITP 93.46 Paris, 1993; or I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.-Y. Thibon, "Noncommutative symmetric functions," DIMACS Technical Reports 94–28, 1994.)

Chapter III.

Noncommutative Lagrange Theorem

Section 3.1. Algebraic Proof.

3.1.1. Let $a_0, a_1, a_2, \ldots$ be non commuting formal variables and R be a ring of formal series generated by these variables.

Theorem 3.1. A functional equation in R

$$f = a_0 + a_1 f + a_2 f^2 + a_3 f^3 + \cdots \tag{3.1.1}$$

has a unique solution $f \in R$ and

a) $f = |A|_{11}^{-1} a_0$

where

$$A = \begin{pmatrix} 1-a_1 & -a_2 & -a_3 & \ldots \\ -a_0 & 1-a_1 & -a_2 & \ldots \\ 0 & -a_0 & 1-a_1 & \ldots \\ 0 & 0 & -a_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.1.2}$$

is an infinite matrix and

b) $f = P \cdot Q^{-1}$

where

$$P = a_0 + \sum_{i_1+i_2+\cdots+i_{l+1}=l} a_{i_1} a_{i_2} \ldots a_{i_{l+1}}$$

$$Q = 1 + \sum_{i_1+i_2+\cdots+i_l=l} a_{i_1} a_{i_2} \ldots a_{i_l}$$

Here $l \geq 1,\ i_k \geq 0,\ k = 1, 2, 3, \ldots$.

Proof. Set $\deg(a_{i_1} \ldots a_{i_l}) = i_1 + \cdots + i_l$. Let

$$f = f_0 + f_1 + f_2 + \cdots$$

be an expansion f into homogeneous components: $\deg f_i = i$. Then (3.1.1) gives us

$$f_0 = a_0, \quad f_1 = a_1 f_0, \quad f_2 = a_1 f_1 + a_2 f_0^2,$$

$$f_3 = a_1 f_2 + a_2(f_0 f_1 + f_1 f_0) + a_3 f_0^3, \ldots$$

So, every f_{i+1} can be written in a unique way via $f_0, \ldots, f_i$. It follows that (3.1.1) has a unique solution.

To prove a), we need the following:

Lemma 3.1.2. Let A be matrix (3.1.2). Then

$$|A|_{1j} = (|A|_{11} \cdot a_0^{-1})^j a_0.$$

Proof. From the "homological relations" for quasideterminants one has

$$|A|_{1j} \cdot |A^{11}|_{2j}^{-1} = -|A|_{11} \cdot |A^{1j}|_{21}^{-1}.$$

Note that

$$|A^{11}|_{2j} = |A|_{1j-1}, \quad |A^{1j}|_{21} = -a_0.$$

So

$$|A|_{1j} = |A|_{11} a_0^{-1} |A|_{1j-1}.$$

By induction we obtain

$$|A|_{1j} = (|A|_{11} a_0^{-1})^j a_0.$$

Let us prove that $f = |A|_{11}^{-1} \cdot a_0$ satisfies (3.1.1). By the definition of quasideterminant

$$|A|_{11} = 1 - a_1 - a_2 |A^{11}|_{22}^{-1} a_0 - a_3 |A^{11}|_{23}^{-1} a_0 - \ldots .$$

So by Lemma 3.1.2

$$|A|_{11} = 1 - a_1 - a_2 |A|_{11}^{-1} a_0 - a_3 (|A|_{11}^{-1} a_0)^2 - \cdots$$

Multiplying by $f = |A|_{11}^{-1} a_0$ from the right we obtain

$$a_0 = f - a_1 f - a_2 f^2 - a_3 f^3 - \ldots$$

and a) is proved.

To prove b) we need the following useful

Proposition 3.1.3. Let $B = (b_{ij})$, $1 \leq i, j \leq n$. Let E_n be the unit $n \times n$ matrix. Then

$$|E_n - B|_{ji}^{-1} = \delta_{ij} + \sum_{l \geq 0} \sum_{1 \leq k_1, \ldots, k_l \leq n} b_{ik_1} b_{k_1 k_2} \cdot \ldots \cdot b_{k_l j}.$$

where δ_{ij} is the Kronecker delta.

Proof. Use the expansion

$$(E_n - B)_{ji}^{-1} = E_n + B + B^2 + B^3 + \cdots$$

Let us return to our proof. We will need one more

Lemma 3.1.4. Let

$$C = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$$

where X, Y, Z, T are $k \times k$, $k \times n$, $n \times k$, and $n \times n$-matrices respectively.

Suppose that

$$Z = \begin{pmatrix} 0 & \dots & 0 & -\alpha \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

Then,

$$|T|_{11}^{-1}\alpha = |C|_{kk+1}^{-1} \cdot |C|_{kk}.$$

<u>Proof</u>. According to the "homological relations"

$$|C|_{kk+1} \cdot |C^{k\,k}|_{k+1\,k+1}^{-1} = -|C|_{k\,k} \cdot |C^{k\,k+1}|_{k+1\,k}^{-1}.$$

Note that

$$|C^{k\,k}|_{k+1\,k+1} = |T|_{11}, \quad |C^{k\,k+1}|_{k+1\,k} = -\alpha.$$

So

$$|T|_{11}^{-1}\alpha = |C|_{k\,k+1}^{-1} \cdot |C|_{k\,k}.$$

Let us return to matrix (3.1.2) and let $A_{(n)}$ be its $n \times n$-submatrix

$$A_{(n)} = \begin{pmatrix} 1-a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ -a_0 & 1-a_1 & \dots & -a_{n-2} & -a_{n-1} \\ 0 & -a_0 & \dots & -a_{n-3} & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -a_0 & 1-a_1 \end{pmatrix}$$

It follows from Lemma 3.1.4 that

$$|A_{(n)}|_{11}^{-1} a_0 = |A_{(n+k)}|_{k\,k+1}^{-1} \cdot |A_{(n+K)}|_{k\,k}.$$

Set

$$P_{n,k} = |A_{(n+k)}|_{k\,k+1}^{-1}; \quad Q = A_{(n,k)}|_{k\,k}^{-1}.$$

It is evident that

$$f = |A|_{11}^{-1} \cdot a_0 = \lim_{n \to \infty} |A_{(n)}|^{-1} a_0 = P_k \cdot Q_k^{-1}$$

where

$$P_k = \lim_{n \to \infty} P_{n,k} = |A|_{k\,k+1}^{-1}, \quad Q_k = \lim_{n \to \infty} Q_{n,k} = |A|_{k\,k}^{-1}.$$

By Proposition 3.1.3

$$P_k = a_0 + \sum_{l>0} \sum_{i_1+i_2+\ldots+i_{l+1}=l} a_{i_1} a_{i_2} \ldots a_{i_{l+1}}$$

where the sum is taken over all sequences $(i_1, \ldots, i_{l+1})$ of nonnegative integers such that for any $1 \le r \le l+1$

$$i_1 + i_2 + \ldots + i_r - r \ge -k + 1$$

By the same proposition

$$Q_k = 1 + \sum_{l>0} \sum_{i_1+i_2+\ldots+i_l=l} a_{i_1} a_{i_2} \ldots a_{i_l}$$

where the sum is taken over all sets $(i_1, \ldots, i_{l+1})$ of nonnegative integers, such that for any $1 \le r \le l$

$$i_1 + i_2 + \ldots + i_r - r \ge -k + 1.$$

As $k \to +\infty$ then $P_k \to P$, $Q_k \to Q$. Hence $f = PQ^{-1}$. The theorem is proven.

3.1.2. <u>q-Analog of Lagrange Theorem</u>. In this subsection we will obtain the Gessel's q-analog of commutative Lagrange Theorem.

Let $\varphi = \varphi(z, q)$ be a function of two variables. Set

$$\varphi^{(n)}(z, q) = \varphi(zq^{n-1}, q)\varphi(zq^{n-2}, q) \cdot \ldots \cdot \varphi(z, q),$$

$$\varphi_{(n)}(z, q) = \varphi(z, q)\varphi(zq^{-1}, q) \cdot \ldots \cdot \varphi(zq^{-n+1}, q)$$

<u>Theorem 3.1.5.</u> (cf. [Gessel]). Let z be a formal variable and $q, g_0, g_1, g_2, \ldots$ be formal parameters commuting with z (they need not commute each other).

Let G be a ring of formal series generated by g_i, $i \ge 0$. Then a functional equation in G

$$F(z, q) = zq(g_0 + g_1 F(z, q) + g_2 F^{(2)}(z, q) + g_3 F^{(3)}(z, q) + \ldots, \tag{3.1.3}$$

has a unique solution $F(z, q)$ and

a) $F(z, q) = |\widetilde{A}|_{11}^{-1}$, where

$$\widetilde{A} = \begin{pmatrix} 1 - g_1 qz & -g_2 q^3 z^2 & -g_3 q^6 z^3 & \ldots & -g_n q^{\binom{n+1}{2}} z^n & \ldots \\ -g_0 & 1 - g_1 q^2 z & -g_2 q^5 z^2 & \ldots & -g_{n-1} q^{\binom{n}{2}+n-1} z^{n-1} & \ldots \\ 0 & -g_0 & 1 - g_1 q^3 z & \ldots & -g_{n-2} q^{\binom{n-1}{2}+2(n-2)} z^{n-2} & \ldots \\ 0 & 0 & -g_0 & \ldots & -g_{n-3} q^{\binom{n-2}{2}+3(n-3)} z^{n-3} & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

b)

$$F(z, q) = \frac{z \cdot \sum_{n\ge 0} q^{\binom{n+2}{z}} z^n \cdot [t^n]\, g_{(n+1)}\left(q^{-1}t, q\right)}{\sum_{n\ge 0} q^{\binom{n+1}{2}} z^n \cdot [t^n]\, g_{(n)}\left(q^{-1}t, q\right)}$$

where

$$g(\xi, q) = g_0 + g_1\xi + g_2\xi^2 + \ldots$$

and

$$[t^n] \cdot \left(\sum_{n\ge 0} \varphi_n t^n\right) := \varphi_n.$$

Proof. We show that Theorem 3.1.5 follows from noncommutative Lagrange Theorem 3.1.1. Let x, y be q-commuting variables, i.e. $xy = qyx$. Set $z = yx$.

In the notations of Theorem 3.1.1 we set

$$a_i = g_i x^i y, \quad i = 0, 1, 2, \dots$$

and suppose that g_i commute with x and y.

Let $F = f \cdot xq$.

Note that F is a function of z and q. It follows from the representation of f as PQ^{-1} and the formulas for the formal series P and Q.

From the obvious identity

$$x\varphi(z, q) = \varphi(zq, q)x$$

it follows that

$$f^n = \left(F \cdot (xq)^{-1}\right)^n = (xq)^{-n+1} F^{(n)} (xq)^{-1}$$

Let us set $f = F \cdot (xq)^{-1}$ and $a_i = q_i x^l y$ in functional equation (3.1.1):

$$F \cdot (xq)^{-1} = g_0 y + g_1 xyF \cdot (xq)^{-1} + g_2 x^2 y \left(F \cdot (xq^{-1})\right)^2 + \dots$$

So

$$F = qz(g_0 + g_1 F + g_2 F^{(2)} + \dots)$$

and we get formula (3.1.3).

Let us prove a). By Theorem 3.1.1 $F = fxq = |A|_{11}^{-1} g_0 yxq = |A|_{11}^{-1} g_0 zq$, where the matrix A is as follows:

$$A = \begin{pmatrix} 1 - g_1 xy & -g_2 x^2 y & -g_3 x^3 y & \dots \\ -g_0 y & 1 - g_1 xy & -g_2 x^2 y & \dots \\ 0 & -g_0 y & 1 - g_1 xy & \dots \\ 0 & 0 & -g_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Multiply j-th column in A by y^{j-1} from the right and i-th row in A by y^{1-i} from the left for all $i, j \geq 2$. We get the matrix $\widetilde{A}$. Note that $|A|_{11} = |\widetilde{A}|_{11}$. Hence $F = |\widetilde{A}|_{11}^{-1} g_0 zq$ and a) is proven.

Now prove b). By Theorem 3.1.1 we have

$$F = fxq = P \cdot Q^{-1} xq,$$

where

$$
\begin{aligned}
P &= a_0 + \sum_{l\geq 1} \sum_{i_1+i_2+\ldots+i_{l+1}=l} a_{i_1} a_{i_2} \ldots a_{i_{l+1}} = \\
&= g_0 y + \sum_{l\geq 1} \sum_{i_1+i_2+\ldots+i_{l+1}=l} (g_{i_1} x^{i_1} y) \ldots (g_{i_{l+1}} x^{i_{l+1}} y) = \\
&= g_0 y + \sum_{l\geq 1} \sum_{i_1+i_2+\ldots+i_{l+1}=l} x^l y^{l+1} q^{-(i_2+2i_3+\ldots+l i_{l+1})} g_{i_1} g_{i_2} \ldots g_{i_{l+1}} = \\
&= \left\{ q g_0 z + \sum_{l\geq 1} \sum_{i_1+i_2+\ldots+i_{l+1}=l} x^{l+1} y^{l+1} q^{-(i_1+2i_2+\ldots+(l+1)i_{l+1})} \cdot g_{i_1} g_{i_2} \ldots g_{i_{l+1}} \right\} (qx)^{-1} = \\
&= \left\{ q g_0 z + \sum_{l\geq 1} \sum_{i_1+i_2+\ldots+i_{l+1}=l} (yx)^{l+1} q^{\binom{l+2}{2}} q^{-i_1} g_{i_1} \cdot q^{-2i_2} g_{i_2} \ldots q^{-(l+1)i_{l+1}} g_{i_{l+1}} \right\} (qx)^{-1} = \\
&= z \cdot \left\{ \sum_{l\geq 1} z^l q^{\binom{l+2}{2}} \left[t^l\right] g_{(l+1)}(y^{-1}t, q) \right\} (qx)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
Q &= 1 + \sum_{l\geq 1} \sum_{i_1+i_2+\ldots+i_l=l} a_{i_1} a_{i_2} \cdots a_{i_l} = \\
&= 1 + \sum_{l\geq 1} \sum_{i_1+i_2+\ldots+i_l=l} (g_{i_1} x^{i_1} y) \cdots (g_{i_l} x^{i_l} y) = \\
&= 1 + \sum_{l\geq 1} \sum_{i_1+i_2+\ldots+i_l=l} x^l y^l g^{-(i_2+2i_3+\cdots+(l-1)i_l)} \cdot g_{i_1} g_{i_2} \ldots g_{i_l} = \\
&= x \left\{ 1 + \sum_{l\geq 1} \sum_{i_1+i_2+\ldots+i_l=l} x^l y^l g^{-(i_2+2i_3+\cdots+(l-1)i_l)} \cdot g_{i_1} g_{i_2} \ldots g_{i_l} \right\} x^{-1} = \\
&= x \left\{ 1 + \sum_{l\geq 1} \sum_{i_1+i_2+\ldots+i_l=l} z^l q^{\binom{l+1}{2}} q^{-i_1} g_{i_1} q^{-i_2} g_{i_2} \ldots q^{-l i_l} g_{i_l} \right\} x^{-1} = \\
&= x \left\{ \sum_{l\geq 0} z^l q^{\binom{l+1}{2}} \left[t^l\right] g_{(l)}(q^{-1}t, q) \right\} x^{-1}.
\end{aligned}
$$

We get the following Gessel's formula for $F = PQ^{-1} \cdot xq$.

$$
F = \frac{z \sum_{l\geq 0} q^{\binom{l+2}{2}} z^l \left[t^l\right] g_{(l+1)}(q^{-1}z, q)}{\sum_{l\geq 0} q^{\binom{l+1}{2}} z^l \left[t^l\right] g_{(l)}(q^{-1}z, q)}.
$$

Section 3.2. Combinatorial Proof of Noncommutative Lagrange Theorem (following [Gessel])

3.2.1. Let $S = \{b_{-1}, b_0, b_1, \ldots, b_u, \ldots\}$ and let S^* be the free monoid generated by $b_i \in S$.

For any word $w = b_{i_1} b_{i_2} \ldots b_{i_l}$ in S^* define its height $v(w) = i_1 + i_2 + \ldots + i_l$. Then $v : S^* \to \mathbf{Z}_+$ will be a semi-group homomorphism. A word $v \in S^*$ is called a left factor of w if $w = v \cdot u$ for some $u \in S^*$.

Consider

$$R = \{w \in S^* \mid v(w) = 0 \text{ and } v(v) \geq 0 \text{ for every left factor } v \text{ of } w\}.$$

Then R is a submonoid in S^*. Set $F = Rb_{-1}$.

Proposition 3.2.1. Let w be a word $w \in F$. Suppose that w starts with b_{j-1}, $j \leq 0$. Then the word w can be uniquely decomposed as $w = b_{j-1}w_1 \dots w_j$, where $w_i \in F$ for $i = 1, \dots, j$. Hence

$$F = \coprod_{j=0}^{\infty} b_{j-1} F^j$$

Let $G_k = \{w \in S^*, v(w) = k\}$, $k \in \mathbf{Z}$.

Proposition 3.2.2. Any $w \in G_k$ has a unique decomposition $w = uv$, where $u \in F$, $v \in G_{k+1}$.

Let us show that both these propositions imply Lagrange Theorem. Set

(*missing equation?*)

where the sum is considered in the semi-group ring of monoid S^*. From Proposition 3.2.1 it follows that

$$f = \sum_{j=0}^{\infty} b_{j-1} f^j \tag{3.2.1}$$

and Proposition 3.2.2 implies

$$f \cdot g_{k+1} = g_k, \ k \in \mathbf{Z}. \tag{3.2.2}$$

Consider the following almost upper triangular infinite matrix

$$B = \begin{pmatrix} b_0 & b_1 & b_2 & \dots \\ b_{-1} & b_0 & b_1 & \dots \\ 0 & b_{-1} & b_0 & \dots \\ 0 & 0 & b_{-1} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.2.3}$$

Then

$$|E - B|_{11} \cdot b_{-1} = \sum_{i=0}^{\infty} |B^i|_{11} \cdot b_{-1} = \left(\sum_{w \in R} w \right) b_{-1} = f \tag{3.2.4}$$

Here E is an infinite unite matrix. Setting $a_j = b_{j+1}$, $j = 0, 1, 2, \dots$ and combining (3.2.1) and (3.2.4) we get Theorem 3.1.1.a).

It is easy to see that for $a_j = b_{j+1}$ we have $g_0 = Q$, $g_1 = P$ in notations of Theorem 3.1.1. It proves b) by (3.2.2). Moreover (3.2.2) gives a generalization of Theorem 3.1.1.b).

Proof of Proposition 3.2.1. Let us fix a word $w = a_{i_1} a_{i_2} \dots a_{i_l} \in F$. Set $j = i_1 + 1$. Consider all left factors $w_m = a_{i_1} a_{i_2} \dots a_{i_m}$, $1 \leq m \leq l$ of the word w. Then $v(w_1) = v(a_{i_1}) = i_1$, $v(w_l) = -1$. Consider the set of indices $2 \leq m_1, \dots, m_j \leq m$ such that $v(w_{m_k} =$

$i_1-k,\ 1\le k\le j$ and $v(w_r)>v(w_{m_k}),\ 1\le r\le m_k$. It is easy to see that this set is correctly defined..

Let $w_{m_k}=w_{m_{k-1}}\cdot u_k,\ 1\le k\le j$. By definition, $u_k\in F_k$ for $1\le k\le j$. Then $w=a_{j-1}u_1u_2\dots u_j$ is the decomposition we need. Its uniqueness follows directly from the uniqueness of the set $(m_k),\ 1\le k\le j$.

Proof of Proposition 3.2.2. Let us fix a word $w=a_{i_1}a_{i_2}\dots a_{i_k}\in G_k$ and consider its left factors $w_m=a_{i_1}a_{i_2}\cdot\ldots\cdot a_{i_m},\ 0\le m\le l$. Then $v(w_0)=0,\ v(w_l)=v(w)=k<0$. Then there exists a unique $r,\ 1\le r\le 1$ such that $v(w_r)=-1,\ v(w_s)\ge 0,\ 0\le s<r$. Then $w_r\in F$. Let $w=w_r\cdot u$. Then $u\in G_{k+1}$, i.e. $w=w_ru$ is a decomposition we looked for. The uniqueness of r implies the uniqueness of the decomposition.

Remark. Propositions 3.2.1 and 3.2.2 have a simple description in terms of paths on the plane (see [GJ]). With a word $w=a_{i_0}a_{i_1}\dots a_{i_n}$, where i_j are integers, we associate a path ϑ_w connecting the vertices $(0,i_0),\ (1,i_1),\dots,(n,i_n)$. Then all paths θ_w in upper half-plane are in $1-1$ correspondence with the words from R. In these terms, Proposition 3.2.1 is in fact the statement (5.2.9) from [GJ].

However, we prefer to use the language of monoids because of possible generalizations and modifications. The idea of using submonoids of words with the left factors of nonnegative height was used earlier in [??].

Section 3.3. Application: An Expression for Ramanujan Continued Fraction.

Let x and y be two non commutating formal variables. Consider the continued fraction

$$K(x,y)=\cfrac{1}{1-x\cfrac{1}{1-x\cfrac{1}{1-x\cfrac{1}{1-\dots}y}y}y}$$

which is equal to $|C|_{11}^{-1}$, where

$$C=\begin{pmatrix}1&x&0&0&\dots\\ y&1&x&0&\dots\\ 0&y&1&x&\dots\\ 0&0&y&1&\dots\\ \vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

is a Jacobian matrix.

Set in Theorem 3.1.1 $a_i=x^iy$.

Proposition 3.3.1. In the notations of Theorem we have 3.1.1

$$f=K(x,y)\cdot y.$$

Proof. It is enough to prove that $|C|_{11}^{-1} = |A|_{11}^{-1}$ where

$$A = \begin{pmatrix} 1-xy & -x^2y & -x^3y & \dots \\ -y & 1-xy & -x^2y & \dots \\ 0 & -y & 1-xy & \dots \\ 0 & 0 & -y & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Let $A_{(n)}$ be $n \times n$ submatrix of A in the first n rows and columns. Subtract from the 1st row in the matrix $A_{(n)}$ the 2d row multiplied by x from the left, then subtract from the 2d row the 3d row multiplied by x from the left, etc. Under these operations the quasideterminant $|A|_{11}$ will not change and matrix $A_{(n)}$ will be transformed into the matrix

$$\begin{pmatrix} 1 & -x & 0 & \dots & 0 \\ -y & 1 & -x & \dots & 0 \\ 0 & -y & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1-xy \end{pmatrix}$$

For a monomial $u = x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}\dots x^{\alpha_n}y^{\beta_n}$ denote by $\alpha(u)$ the sum $\alpha_1 + \alpha_2 + \dots + \alpha_n$ and by $\beta(u)$ the sum $\beta_1 + \beta_2 + \dots + \beta_n$. Let

$$P = y + xy^2 + yxy + y^2x + \cdots$$

be formal sum of all monomials u such that $\alpha(u) + 1 = \beta(u)$ and

$$Q = 1 + xy + yx + x^2y^2 + \cdots$$

be the formal sum of all monomials v such that $\alpha(v) = \beta(v)$.

It follows from Theorem 3.1.1 and Proposition 3.3.1 that

$$K(x,y) = PQ^{-1}y^{-1}.$$

Suppose that $xy = qyx$ and q commutes with $z = yx$. Then

$$K = \cfrac{1}{1 - qz\cfrac{1}{1 - q^2z\cfrac{1}{1 - q^3z\cfrac{1}{1-\cdots}}}}$$

is the Ramanujan continued fraction. Now

$$P = \sum_{n\geq 0} y^{n+1}x^n \binom{2n+1}{n}_q = \left\{\sum_{n\geq 0}(yx)^n q^{-\binom{n+1}{2}}\binom{2n+1}{n}_q\right\} y = \left\{\sum_{n\geq 0} z^n q^{-\binom{n+1}{2}}\binom{2n+1}{n}_q\right\} y,$$

$$Q = \sum_{n\geq 0} y^n x^n \binom{2n}{n}_q = y^{-1}\left\{\sum_{n\geq 0} y^{n+1}x^n y^{-1}\binom{2n}{n}_q\right\} y = y^{-1}\left\{\sum_{n\geq 0} z^n q^{-\binom{n+1}{2}}\binom{2n}{n}_q\right\} y.$$

We get the following theorem.

Theorem 3.3.2.

$$\frac{1}{1-\dfrac{qz}{1-\dfrac{q^2z}{1-\ldots}}}=\frac{\sum\limits_{n\geq 0} z^n q^{-\binom{n+1}{2}}\binom{2n+1}{n}_q}{\sum\limits_{n\geq 0} z^n q^{-\binom{n+1}{2}}\binom{2n}{n}_q}$$

Section 3.4. Noncommutative Inversion Polynomials.

3.4.1. Let T_n, $n \geq 0$ be the set of all trees with $n+1$ vertices $0, 1, 2, \ldots, n$. A pair of vertices (i, j), $i < j$ of a tree t is called an *inversion* of t if the vertex j belongs to the path connecting the vertices 0 and i. Let $\operatorname{inv}(t)$ be the number of all inversion of $t \in T_n$. The polynomial

$$J_n(q) = \sum_{t\in T_n} q^{\operatorname{inv}(t)}$$

is called the *inversion polynomial.* It is know that $J_n(q)$ satisfies the following recurrence relation.

$$J_n(q) = \sum_{k=1}^{n} \binom{n-1}{k-1} (k+1)_q J_k(q) J_{n-k-1}.$$

A sequence of integers $a = (a_1, a_2, \ldots, a_n)$, $1 \leq a_1, \ldots, a_n \leq n$ such that $|\{i \mid a_i \leq k\}| \leq k$ for $k = 1, 2, \ldots, n$ is called a *majorant sequence* of length n.

Denote by $\widetilde{M}_n$ the set of all majorant sequences of length n. The number $s(b) = b_1 + b_2 + \ldots + b_n - \binom{n+1}{2}$ is called a *weight* of $b = (b_1, \ldots, b_n) \in \widetilde{M}_n$. It is easy to see that $s(b) \geq 0$. Let $M_n(q) = \sum\limits_{b\in\widetilde{M}_n} q^{s(b)}$.

Theorem 3.4.1. (Kreweras).

$$J_n(q) = M_n(q).$$

3.4.2. Noncommutative Inversion Polynomials.

Let x, y be two noncommutative formal variables. Set $M_i(b) = |\{j \mid b_j = n - i + 1\}|$ for $b = (b_1, b_2, \ldots, b_n) \in \widetilde{M}_n$.

Definition 3.4.1.The following polynomial is call the *noncommutative inversion polynomial.*

$$I_n(x, y) = \sum_{b\in\widetilde{M}_n} x^{M_1(b)} y\, x^{M_2(b)} y \ldots x^{M_n(b)} y = I_n(x, y).$$

The relation of $I_n(x, y)$ with the polynomials M_n and J_n is given by the following proposition.

Proposition 3.4.1. Let $xy = qxy$ and $yx = z$. Then

$$I_n(x,y) = M_n(q)(qz)^n = J_n(q)(qz)^n$$

Proof. It is sufficient to check that

$$X^{M_1(b)} y\, x^{M_2(b)} y \ldots x^{M_n(b)} y = q^{s(b)}.(xy)^n$$

This is true because

$$s(b) = M_1 \cdot n + M_2(n-1) + \ldots + M_n \cdot 1 - \binom{n+1}{2}.$$

A generating function for noncommutative inversion polynomials is given by the following theorem.

Theorem 3.4.2.

$$\sum_{n\geq 0} \frac{1}{n!} I_n(x,y) y = P \cdot Q^{-1},$$

where

$$P = \sum_{m_1+m_2+\ldots+m_{l+1}=l} \frac{1}{m_1!m_2!\cdots m_{l+1}!} x^{m_1} y\, x^{m_2} y \ldots x^{m_{l+1}} y,$$

$$Q = \sum_{m_1+\ldots+m_l=l} \frac{1}{m_1!m_2!\ldots m_l!} x^{m_1} y\, x^{m_2} y \ldots x^{m_l} y.$$

Proof. Use Theorem 3.1 for $a_m = \frac{1}{m!} x^m y$.

As a corollary we get the following theorem.

Theorem 3.4.3.

$$\sum_{n\geq 0} \frac{1}{n!} J_n(q)(qz)^n z = \frac{\displaystyle\sum_{l\geq 0} \frac{1}{l!} (qz)^{l+1} q^{-\binom{l+1}{2}} (1+q+\ldots+q^l)^l}{\displaystyle\sum_{l\geq 0} \frac{1}{l!} (qz)^l q^{-\binom{l+1}{2}} (1+q+\ldots+q^{l-1})^l}.$$

Proof. Set $xy = qyx$ and $yx = z$ in the notations of Theorem 3.4.2. Then

$$\begin{aligned}
P &= \sum_{m_1+m_2+\ldots+m_{l+1}=l} \frac{1}{l!} \binom{l}{m_1 m_2 \ldots m_{l+1}} y^{l+1} x^l \cdot q^{m_1(l+1)+m_2\cdot l+\ldots+m_{l+1}\cdot 1} = \\
&= \sum_{m_1+\ldots+m_{l+1}=l} \frac{1}{e!} \binom{l}{m_1 m_2 \ldots m_{l+1}} (yx)^{l+1} q^{-\binom{l+1}{2}} \cdot q^{m_1(l+1)+m_2 l+\ldots+m_{l+1}\cdot 1} x^{-1} = \\
&= \sum_{l\geq 0} \frac{1}{e!} (qz)^{l+1} q^{-\binom{l+1}{2}} (1+q+\ldots+q^l)^l x^{-1};
\end{aligned}$$

$$
\begin{aligned}
Q &= \sum_{m_1+\ldots+m_l=l} \frac{1}{l!}\binom{l}{m_1 m_2 \ldots m_l} y^l x^l q^{m_1\cdot l+m_2\cdot(l-1)+\ldots+m_l}\\
&= \sum_{m_1+\ldots+m_l=l} \frac{1}{l!}\binom{l}{m_1 m_2 \ldots m_l} (qyx)^l q^{-\binom{l}{2}} \cdot q^{m_1(l-1)+m_2(l-2)+\ldots+m_l\cdot 0} =\\
&= \sum_{l\geq 0} \frac{1}{l!}(qz)^l a^{-\binom{l}{2}}(1+q+\ldots+q^{l-1})^l =\\
&= x\left(\sum_{l\geq 0} \frac{1}{l!}(qz)^l a^{-\binom{l+1}{2}}(1+q+\ldots+q^{l-1})^l\right) x^{-1}.
\end{aligned}
$$