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SYZYGIES OF ORIENTED MATROIDS

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Abstract. We construct minimal cellular resolutions of squarefree monomial ideals arising from hyperplane arrangements, matroids and oriented matroids. These are Stanley-Reisner ideals of complexes of independent sets, and of triangulations of Lawrence matroid polytopes. Our resolution provides a cellular realization of Stanley's formula for their Betti numbers. For unimodular matroids our resolutions are related to hyperplane arrangements on tori, and we recover the resolutions constructed by Bayer, Popescu and Sturmfels [2]. We resolve the combinatorial problems posed in [2] by computing Möbius invariants of graphic and cographic arrangements in terms of Hermite polynomials.

1. Cellular Resolutions from Hyperplane Arrangements

A basic problem of combinatorial commutative algebra is to find the syzygies of a monomial ideal $M = \langle m_1, \ldots, m_r \rangle$ in the polynomial ring $\mathbf{k}[\mathbf{x}] = \mathbf{k}[x_1, \ldots, x_n]$ over a field $\mathbf{k}$. One approach involves constructing *cellular resolutions*, where the i-th syzygies of M are indexed by the i-dimensional faces of a CW-complex on r vertices. After reviewing the general construction of cellular resolutions from [4], we shall define the monomial ideals and resolutions that are studied in this paper.

Let Δ be a *CW-complex* [12, §38] with r vertices $v_1, \ldots, v_r$, which are labeled by the monomials $m_1, \ldots, m_r$. We write $c \geq c'$ whenever a cell c' belongs to the closure of another cell c of Δ. This defines the face poset of Δ. We label each cell c of Δ with the monomial $m_c = \operatorname{lcm}\{m_i \mid v_i \leq c\}$, the least common multiple of the monomials labeling the vertices of c. Also set $m_\emptyset = 1$ for the empty cell of Δ. Clearly, $m_{c'}$ divides m_c whenever $c' \leq c$. The principal ideal $\langle m_c \rangle$ is identified with the free $\mathbb{N}^n$-graded $\mathbf{k}[\mathbf{x}]$-module of rank 1 with generator in degree $\deg m_c$. For a pair of cells $c \geq c'$, let $p_c^{c'} : \langle m_c \rangle \to \langle m_{c'} \rangle$ be the inclusion map of ideals. It is a degree-preserving homomorphism of $\mathbb{N}^n$-graded modules.

Fix an orientation of each cell in Δ, and define the *cellular complex* $C_\bullet(\Delta, M)$

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} = \mathbf{k}[\mathbf{x}]$$

as follows. The $\mathbb{N}^n$-graded $\mathbf{k}[\mathbf{x}]$-module of i-chains is

$$C_i \quad = \bigoplus_{c\,:\,\dim c = i} \langle m_c \rangle \ ,$$

where the direct sum is over all i-dimensional cells c of Δ. The differential $\partial_i : C_i \to C_{i-1}$ is defined on the component $\langle m_c \rangle$ as the alternating sum of the maps $p_c^{c'}$:

$$\partial_i \quad = \sum_{c' \leq c,\, \dim c' = i-1} [c : c']\ p_c^{c'} \ ,$$

where $[c : c'] \in \mathbb{Z}$ is the *incidence coefficient* of oriented cells c and c' in the usual topological sense. For a regular CW-complex, the incidence coefficient $[c : c']$ is $+1$ or -1 depending on the orientation of cell c' in the boundary of c. The differential

∂_i preserves the $\mathbb{N}^n$-grading of $\mathbf{k}[\mathbf{x}]$-modules. Note that if $m_1 = \cdots = m_r = 1$ then $C_\bullet(\Delta, M)$ is the usual chain complex of Δ over $\mathbf{k}[\mathbf{x}]$. For any monomial $m \in \mathbf{k}[\mathbf{x}]$, we define $\Delta_{\leq m}$ to be the subcomplex of Δ consisting of all cells c whose label m_c divides m. We call any such $\Delta_{\leq m}$ an *M-essential* subcomplex of Δ.

Proposition 1.1. [4, Proposition 1.2] *The cellular complex $C_\bullet(\Delta, M)$ is exact if and only if every M-essential subcomplex $\Delta_{\leq m}$ of Δ is acyclic over $\mathbf{k}$. Moreover, if $m_c \neq m_{c'}$ for any $c > c'$, then $C_\bullet(\Delta, M)$ gives a minimal free resolution of M.*

Proposition 1.1 is derived from the observation that, for a monomial m, the $(\deg m)$-graded component of $C_\bullet(\Delta, M)$ equals the chain complex of $\Delta_{\leq m}$ over $\mathbf{k}$. If both of the hypotheses in Proposition 1.1 are met, then we say that Δ is an *M-complex*, and we call $C_\bullet(\Delta, M)$ a *minimal cellular resolution* of M. Thus each M-complex Δ produces a minimal free resolution of the ideal M. In particular, for an M-complex Δ, the number $f_i(\Delta)$ of i-dimensional cells of Δ is exactly the *i-th Betti number* of M, i.e., the rank of the i-th free module in a minimal free resolution. Thus, for fixed M, all M-complexes have the same f-vector.

Examples of M-complexes appearing in the literature include planar maps [11], Scarf complexes [3] and hull complexes [4]. A general construction of M-complexes using discrete Morse theory was proposed by Batzies and Welker [1]. We next introduce a family of M-complexes which generalizes those in [2, Theorem 4.4].

Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be an arrangement of n affine hyperplanes in $\mathbb{R}^d$,

$$H_i \quad = \quad \{v \in \mathbb{R}^d \mid h_i(v) = c_i\}, \quad i = 1, \ldots, n, \tag{1}$$

where $c_1, \ldots, c_n \in \mathbb{R}$ and $h_1, \ldots, h_n$ are nonzero linear forms that span $(\mathbb{R}^d)^*$.

We fix two sets of variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, and we associate with the arrangement $\mathcal{A}$ two functions m_x and m_{xy} from $\mathbb{R}^d$ to sets of monomials:

$$m_x : v \longmapsto \prod_{i\,:\,h_i(v) \neq c_i} x_i \quad \text{and} \quad m_{xy} : v \longmapsto \left(\prod_{i\,:\,h_i(v) > c_i} x_i \right) \cdot \left(\prod_{j\,:\,v_j(v) < c_j} y_j \right).$$

Note that $m_x(v)$ is obtained from $m_{xy}(v)$ by specializing y_i to x_i for all i.

Definition 1.2. The *matroid ideal* of $\mathcal{A}$ is the ideal $M_\mathcal{A}$ of $\mathbf{k}[\mathbf{x}] = \mathbf{k}[x_1, \ldots, x_n]$ generated by the monomials $\{m_x(v) : v \in \mathbb{R}^d\}$. The *oriented matroid ideal* of $\mathcal{A}$ is the ideal $O_\mathcal{A}$ of $\mathbf{k}[\mathbf{x}, \mathbf{y}] = \mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ generated by $\{m_{xy}(v) : v \in \mathbb{R}^d\}$.

The hyperplanes $H_1, \ldots, H_n$ partition $\mathbb{R}^d$ into relatively open convex polyhedra, called the *cells* of $\mathcal{A}$. Two points $v, v' \in \mathbb{R}^d$ lie in the same cell c if and only if $m_{xy}(v) = m_{xy}(v')$. We write $m_{xy}(c)$ for that monomial, and similarly $m_x(c)$ for its image under $y_i \mapsto x_i$. Note that $m_x(c')$ divides $m_x(c)$, and $m_{xy}(c')$ divides $m_{xy}(c)$, provided $c' \leq c$. The cells of dimension 0 and d are called *vertices* and *regions,* respectively. A cell is *bounded* if it is bounded as a subset of $\mathbb{R}^d$. The set of all bounded cells forms a regular CW-complex $B_\mathcal{A}$ called the *bounded complex* of $\mathcal{A}$.

Figure 1 shows an example of a hyperplane arrangement $\mathcal{A}$ with $d = 2$ and $n = 4$, together with monomials that label its bounded cells. The bounded complex $B_\mathcal{A}$ of this arrangement consists of 4 vertices, 5 edges, and 2 regions.

Theorem 1.3. (a) *The ideal $M_\mathcal{A}$ is minimally generated by the monomials $m_x(v)$, where v ranges over the vertices of $\mathcal{A}$. The bounded complex $B_\mathcal{A}$ is an $M_\mathcal{A}$-complex. Thus its cellular complex $C_\bullet(B_\mathcal{A}, M_\mathcal{A})$ gives a minimal free resolution for $M_\mathcal{A}$.*

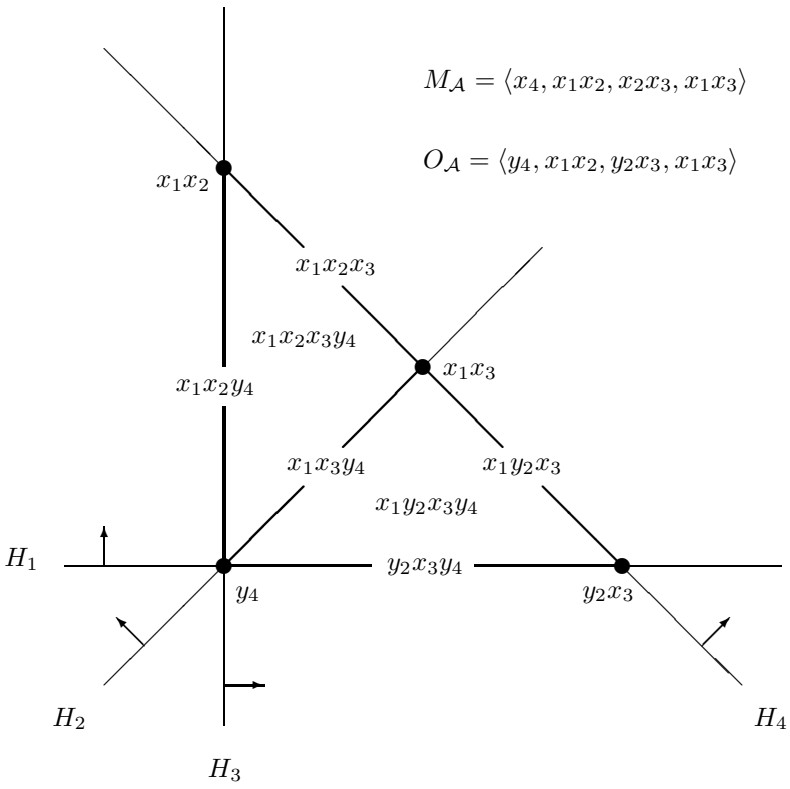

FIGURE 1. The bounded complex $B_{\mathcal{A}}$ with monomial labels.

(b) *The ideal $O_{\mathcal{A}}$ is minimally generated by the monomials $m_{xy}(v)$, where v ranges over the vertices of $\mathcal{A}$. The bounded complex $B_{\mathcal{A}}$ is an $O_{\mathcal{A}}$-complex. Thus its cellular complex $C_\bullet(B_{\mathcal{A}}, O_{\mathcal{A}})$ gives a minimal free resolution for $O_{\mathcal{A}}$.*

To prove Theorem 1.3, we must check that for both ideals, the two hypotheses of Proposition 1.1 are satisfied. The second hypothesis is immediate: for a pair of cells $c > c'$, there is a hyperplane $H_i \in \mathcal{A}$ that contains c' but does not contain c, in which case $m_x(c)$ is divisible by x_i and $m_x(c')$ is not divisible by x_i. Analogously, for the oriented matroid ideal $O_{\mathcal{A}}$. The essence of Theorem 1.3 is the acyclicity condition, which states that all $M_{\mathcal{A}}$-essential and $O_{\mathcal{A}}$-essential subcomplexes of $B_{\mathcal{A}}$ are acyclic. For the whole bounded complex this is known:

Proposition 1.4. (Björner and Ziegler, see [6, Theorem 4.5.7])
The complex $B_{\mathcal{A}}$ of bounded cells of a hyperplane arrangement $\mathcal{A}$ is contractible.

The acyclicity of all $M_{\mathcal{A}}$-essential subcomplexes of $B_{\mathcal{A}}$ is an easy consequence of Proposition 1.4: each $M_{\mathcal{A}}$-essential subcomplex is a bounded complex of a hyperplane arrangement induced by $\mathcal{A}$ in one of the flats of $\mathcal{A}$. The acyclicity of all $O_{\mathcal{A}}$-essential subcomplexes will follow from a generalization of Proposition 1.4 stated in Proposition 2.4 below. We give more details in Section 2, where Theorem 1.3 is restated and proved in the more general setting of oriented matroids.

The main result in this paper is the construction of the minimal free resolution of an arbitrary matroid ideal (Theorems 3.3 and 3.9) and an arbitrary oriented matroid ideal (Theorem 2.2). A numerical consequence of this result is a refinement of Stanley's formula, given in [15, Theorem 9], for their Betti numbers (Corollary 2.3, Corollary 3.4; see also the last paragraph of Section 3). The simplicial complexes corresponding to matroid ideals and oriented matroid ideals are the complexes of independent sets in matroids (Remark 3.1) and the triangulations of Lawrence matroid polytopes (Theorem 2.9), respectively. In the unimodular case, oriented matroid ideals arise as initial ideals of toric varieties in $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, by work of Bayer, Popescu and Sturmfels [2, §4], and their Betti numbers can be interpreted as face numbers of hyperplane arrangements on a torus (Theorem 4.1). Every ideal considered in this paper is Cohen-Macaulay, its Cohen-Macaulay type (= highest Betti number) is the Möbius invariant of the underlying matroid, and all other Betti numbers are sums of Möbius invariants of matroid minors (§4, (8)). In Section 5 we resolve the enumerative problems concerning graphic and cographic matroids which were left open in [2, §5]. Propositions 5.3 and 5.7 give combinatorial expressions for the Möbius invariant of any graph. More precise and efficient formulas, in terms of Hermite polynomials, are established for the Möbius coinvariants of complete graphs (Theorem 5.8) and of complete bipartite graphs (Theorem 5.14).

2. Oriented Matroid Ideals

The axiomatic theory of oriented matroids provides a foundation for geometric combinatorics, much in the same way as the axiomatic theory of schemes provides a foundation for algebraic geometry. Oriented matroid techniques are ubiquitous in the study of hyperplane arrangements, point configurations, and convex polytopes. In this section we establish a link between oriented matroids and commutative algebra. In the resulting combinatorial context, the algebraists' classical question, "What makes a complex exact?" [7], receives a surprising answer: it is the Topological Representation Theorem of Folkman and Lawrence [6, Chapter 5].

We start by briefly reviewing one of the axiom systems for oriented matroids [6]. Fix a finite set E. A *sign vector* X is an element of $\{+,-,0\}^E$. The *positive part* of X is denoted $X^+ = \{i \in E : X_i = +\}$, and similarly X^- and X^0. The support of X is $\underline{X} = \{i \in E : X_i \neq 0\}$. The *opposite* $-X$ of a vector X is given by $(-X)_i = -X_i$. The *composition* $X \circ Y$ of two vectors X and Y is the sign vector defined by

$$(X \circ Y)_i = \begin{cases} X_i & \text{if } X_i \neq 0, \\ Y_i & \text{if } X_i = 0. \end{cases}$$

The *separation set* of sign vectors X and Y is $S(X,Y) = \{i \in E \mid X_i = -Y_i \neq 0\}$.

A set $\mathcal{L} \subseteq \{+,-,0\}^E$ is the set of *covectors* of an *oriented matroid on* E if and only if it satisfies the following four axioms [6, § 4.1.1]:

1. the zero sign vector 0 is in $\mathcal{L}$;
2. (symmetry) if $X \in \mathcal{L}$ then $-X \in \mathcal{L}$;
3. (composition) if $X, Y \in \mathcal{L}$ then $X \circ Y \in \mathcal{L}$;
4. (elimination) if $X, Y \in \mathcal{L}$ and $i \in S(X,Y)$ then there exists $Z \in \mathcal{L}$ such that $Z_i = 0$ and $Z_j = (X \circ Y)_j = (Y \circ X)_j$ for all $j \notin S(X,Y)$.

Somewhat informally, we say that such a pair $(E, \mathcal{L})$, *is* an oriented matroid. An *affine oriented matroid* [6, §10.1], denoted $\mathcal{M} = (E, \mathcal{L}, g)$, is an oriented matroid

with a distinguished element $g \in E$ such that g is not a *loop,* i.e., $X_g \neq 0$ for at least one covector $X \in \mathcal{L}$. The *positive part* of $\mathcal{L}$ is $\mathcal{L}^+ = \{X \in \mathcal{L} \,:\, X_g = +\}$.

The set $\{+,-,0\}^E$ is partially ordered by the product of partial orders

$$0 < + \quad \text{and} \quad 0 < - \qquad (+ \text{ and } - \text{ are incomparable}).$$

This induces a partial order on the set of covectors $\mathcal{L}$. A covector X is called *bounded* if every nonzero covector $Y \leq X$ is in the positive part $\mathcal{L}^+$.

The Topological Representation Theorem for Oriented Matroids, see [6, Theorem 5.2.1], states that $\widehat{\mathcal{L}} = \mathcal{L} \cup \{\hat{1}\}$ is the face lattice of an arrangement of pseudospheres, and $\widehat{\mathcal{L}}^+ = \mathcal{L}^+ \cup \{\hat{0}, \hat{1}\}$ is the face lattice of an arrangement of pseudohyperplanes [6, Exercise 5.8]. These are regular CW-complexes homeomorphic to a sphere and a ball, respectively. (This is why $\widehat{\mathcal{L}}$ is called the *face lattice*, and $\widehat{\mathcal{L}}^+$ is called the *affine face lattice* of $\mathcal{M}$). The *bounded complex* $B_{\mathcal{M}}$ of $\mathcal{M}$ is their subcomplex formed by the cells associated with the bounded covectors. The bounded complex is uniquely determined by its face lattice—the poset of bounded covectors. Slightly abusing notation, we denote this poset by the same symbol $B_{\mathcal{M}}$.

We write $\mathrm{rk}(\,\cdot\,)$ for the rank function of the lattice $\widehat{\mathcal{L}}$. The atoms of $\widehat{\mathcal{L}}$, i.e., the elements of rank 1, are called *cocircuits* of $\mathcal{M}$. The vertices of the bounded complex $B_{\mathcal{M}}$ are exactly the cocircuits of $\mathcal{M}$ that belong to the positive part $\mathcal{L}^+$.

Example 2.1. (Affine oriented matroids from hyperplane arrangements)
Let $\mathcal{C} = \{H_1, \ldots, H_n, H_g\}$ be a central hyperplane arrangement in $\mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$, written as $H_i = \{(v,w) \in \mathbb{R}^d \times \mathbb{R} : h_i(v) = c_i w\}$ and $H_g = \{(v,w) : w = 0\}$. The restriction of $\mathcal{C}$ to the hyperplane $\{(v,w) : w = 1\}$ is precisely the affine arrangement $\mathcal{A}$ in Section 1. Fix $E = \{1, \ldots, n, g\}$. The image of the map

$$\mathbb{R}^{d+1} \to \{+,-,0\}^E,\ (v,w) \mapsto \big(\operatorname{sign}(h_1(v) - c_1 w),\ \ldots,\ \operatorname{sign}(h_n(v) - c_n w),\ \operatorname{sign}(w)\big)$$

is the set $\mathcal{L}$ of covectors of an oriented matroid on E. The affine face lattice $\widehat{\mathcal{L}}^+$ of $\mathcal{M} = (E, \mathcal{L}, g)$ equals the face lattice of the affine hyperplane arrangement $\mathcal{A}$. The bounded complex $B_{\mathcal{M}}$ coincides with the bounded complex $B_{\mathcal{A}}$ in Proposition 1.4.

Let $\mathcal{M} = (E, \mathcal{L}, g)$ be an affine oriented matroid on $E = \{1, \ldots, n, g\}$. With every sign vector $Z \in \{0,+,-\}^E$ we associate a monomial

$$m_{xy}(Z) = \left(\prod_{i\,:\,Z_i = +} x_i \right) \cdot \left(\prod_{j\,:\,Z_i = -} y_i \right), \qquad \text{where } x_g = y_g = 1.$$

The *oriented matroid ideal* O is the ideal in the polynomial ring $\mathbf{k}[\mathbf{x}, \mathbf{y}] = \mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ generated by all monomials corresponding to covectors $Z \in \mathcal{L}^+$. The *matroid ideal* M associated with $\mathcal{M} = (E, \mathcal{L}, g)$ is the ideal of $\mathbf{k}[\mathbf{x}]$ obtained from O by specializing y_i to x_i for all i. These ideals are treated in Section 3. The main result of this section concerns the syzygies of the oriented matroid ideal O.

Theorem 2.2. *The oriented matroid ideal O is minimally generated by the monomials corresponding to the vertices of $B_{\mathcal{M}}$. The bounded complex $B_{\mathcal{M}}$ is an O-complex. Thus its cellular complex $C_\bullet(B_{\mathcal{M}}, O)$ gives a minimal $\mathbb{N}^{2n}$-graded free $\mathbf{k}[\mathbf{x}, \mathbf{y}]$-resolution of O.*

Recall that for a monomial m in $\mathbf{k}[\mathbf{x}, \mathbf{y}]$, the corresponding $\mathbb{N}^{2n}$-graded Betti number of O, $\beta_m(O)$, is the multiplicity of the summand $\langle m \rangle$ in a minimal $\mathbb{N}^{2n}$-graded $\mathbf{k}[\mathbf{x}, \mathbf{y}]$-resolution of O. Theorem 2.2 implies the following numerical result.

Corollary 2.3. *The $\mathbb{N}^{2n}$-graded Betti numbers of O are all 0 or 1. They are given by the coefficients in the numerator of the $\mathbb{N}^{2n}$-graded Hilbert series of O:*

$$\Big(\sum_{Z\in B_{\mathcal{M}}} (-1)^{\mathrm{rk}(Z)} m_{xy}(Z)\Big) \Big/ \prod_{i=1}^{n}(1-x_i)(1-y_i). \tag{2}$$

Proof of Theorem 2.2: Distinct cells Z and Z' of the bounded complex $B_{\mathcal{M}}$ have distinct labels $m_{xy}(Z) \neq m_{xy}(Z')$. This implies minimality of the complex $C_\bullet(B_{\mathcal{M}}, O)$. In order to prove exactness of $C_\bullet(B_{\mathcal{M}}, O)$, we must verify the first hypothesis in Proposition 1.1. To this end, we shall digress and first present a generalization of Proposition 1.4.

The *regions* of an oriented matroid $(E, \mathcal{L})$ are the maximal covectors, i.e, the maximal elements of the poset $\mathcal{L}$. For a covector $X \in \mathcal{L}$ and a subset E' of E, denote by $X|_{E'} \in \{+,-,0\}^{E'}$ the restriction of X to E': $(X|_{E'})_i = X_i$, for every $i \in E'$. The restriction of $(E, \mathcal{L})$ to a subset E' of E is the oriented matroid on E' with the set of covectors $\mathcal{L}|_{E'} = \{X|_{E'} : X \in \mathcal{L}\}$.

The following result, which was cited without proof in [2, Theorem 4.4], is implicit in the derivation of [6, Theorem 4.5.7]. We are grateful to Günter Ziegler for making this explicit by showing us the following proof. Ziegler's proof does not rely on the Topological Representation Theorem for Oriented Matroids. If one uses that theorem, then Proposition 2.4 can also be proved by a topological argument.

Proposition 2.4. (G. Ziegler) *Let $\mathcal{M} = (E, \mathcal{L}, g)$ be an affine oriented matroid and $B_{\mathcal{M}}$ its bounded complex. For any subset E' of E and any region R' of $(E', \mathcal{L}|_{E'})$, the CW-complex with the face poset $B' = \{X \in B_{\mathcal{M}} : X|_{E'} \leq R'\}$ is contractible.*

Proof: Let $\mathbb{T}$ denote the set of regions of $\mathcal{L}$. A subset $A \subseteq \mathbb{T}$ is said to be *T-convex* if it is an intersection of "half-spaces", i.e., sets of the form $\mathbb{T}_e^+ = \{T \in \mathbb{T} : T_e = +\}$ and $\mathbb{T}_e^- = \{T \in \mathbb{T} : T_e = -\}$. Each region $R \in \mathbb{T}$ defines a partial order on $\mathbb{T}$:

$$T_1 \leq T_2 \quad :\Longleftrightarrow \quad \{e \in E : R_e = -(T_1)_e\} \subseteq \{e \in E : R_e = -(T_2)_e\}.$$

Denote this poset by $\mathbb{T}(\mathcal{L}, R)$. We also abbreviate $\mathbb{T}^+ := \mathbb{T}_g^+ = \mathbb{T} \cap \mathcal{L}^+$.

We may assume that B' is non-empty. Then $\mathcal{R} := \{X \in \mathbb{T}^+ : X|_{E'} = R'\}$ is a non-empty, T-convex set. Lemma 4.5.5 in [6] states that $\mathcal{R}$ is an order ideal of $\mathbb{T}(\mathcal{L}, R)$, and, moreover, it is an order ideal of $\mathbb{T}^+ \subseteq \mathbb{T}(\mathcal{L}, R)$. By [6, Proposition 4.5.6], there exists a recursive coatom ordering of $\widehat{\mathcal{L}}^+$ in which the elements of $\mathcal{R}$ come first. The restriction of this ordering to $\mathcal{R}$ is a recursive coatom ordering of the poset $\widehat{\mathcal{L}}_{\mathcal{R}}^+ = \{X \in \mathcal{L}^+ : X \leq T \text{ for some } T \in \mathcal{R}\} \cup \{\hat{1}\}$. This implies (using [6, Lemma 4.7.18]) that the order complex $\Delta_{\mathrm{ord}}(\mathcal{L}_{\mathcal{R}}^+)$ of $\mathcal{L}_{\mathcal{R}}^+$ is a shellable $(r-1)$-ball. It is a subcomplex of $\Delta_{\mathrm{ord}}(\mathcal{L}^+)$, which is also an $(r-1)$-ball, by [6, Theorem 4.5.7]. Let $U = \mathcal{L}_{\mathcal{R}}^+ \backslash B_{\mathcal{M}}$ be the set of "unbounded covectors". Then the subcomplex Δ_U of $\Delta_{\mathrm{ord}}(\mathcal{L}_{\mathcal{R}}^+)$ induced on the vertex set of U lies in the boundary of $\Delta_{\mathrm{ord}}(\mathcal{L}^+)$, and hence also in the boundary of $\Delta_{\mathrm{ord}}(\mathcal{L}_{\mathcal{R}}^+)$. Thus $||\Delta_{\mathrm{ord}}(\mathcal{L}_{\mathcal{R}}^+)|| \setminus ||\Delta_U||$ is a contractible space. By [6, Lemma 4.7.27], the space $||\Delta_{\mathrm{ord}}(B')||$ is a strong deformation retract of $||\Delta_{\mathrm{ord}}(\mathcal{L}_{\mathcal{R}}^+)|| \setminus ||\Delta_U||$, and is hence contractible as well. □

We now finish the proof of Theorem 2.2. Consider any O-essential subcomplex $(B_{\mathcal{M}})_{\leq \mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}}$ of $B_{\mathcal{M}}$, with $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$. This complex consists of all cells Z whose label

$m_{xy}(Z)$ divides $\mathbf{x}^{\mathbf{a}}\,\mathbf{y}^{\mathbf{b}}$. Set

$$\begin{aligned} E'' &= \{1 \le i \le n : a_i = 0 \text{ and } b_i = 0\}, \\ E' &= \{1 \le i \le n : \text{ exactly one of } a_i \text{ and } b_i \text{ is positive}\} \subseteq E \setminus E''. \end{aligned}$$

We first replace our affine oriented matroid $(E, \mathcal{L}, g)$ by the affine oriented matroid $(E\backslash E'', \mathcal{L}/E'', g)$ gotten by contraction at E''. Next we define $R' \in \{+, -, 0\}^{E'}$ by

$$R'_i = \begin{cases} + & \text{if } a_i > 0 \\ - & \text{if } b_i > 0 \end{cases} \quad \text{for every } i \in E'$$

We apply Proposition 2.4 with this R' to $(E\backslash E'', \mathcal{L}/E'', g)$. Then B' is the face poset of $(B_{\mathcal{M}})_{\le \mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}}$, which is therefore contractible. □

The oriented matroid ideal O is square-free, and hence is the Stanley-Reisner ideal of a simplicial complex $\Delta_{\mathcal{M}}$ on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$, whose faces correspond to square-free monomials of $\mathbf{k}[\mathbf{x}, \mathbf{y}]$ that do not belong to O, i.e.,

$$\{i_1, \dots, i_k, j'_1, \dots, j'_m\} \in \Delta_{\mathcal{M}} \quad \text{if and only if} \quad x_{i_1} \dots x_{i_k} y_{j_1} \dots y_{j_m} \notin O.$$

In what follows we give a geometric description of that simplicial complex.

Lemma 2.5. *We have $F \cap \{i, i'\} \neq \emptyset$ for any facet F of $\Delta_{\mathcal{M}}$ and $i \in \{1, \dots, n\}$.*

Proof: Let F be a face of $\Delta_{\mathcal{M}}$ such that $F \cap \{i, i'\} = \emptyset$. Suppose that neither $F' = F \cup \{i\}$ nor $F'' = F \cup \{i'\}$ is a face of $\Delta_{\mathcal{M}}$. Then there exist cocircuits $Z', Z'' \in B_{\mathcal{M}}$ such that

$$Z'_i = +, \quad (Z')^+ \setminus \{i\} \subseteq \{1 \le j \le n : j \in F\} \cup \{g\}, \quad (Z')^- \subseteq \{1 \le j \le n : j' \in F\}$$

$$Z''_i = -, \quad (Z'')^+ \subseteq \{1 \le j \le n : j \in F\} \cup \{g\}, \quad (Z'')^- \setminus \{i\} \subseteq \{1 \le j \le n : j' \in F\}.$$

By the strong elimination axiom applied to (Z', Z'', i, g), there is a cocircuit Z such that $Z_i = 0$, $Z_g = +$, $Z^+ \subseteq (Z')^+ \cup (Z'')^+$, $Z^- \subseteq (Z')^- \cup (Z'')^-$. Thus $Z \in B_{\mathcal{M}}$, and the monomial $m_{xy}(F)$ is divisible by $m_{xy}(Z) \in O$. This contradicts $F \in \Delta_{\mathcal{M}}$. □

Suppose now that an affine oriented matroid $\mathcal{M} = (E, \mathcal{L}, g)$ is a single element extension of the matroid $\mathcal{M}\backslash g = (E\backslash g, \mathcal{L}\backslash g)$ by an element g in *general position,* in the sense of [6, Proposition 7.2.2]. For the affine arrangement $\mathcal{A}$ in Section 1 or Example 2.1, this means that $\mathcal{A}$ has no vertices at infinity. In such a case, Theorem 2.2 implies the following properties of O. In the rest of this section we denote by r the rank of $\mathcal{M}\backslash g$.

Corollary 2.6. $\mathbf{k}[\Delta_{\mathcal{M}}] = \mathbf{k}[\mathbf{x}, \mathbf{y}]/O$ *is a Cohen-Macaulay ring of dimension* $2n-r$.

Proof: Since $\mathrm{rk}(\mathcal{M}\backslash g) = r$, every $(n - r + 1)$-element subset $\{i_1, \dots, i_{n-r+1}\}$ of $\{1, \dots, n\}$ contains the support of a (signed) cocircuit. This implies that every monomial of the form $x_{i_1} \dots x_{i_{n-r+1}} y_{i_1} \dots y_{i_{n-r+1}}$ belongs to O. The variety defined by these monomials is a subspace arrangement of codimension r. Hence O has codimension $\le r$, which means that the ring $\mathbf{k}[\Delta_{\mathcal{M}}] = \mathbf{k}[\mathbf{x}, \mathbf{y}]/O$ has Krull dimension $\le 2n-r$. By Theorem 2.2, the bounded complex $B_{\mathcal{M}}$ supports a minimal free resolution of O, and therefore

$$\mathrm{depth}\,(\mathbf{k}[\Delta_{\mathcal{M}}]) \;=\; 2n - (\text{the length of this resolution}) \;=\; 2n - r.$$

Hence $\mathrm{depth}\,(\mathbf{k}[\Delta_{\mathcal{M}}]) = \dim(\mathbf{k}[\Delta_{\mathcal{M}}]) = 2n - r$, and $\mathbf{k}[\Delta_{\mathcal{M}}]$ is Cohen-Macaulay. □

Corollary 2.7. *$\{x_1-y_1,\ldots,x_n-y_n\}$ is a regular sequence on $\mathbf{k}[\Delta_{\mathcal{M}}] = \mathbf{k}[\mathbf{x},\mathbf{y}]/O$.*

Proof: Since $\mathbf{k}[\Delta_{\mathcal{M}}]$ is Cohen-Macaulay, it suffices to show that $\{x_1-y_1,\ldots,x_n-y_n\}$ is a part of a linear system of parameters (l.s.o.p.). This follows from Lemma 2.5 and the l.s.o.p. criterion due to Kind and Kleinschmidt [18, Lemma III.2.4]. □

Consider any signed circuit $C = (C^+, C^-)$ of our oriented matroid such that g lies in C^-. By the *general position* assumption on g, the complement of g in that circuit is a basis of the underlying matroid. We write P_C for the ideal generated by the variables x_i for each $i \in C^+$ and the variables y_j for each $j \in C^-\backslash\{g\}$.

Proposition 2.8. *The minimal prime decomposition of the oriented matroid ideal equals $O = \bigcap_C P_C$ where the intersection is over all circuits C such that $g \in C^-$.*

Proof: The right hand side is easily seen to contain the left hand side. For the converse it suffices to divide by the regular sequence $x_1-y_1,\ldots,x_n-y_n$ and note that the resulting decomposition for the matroid ideal M is easy (Remark 3.1). □

Our final result relates the ideal O to matroid polytopes and their triangulations. The monograph of Santos [14] provides an excellent state-of-the art introduction. We refer in particular to [14, Section 4], where Santos introduces triangulations of Lawrence (matroid) polytopes, and he shows that these are in bijection with one-element liftings of the underlying matroid. Under matroid duality, one-element liftings correspond to one-element extensions. In our context, these extensions correspond to adding the special element g which plays the role of the pseudohyperplane at infinity. From Santos' result we infer the following theorem.

Theorem 2.9. *The oriented matroid ideal O is the Stanley-Reisner ideal of the triangulation of the Lawrence matroid polytope induced by the lifting dual to the extension by g. In particular, O is the Stanley-Reisner ideal of a triangulated ball.*

The second assertion holds because lifting triangulations of matroid polytopes are triangulated balls and, by Santos' work, every triangulation of a Lawrence matroid polytope is a lifting triangulation. We remark that it is unknown whether arbitrary triangulations of matroid polytopes are topological balls [14, page 7].

3. Matroid Ideals

Let $\underline{\mathcal{M}}$ be an (unoriented) matroid on the set $\{1,\ldots,n\}$ and let L be its lattice of flats. We encode $\underline{\mathcal{M}}$ by the *matroid ideal* M generated by the monomials $m_x(F) = \prod_{i:i\notin F} x_i$ for every proper flat $F \in L$. The minimal generators of M are the squarefree monomials representing cocircuits of $\underline{\mathcal{M}}$, that is, the monomials $m_x(H)$ where H runs over all hyperplanes of $\underline{\mathcal{M}}$. Equivalently, M is the Stanley-Reisner ideal of the simplicial complex of independent sets of the dual matroid $\underline{\mathcal{M}}^*$. This explains what happens when we substitute $y_i \mapsto x_i$ in Proposition 2.8:

Remark 3.1. *The matroid ideal M has the minimal prime decomposition*

$$M \quad = \bigcap_{B \text{ basis of } \underline{\mathcal{M}}} \langle\, x_i \mid i \in B \,\rangle.$$

The following characterization of our ideals can serve as a definition of the word "matroid". It is a translation of the (co)circuit axiom into commutative algebra.

Remark 3.2. A proper square-free monomial ideal M of $\mathbf{k}[\mathbf{x}]$ is a *matroid ideal* if and only if for every pair of monomials $m_1, m_2 \in M$ and any $i \in \{1, \ldots, n\}$ such that x_i divides both m_1 and m_2, the monomial $\operatorname{lcm}(m_1, m_2)/x_i$ is in M as well.

Matroid ideals have been studied since the earliest days of combinatorial commutative algebra, as a paradigm for shellability and Cohen-Macaulayness. Stanley computed their Betti numbers in [15, Theorem 9]. The purpose of this section is to construct an explicit minimal $\mathbf{k}[\mathbf{x}]$-free resolution for any matroid ideal M.

We note that Reiner and Welker [13] used the term "matroid ideal" for the square-free monomial ideals which are Alexander dual to our ideals. The matroid ideals in [13] have linear resolution but are generally not Cohen-Macaulay, while our matroid ideals are Cohen-Macaulay but their resolution (given below) is generally not linear. In particular, the Alexander dual of a matroid ideal is completely different from the matroid ideal of the dual matroid.

We first consider the case where $\underline{\mathcal{M}}$ is an *orientable matroid*. This means that there exists an oriented matroid $\mathcal{M}$ whose underlying matroid is $\underline{\mathcal{M}}$. Let $\mathcal{L}$ be the set of covectors of a single element extension of $\mathcal{M}$ by an element g in general position (see [6, Proposition 7.2.2]). Consider the affine oriented matroid $\widetilde{\mathcal{M}} = (E, \mathcal{L}, g)$, where $E = \{1, \ldots, n\} \cup \{g\}$, and its bounded complex $B_{\widetilde{\mathcal{M}}}$. Note that, for each sign vector Z in $B_{\widetilde{\mathcal{M}}}$, the zero set Z^0 is a flat in L. Moreover, by the genericity hypothesis on g, all flats arise in this way. We label each cell Z of the bounded complex $B_{\widetilde{\mathcal{M}}}$ by the monomial $m_x(Z) = \prod\{x_i : 1 \le i \le n \text{ and } Z_i \neq 0\}$.

Theorem 3.3. *Let M be the matroid ideal of an orientable matroid. Then the bounded complex $B_{\widetilde{\mathcal{M}}}$ of any corresponding affine oriented matroid is an M-complex, and its cellular complex $C_\bullet(B_{\widetilde{\mathcal{M}}}, M)$ gives a minimal free resolution of M over $\mathbf{k}[\mathbf{x}]$.*

Proof: Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and consider M-essential subcomplex $(B_{\widetilde{\mathcal{M}}})_{\le \mathbf{x}^\mathbf{a}}$. This complex (if not empty) is the bounded complex of the contraction of $(E, \mathcal{L}, g)$ by $\{1 \le i \le n : a_i = 0\}$, and, hence is acyclic by Proposition 2.4. Since $m_x(Z')$ is a proper divisor of $m_x(Z)$ whenever $Z' < Z$, and $Z', Z \in B_{\widetilde{\mathcal{M}}}$, it follows that $B_{\widetilde{\mathcal{M}}}$ is an M-complex. □

We remark that $C_\bullet(B_{\widetilde{\mathcal{M}}}, M)$ is obtained from the complex $C_\bullet(B_{\widetilde{\mathcal{M}}}, O)$, where O is the oriented matroid ideal of $\widetilde{\mathcal{M}} = (E, \mathcal{L}, g)$, by specializing y_i to x_i for all i. Hence Theorem 2.2 and Corollary 2.7 give a second proof of Theorem 3.3.

Corollary 3.4. *The $\mathbb{N}^n$-graded Hilbert series of any matroid ideal M equals*

$$\left(\sum_{F \in L} \mu_L(F, \hat{1}) \cdot \prod\{x_j : j \notin F\} \right) / \prod_{i=1}^{n} (1 - x_i) \tag{3}$$

where L is the lattice of flats of $\underline{\mathcal{M}}$, and μ_L is its Möbius function.

There are several ways of deriving this corollary. First, it follows from [15, Theorem 9]. A second possibility is to observe that the geometric lattice L coincides with the lcm lattice (in the sense of [8]) of the ideal M, and then [8, Theorem 2.1] implies the claim. Finally, in the orientable case, Corollary 3.4 follows from Theorem 3.3 and the oriented matroid version of Zaslavksy's face-count formula.

Proposition 3.5. (Zaslavsky's Formula) [21], [6, Theorem 4.6.5] *The number of bounded regions of a rank r affine oriented matroid $\widetilde{\mathcal{M}} = (E, \mathcal{L}, g)$ equals $(-1)^r \mu_L(\hat{0}, \hat{1})$.*

We next treat the case of non-orientable matroids. It would be desirable to construct an M-complex for an arbitrary matroid ideal M, and to explore the "space" of all possible M-complexes. Currently we do not know how to construct them. Therefore we introduce a different technique for resolving M minimally.

Let P be any graded poset which has a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. (Later on, we will take P to be the order dual of our geometric lattice L). Let $\Delta(P)$ denote the order complex of P, that is, the simplicial complex whose simplices $[F_0, F_1, \ldots, F_i]$ are decreasing chains $\hat{1} > F_0 > F_1 > \ldots > F_i > \hat{0}$. For $F \in P$ denote by $\Delta(F)$ the order complex of the lower interval $[\hat{0}, F]$. Note that $\dim \Delta(F) = \mathrm{rk}(F) - 2$. Let $C_i(\Delta(F))$ be the $\mathbf{k}$-vector space of i-dimensional chains of $\Delta(F)$, and let

$$0 \longrightarrow C_{\mathrm{rk}(F)-2}(\Delta(F)) \longrightarrow \ldots \xrightarrow{\partial_2} C_1(\Delta(F)) \xrightarrow{\partial_1} C_0(\Delta(F)) \xrightarrow{\partial_0} C_{-1}(\Delta(F)) \longrightarrow 0\,.$$

be the usual (augmented) chain complex, i.e., the differential is given by

$$\partial_i[F_0, F_1, \ldots, F_i] = \sum_{j=0}^{i} (-1)^j [F_0, \ldots, \widehat{F_j}, \ldots, F_i] \quad \text{for } i > 0 \quad \text{and } \partial_0[F_0] = \emptyset.$$

Denote by $Z_i(\Delta(F)) = \ker(\partial_i)$ the space of i-cycles, and by $\widetilde{H}_i(\Delta(F))$ the ith (reduced) homology of $\Delta(F)$. For relevant background on poset homology see [5].

For each pair $F, F' \in P$ such that $\mathrm{rk}(F) - \mathrm{rk}(F') = 1$, we define a map

$$\phi : C_i(\Delta_F) \longrightarrow C_{i-1}(\Delta_{F'}) \text{ by } [F_0, F_1, \ldots, F_i] \mapsto \begin{cases} 0, & \text{if } F_0 \neq F' \\ [F_1, \ldots, F_i] & \text{if } F_0 = F'. \end{cases}$$

The map ϕ is zero unless $F' \lessdot F$ (in words: F covers F'). Note that $\partial \circ \phi = -\phi \circ \partial$, and hence the restriction of ϕ to cycles gives a map $\phi : Z_i(\Delta(F)) \longrightarrow Z_{i-1}(\Delta(F'))$. Combining these maps together we obtain a complex of $\mathbf{k}$-vector spaces:

$$\begin{aligned} \mathcal{Z}(P): \quad 0 \longrightarrow Z_{r-2}(\Delta(P)) &\xrightarrow{\phi} \bigoplus_{\mathrm{rk}(F)=r-1} Z_{r-3}(\Delta(F)) \xrightarrow{\phi} \cdots \\ \cdots &\xrightarrow{\phi} \bigoplus_{\mathrm{rk}(F)=2} Z_0(\Delta(F)) \xrightarrow{\phi} \bigoplus_{\mathrm{rk}(F)=1} Z_{-1}(\Delta(F)) \longrightarrow \mathbf{k}\,. \end{aligned}$$

The complex property $\phi^2 = 0$ is verified by direct calculation using the equation (4) stated below. Let $P_{(j)}$ denote the poset obtained from P by removing all rank levels $\geq j$, and let $\Delta(P_{(j)})$ be the order complex of $P_{(j)} \cup \{\hat{1}\}$.

Proposition 3.6. *The complex $\mathcal{Z}(P)$ is exact if $\widetilde{H}_i(\Delta(P_{(i+3)})) = 0$ for all $i \leq r-3$.*

To prove Proposition 3.6 we need some notation. If $x \in \bigoplus_{\mathrm{rk}(F)=i} Z_{i-2}(\Delta(F))$, we denote its F-component by x_F. For a simplex $\sigma = [F_0, F_1, \ldots, F_i]$ we also write $\sigma = F_0 * [F_1, \ldots, F_i]$, and the operation "$*$" extends to $\mathbf{k}$-linear combinations.

Remark 3.7. Suppose that $z \in C_i(\Delta(P_{(i+2)}))$. Then z can be expressed as

$$z \quad = \sum_{\mathrm{rk}(F')=i+1} F' * y_{F'} \quad = \sum_{\mathrm{rk}(F')=i+1} \sum_{F'' \lessdot F'} F' * F'' * x_{F',F''},$$

where $y_{F'} \in C_{i-1}(\Delta(F'))$ and $x_{F',F''} \in C_{i-2}(\Delta(F''))$. Its boundary equals

$$\begin{aligned}\partial(z) &= \textstyle\sum_{\mathrm{rk}(F'')=i} F'' * \sum_{F'>F''} x_{F',F''} \\ &\quad - \textstyle\sum_{\mathrm{rk}(F')=i+1} F' * \sum_{F''<F'} x_{F',F''} \quad + \sum_{F',F''} F' * F'' * \partial(x_{F',F''}).\end{aligned}$$

We conclude that z is a cycle if and only if the following conditions are satisfied:

$$\sum_{F'>F''} x_{F',F''} = 0 \qquad \text{for all } F'' \text{ with } \mathrm{rk}(F'') = i \tag{4}$$

$$\sum_{F''<F'} x_{F',F''} = 0 \qquad \text{for all } F' \text{ with } \mathrm{rk}(F') = i+1 \tag{5}$$

$$\partial(x_{F',F''}) = 0 \qquad \text{for all } F', F'' \text{ such that } F'' \lessdot F'. \tag{6}$$

Proof of Proposition 3.6: To show that $\mathcal{Z}(P)$ is exact, consider $y = (y_{F'}) \in \bigoplus_{\mathrm{rk}(F')=i+1} Z_{i-1}(\Delta(F'))$ such that $\phi(y) = 0$. There are several cases: If $i = r-1$, then $y = y_{\hat{1}}$ can be expressed as $\sum_{\mathrm{rk}(F)=r-2} F * x_F$, where $x_F \in C_{r-3}(\Delta(F))$. Then $0 = \phi(y)_F = x_F$ and therefore $y = 0$. Hence the leftmost map ϕ is an inclusion.

Let $0 < i < r-1$ and define $z = \sum_{\mathrm{rk}(F')=i+1} F' * y_{F'} \in C_i(\Delta(P_{(i+2)}))$. We claim that z is a cycle, that is, $z \in Z_i(\Delta(P_{(i+2)}))$. Indeed, if $i > 0$, then $y_{F'}$ can be expressed as $\sum_{F''<F'} F'' * x_{F',F''}$, where $x_{F',F''} \in C_{i-2}(\Delta(F''))$. Hence

$$(\phi(y))_{F''} = \sum_{F'>F''} x_{F',F''} \quad \forall F'' \text{ with } \mathrm{rk}(F'') = i, \text{ and}$$

$$\partial(y_{F'}) = \sum_{F''<F'} x_{F',F''} - \sum_{F''<F'} F'' * \partial(x_{F',F''}) \qquad \forall F' \text{ with } \mathrm{rk}(F') = i+1.$$

Since $\phi(y) = 0$ and $\partial(y_{F'}) = 0$ for any F' of rank $i+1$, we infer that z satisfies conditions (4)–(6) in Remark 3.7, and therefore is a cycle. In the case $i = 0$ the proof is very similar. Now, if $i = r-2$ then $z \in Z_{r-2}(\Delta(P))$, and $\phi(z) = \phi(\sum F' * y_{F'}) = (y_{F'}) = y$. Hence we are done in this case. If $i < r-2$, then, since $Z_i(\Delta(P_{(i+2)})) \subseteq Z_i(\Delta(P_{(i+3)}))$, and $\widetilde{H}_i(\Delta(P_{(i+3)})) = 0$, it follows that there exists $w \in C_{i+1}(\Delta(P_{(i+3)}))$ such that $\partial(w) = z$. Express w as $\sum_{\mathrm{rk}(F)=i+2} F * v_F$, where $v_F \in C_i(\Delta(F))$. Since $z = \partial(w) = \sum_{\mathrm{rk}(F)=i+2} v_F + \sum_{\mathrm{rk}(F)=i+2} F * \partial(v_F)$, we conclude that $\partial(v_F) = 0$ for all F of rank $i+2$, and that $\sum_F v_F = z = \sum_{F'} F' * y_{F'}$. Thus $v = (v_F) \in \bigoplus_{\mathrm{rk}(F)=i+2} Z_i(\Delta(F))$, and $\phi(v) = y$. □

Corollary 3.8. *If P is a Cohen-Macaulay poset, then $\mathcal{Z}(P)$ is exact.*

Proof: If $\Delta(P)$ is Cohen-Macaulay, then $\Delta(P_{(i)})$ is Cohen-Macaulay for every i (see [16, Theorem 4.3]). This means that all homologies of $\Delta(P_{(i)})$ vanish, except possibly for the top one. Thus the conditions of Proposition 3.6 are satisfied. □

Suppose now that every atom A of P is labeled by a monomial $m_A \in \mathbf{k}[\mathbf{x}]$. The *poset ideal* I_P is the ideal generated by these monomials. Associate with every element F of P a monomial m_F as follows:

$$m_F := \mathrm{lcm}\,\{m_A : \mathrm{rk}(A) = 1, A \le F\} \quad \text{if } F \neq \hat{0} \quad \text{and } m_{\hat{0}} := 1.$$

We say that the labeled poset P is *complete* if all monomials m_F are distinct, and for every $\mathbf{a} \in \mathbb{N}^n$ the set $\{F \in P : \deg(m_F) \le \mathbf{a}\}$ has a unique maximal element.

We identify the principal ideal $\langle m_F \rangle$ with the free $\mathbb{N}^n$-graded $\mathbf{k}[\mathbf{x}]$-module of rank 1 with generator in degree $\deg m_F$. If $F, G \in P$ and $F < G$, then m_F is a divisor of

m_G. Thus there is an inclusion of the corresponding ideals $\langle m_G \rangle \longrightarrow \langle m_F \rangle$. Recall that there is a complex $\mathcal{Z}(P)$ of $\mathbf{k}$-vector spaces associated with P. Tensoring summands of this complex with the ideals $\{\langle m_F \rangle : F \in P\}$, we obtain a complex of $\mathbb{N}^n$-graded free $\mathbf{k}[\mathbf{x}]$-modules :

$$\mathcal{C}(P) \;=\; \bigoplus_{F \in P} Z_{\mathrm{rk}(F)-2}(\Delta(F)) \otimes_{\mathbf{k}} \langle m_F \rangle \qquad \text{with differential } \partial = \phi \otimes i. \tag{7}$$

Theorem 3.9. *Suppose that the labeled poset P is complete and that the homology $\widetilde{H}_i(\Delta(F_{(i+3)}))$ vanishes for any $0 \leq i \leq r-3$ and any $F \in P$ of rank $\geq i+3$, then $(\mathcal{C}(P), \partial)$ is a minimal $\mathbb{N}^n$-graded free $\mathbf{k}[\mathbf{x}]$-resolution of the poset ideal I_P.*

Proof: $(\mathcal{C}(P), \partial)$ is a complex of $\mathbb{N}^n$-graded free $\mathbf{k}[\mathbf{x}]$-modules. To show that it is a resolution, we have to check that for any $\mathbf{a} \in \mathbb{N}^n$, the $\mathbf{a}$th graded component $(\mathcal{C}(P), \partial)_{\mathbf{a}}$ is an exact complex of $\mathbf{k}$-vector spaces. Let $\mathbf{a} \in \mathbb{N}^n$, and let $F \in P$ be the maximal element among all elements $G \in P$ such that $\deg(m_G) \leq \mathbf{a}$. Such an element F exists since the labeled poset P is complete. Then $(\mathcal{C}(P), \partial)_{\mathbf{a}}$ is isomorphic to the complex $\mathcal{Z}([\hat{0}, F])$ of the poset $[\hat{0}, F]$, and hence is exact over $\mathbf{k}$ (by Proposition 3.6). Thus $(\mathcal{C}(P), \partial)$ is exact over $\mathbf{k}[\mathbf{x}]$. Finally, since m_F and m_G are distinct monomials for any pair $F \lessdot G$, the resolution $(\mathcal{C}(P), \partial)$ is minimal. $\square$

From Corollary 3.8 we obtain:

Corollary 3.10. *If P is a complete labeled poset such that every lower interval of P is Cohen-Macaulay, then $(\mathcal{C}(P), \partial)$ is a minimal $\mathbb{N}^n$-graded free resolution of I_P.*

Returning to our matroid $\underline{\mathcal{M}}$, let P be a lattice of flats ordered by reverse inclusion. Hence P is the order dual of the geometric lattice L above. In particular, $\hat{0}$ corresponds to the set $\{1, 2, \ldots, n\}$, and $\hat{1}$ corresponds to the empty set. Label each atom H of P (that is, hyperplane of $\underline{\mathcal{M}}$) by the monomial $m_x(H)$ as in the beginning of this section. Identifying the variables x_i with the coatoms of P, we see that $m_x(H)$ is the product over all coatoms not above H. Then P is a complete labeled poset and its poset ideal I_P is precisely the matroid ideal M. Moreover, all lower intervals of the poset P are Cohen-Macaulay [15, Section 8]. From Corollary 3.10 we obtain the following alternative to Theorem 3.3.

Theorem 3.11. *Let $\underline{\mathcal{M}}$ be any matroid. Then the complex $(\mathcal{C}(P), \partial)$ is a minimal $\mathbb{N}^n$-graded free $\mathbf{k}[\mathbf{x}]$-resolution of the matroid ideal M.*

The two resolutions presented in this section provide a syzygetic realization of Stanley's formula [15, Theorem 9] for the Betti numbers of matroid ideals. That formula states that the number of minimal i-th syzygies of $\mathbf{k}[\mathbf{x}]/M$ is equal to

$$\beta_i(M) \quad = \quad \sum_{F} |\, \mu_L(F, \hat{1}) \,|,$$

where the sum is over all flats F of corank i in $\underline{\mathcal{M}}$. The generating function

$$\psi_{\underline{\mathcal{M}}}(q) = \sum_{i=0}^{\mathrm{rk}(M)} \beta_i(M) \cdot q^i \quad = \quad \sum_{F \text{ flat of } \underline{\mathcal{M}}} |\, \mu_L(F, \hat{1}) \,| \cdot q^{\mathrm{corank}(F)} \tag{8}$$

for the Betti numbers of M is called the *cocharacteristic polynomial* of $\underline{\mathcal{M}}$. In the next two sections we will examine this polynomial for some special matroids.

4. Unimodular toric arrangements

A *toric arrangement* is a hyperplane arrangement which lives on a torus $\mathbb{T}^d$ rather than in $\mathbb{R}^d$. One construction of such arrangements appears in recent work of Bayer, Popescu and Sturmfels [2]. Experts on geometric combinatorics might appreciate the following description: Fix a unimodular matroid $\underline{\mathcal{M}}$, form the associated tiling of Euclidean space by zonotopes [20, Proposition 3.3.4], dualize to get an infinite arrangement of hyperplanes, and divide out by the group of lattice translations.

Here is the same construction again, but now in slow motion. Fix a central hyperplane arrangement $\mathcal{C} = \{H_1, \ldots, H_n\}$ in $\mathbb{R}^d$ where $H_i = \{v \in \mathbb{R}^d : h_i \cdot v = 0\}$ for some $h_i \in \mathbb{Z}^d$. Let L denote the intersection lattice of $\mathcal{C}$ ordered by reverse inclusion. We assume that $\mathcal{C}$ is *unimodular,* which means that the $d \times n$ matrix $(h_1, \ldots, h_n)$ has rank d, and all its $d \times d$-minors lie in the set $\{0, 1, -1\}$. We retain this hypothesis throughout this section. See [20] and [2, Theorem 1.2] for details on unimodularity. The set of all integral translates of hyperplanes of $\mathcal{C}$,

$$H_{ij} = \{v \in \mathbb{R}^d : h_i \cdot v = j\} \quad \text{for } i \in \{1, \ldots, n\} \text{ and } j \in \mathbb{Z},$$

forms an infinite arrangement $\widetilde{\mathcal{C}}$ in $\mathbb{R}^d$. The unimodularity hypothesis is equivalent to saying that the set of vertices of $\widetilde{\mathcal{C}}$ is precisely the lattice $\mathbb{Z}^d$, that is, no new vertices can be formed by intersecting the hyperplanes H_{ij}. Define the *unimodular toric arrangement* $\widetilde{\mathcal{C}}/\mathbb{Z}^d$ to be the set of images of the H_{ij} in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Slightly abusing notation, we refer to these images as hyperplanes on the torus. The images of cells of $\widetilde{\mathcal{C}}$ in $\mathbb{T}^d$ are called *cells* of $\widetilde{\mathcal{C}}/\mathbb{Z}^d$. These cells form a cellular decomposition of $\mathbb{T}^d$. Denote by $f_i = f_i(\widetilde{\mathcal{C}}/\mathbb{Z}^d)$ the number of i-dimensional cells in this decomposition. The next result concerns the *f-vector* $(f_0, f_1, \ldots, f_d)$ of $\widetilde{\mathcal{C}}/\mathbb{Z}^d$.

Theorem 4.1. *If $\widetilde{\mathcal{C}}/\mathbb{Z}^d$ is a unimodular toric arrangement, then*

$$\sum_{i=0}^{d} f_i(\widetilde{\mathcal{C}}/\mathbb{Z}^d) \cdot q^i \;=\; \psi_{\mathcal{C}}(q), \qquad \textit{where } \psi_{\mathcal{C}}(q) \,=\, \sum_{F \in L} \mu_L(F, \hat{1}) \cdot (-q)^{\dim F}$$

is the cocharacteristic polynomial of the underlying hyperplane arrangement $\mathcal{C}$.

Proof: Choose a vector $w \in \mathbb{R}^d$ which is not perpendicular to any 1-dimensional cell of the arrangement $\mathcal{C}$. Consider the affine hyperplane $\{\, v \in \mathbb{R}^d \,:\, w \cdot v = 1\}$. Let $\mathcal{A} = \mathcal{C} \cap H$ be a restriction of $\mathcal{C}$ to H. Then $\mathcal{A}$ is an affine arrangement in H. For any $i \geq 0$, there is a one-to-one correspondence between the $(i-1)$-dimensional bounded cells of $\mathcal{A}$ and the i-dimensional cells of toric arrangement $\widetilde{\mathcal{C}}/\mathbb{Z}^d$. To see this, consider the cells in the infinite arrangement $\widetilde{\mathcal{C}}$ whose minimum with respect to the linear functional $v \mapsto w \cdot v$ is attained at the origin. These cells form a system of representatives modulo the $\mathbb{Z}^d$-action. But they are also in bijection with the bounded cells of $\mathcal{A}$. Using Proposition 3.5 (see also Example 2.1), we conclude

$$f_i(\widetilde{\mathcal{C}}/\mathbb{Z}^d) \quad = \quad f_{i-1}(B_{\mathcal{A}}) \quad = \quad (-1)^i \cdot \sum_{\dim(F)=i} \mu_L(F, \hat{1}),$$

where the sum is over elements of L of corank i. This completes the proof. □

Theorem 4.1 was found independently by Vic Reiner who suggested that we include the following alternative proof. His proof has the advantage that it does not rely on Zaslavsky's formula.

Second proof of Theorem 4.1: Starting with the unimodular toric arrangement $\tilde{\mathcal{C}}/\mathbb{Z}^d$, for each intersection subspace F in the intersection lattice L, let T_F denote the subtorus obtained by restricting $\tilde{\mathcal{C}}/\mathbb{Z}^d$ to F. So T_0 is just $\tilde{\mathcal{C}}/\mathbb{Z}^d$ itself, and T_1 is not actually a torus but rather a point. Our assertion is equivalent to

$$\mu(F,1) \quad = \quad (-1)^{\dim F} \cdot \#\{\text{max cells in } T_F\} \tag{9}$$

Let $\mu'(F)$ denote the right-hand side above. By the definition of the Möbius function of a poset, the equation (9) is equivalent to

$$\sum_{F \le G \le 1} \mu'(G) \;=\; \delta_{F,1}. \qquad \text{(Kronecker delta)}$$

The left hand side of this equation is the (non-reduced) Euler characteristic of T_F. This is 0 since T_F is a torus, unless $F = 1$ so that T_F is a point and then it is 1. □

We remark that Theorem 4.1 can be generalized to arbitrary toric arrangements $\widetilde{\mathcal{C}}/\mathbb{Z}^d$ without the unimodularity hypothesis. The face count formula is a sum of local Möbius function values over all (now more than one) vertices of $\widetilde{\mathcal{C}}/\mathbb{Z}^d$. That generalization has interesting applications to hypergeometric functions, which will be the subject of a future publication. The enumerative applications in the next section all involve unimodular arrangements, so we restrict ourselves to this case. We shall need the following recursion for computing cocharacteristic polynomials.

Proposition 4.2. *Let H be a hyperplane of the arrangement $\mathcal{C}$. Then*

$$\psi_{\mathcal{C}}(q) \quad = \quad \psi_{\mathcal{C}\cap H}(q) \,+\, q \cdot \sum_{c} \psi_{\mathcal{C}/c}(q),$$

where the sum is over all lines c of the arrangement $\mathcal{C}$ that are not contained in H.

The lines c of the arrangement $\mathcal{C}$ are the coatoms of the intersection lattice L. The arrangement $\mathcal{C}/c$ is the hyperplane arrangement $\{\, H_i/c \;:\; c \in H_i \,\}$ in the $(d-1)$-dimensional vectorspace $\mathbb{R}^d/c$. Note that if c is a simple intersection, that is, if c lies on only $d-1$ hyperplanes H_i, then $\psi_{\mathcal{C}/c}(q) = (1+q)^{d-1}$. Note that Proposition 4.2, together with the condition $\psi_{\mathcal{C}}(q) = 1$ for the 0-dimensional arrangement $\mathcal{C}$, uniquely defines the cocharacteristic polynomial.

Proof: The intersection lattice L of any central hyperplane arrangement $\mathcal{C}$ is *semi-modular*, that is, if both F and G cover $F \wedge G$, then $F \vee G$ covers both F and G, see [17, Section 3.3.2]. The assertion follows from the relation [17, formula 3.10.(27)] for the Möbius functions of any semi-modular lattice. □

In the remainder of this section we review the algebraic context in which unimodular toric arrangements arise in [2]. This provides a Gröbner basis interpretation for our proof of Theorem 4.1 and it motivates our enumerative results in Section 5.

Denote by B the $n \times d$ matrix whose rows are $h_1, \ldots, h_n$. All $d \times d$-minors of B are $-1, 0$, or $+1$. The *unimodular Lawrence ideal* of $\mathcal{C}$ is the binomial prime ideal

$$J_{\mathcal{C}} \;:=\; \langle\, \mathbf{x}^a\mathbf{y}^b - \mathbf{y}^a\mathbf{x}^b \;|\; a,b \in \mathbb{N}^n\,,\; a-b \in \mathrm{Image}(B) \,\rangle \quad \text{in} \quad \mathbf{k}[x_1,\ldots,x_n,y_1,\ldots,y_n].$$

The main result of [2] states that the toric arrangement $\mathcal{C}/\mathbb{Z}^d$ supports a cellular resolution of $J_{\mathcal{C}}$. In particular, the Betti numbers of the unimodular Lawrence ideal $J_{\mathcal{C}}$ are precisely the coefficients of the cocharacteristic polynomial $\psi_{\mathcal{C}}(q)$.

The construction in the proof of Theorem 4.1 has a Gröbner basis interpretation. Indeed, the generic vector $w \in \mathbb{R}^d$ defines a term order $\succ$ for the ideal $J_{\mathcal{C}}$ as follows:

$$\mathbf{x}^a\mathbf{y}^b \succ \mathbf{y}^a\mathbf{x}^b \quad \text{if} \quad a - b = B \cdot u \text{ for some } u \in \mathbb{R}^d \text{ with } w \cdot u > 0.$$

It is shown in [2, §4] that the initial monomial ideal $in_{\succ}(J_{\mathcal{C}})$ of $J_{\mathcal{C}}$ with respect to these weights is the oriented matroid ideal associated with the restriction of the central arrangement $\mathcal{C}$ to the affine hyperplane $\{ v \in \mathbb{R}^d : w \cdot v = 1\}$. In symbols,

$$in_{\succ}(J_{\mathcal{C}}) = O_{\mathcal{A}}.$$

In fact, in the unimodular case, Theorem 1.3 (b) is precisely Theorem 4.4 in [2].

Corollary 4.3. *The Betti numbers of the unimodular Lawrence ideal $J_{\mathcal{C}}$, and all its initial ideals $in_{\succ}(J_{\mathcal{C}})$, are the coefficients of the cocharacteristic polynomial $\psi_{\mathcal{C}}$.*

We close this section with a non-trivial example. Let $n = 9, d = 4$ and consider

$$B^T = \begin{array}{c} \begin{matrix} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} & x_{31} & x_{32} & x_{33} \end{matrix} \\ \begin{pmatrix} 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

All nonzero 4×4-minors of this matrix are -1 or $+1$, and hence we get a unimodular central arrangement $\mathcal{C}$ of nine hyperplanes in $\mathbb{R}^4$. This is the *cographic arrangement* associated with the complete bipartite graph $K_{3,3}$. The nine hyperplane variables x_{ij} represent edges in $K_{3,3}$. The associated Lawrence ideal can be computed by saturation (e.g. in `Macaulay2`) from (binomials representing) the four rows of B^T:

$$J_B = \langle\, x_{11}x_{22}y_{12}y_{21} - x_{12}x_{21}y_{11}y_{22}\,,\ x_{12}x_{23}y_{13}y_{22} - x_{13}x_{22}y_{12}y_{23}\,,$$
$$x_{21}x_{32}y_{22}y_{31} - x_{22}x_{31}y_{21}y_{32}, x_{22}x_{33}y_{23}y_{32} - x_{23}x_{32}y_{22}y_{33}\,\rangle : (\textstyle\prod_{1\le i,j\le 3} x_{ij}y_{ij})^{\infty}$$

This ideal has 15 minimal generators, corresponding to the 15 circuits in the directed graph $K_{3,3}$. A typical initial monomial ideal $in_{\prec}(J_B) = O_{\mathcal{A}}$ looks as follows:

$$\big\langle x_{11}x_{22}y_{12}y_{21},\ x_{11}x_{23}y_{13}y_{21},\ x_{11}x_{32}y_{12}y_{31},\ x_{11}x_{33}y_{13}y_{31},\ x_{12}x_{23}y_{13}y_{22},$$
$$x_{12}x_{33}y_{13}y_{32},\ x_{21}x_{32}y_{22}y_{31},\ x_{21}x_{33}y_{23}y_{31},\ x_{22}x_{33}y_{23}y_{32},$$
$$x_{11}x_{22}x_{33}y_{13}y_{21}y_{32},\ x_{11}x_{22}x_{33}y_{12}y_{23}y_{31},\ x_{11}x_{23}x_{32}y_{13}y_{22}y_{31},$$
$$x_{12}x_{21}x_{33}y_{11}y_{23}y_{32},\ x_{12}x_{21}x_{33}y_{13}y_{22}y_{31},\ x_{13}x_{21}x_{32}y_{12}y_{23}y_{31}\big\rangle.$$

This is the oriented matroid ideal of the 3-dimensional affine arrangement $\mathcal{A}$ gotten from $\mathcal{C}$ by taking a vector $w \in \mathbb{R}^4$ with strictly positive coordinates. This ideal is the intersection of 81 monomial primes, one for each spanning tree of $K_{3,3}$. By Theorem 2.9, they form a triangulation of a 13-dimensional Lawrence polytope, which is given by its centrally symmetric Gale diagram $(B^T, -B^T)$ as in [6, Proposition 9.3.2 (b)]. Resolving this ideal (e.g. in `Macaulay2`), we obtain the cocharacteristic polynomial:

$$\psi_{\mathcal{C}}(q) = 1 + 15q + 48q^2 + 54q^3 + 20q^4. \tag{10}$$

It was asked in [2, §5] what such Betti numbers arising from graphic and cographic ideals are in general. This question will be answered in the following section.

5. Graphic and Cographic matroids

Two classes of (central) unimodular arrangements arise from graphs: *graphic* and *cographic* arrangements. Our aim is to compute their cocharacteristic polynomial. This task is easier for graphic arrangements which will be treated first. Cographic arrangements are more challenging and will be discussed further below.

Fix a connected graph G with vertices $[d] = \{1, \ldots, d\}$ and edges $E \subset [d] \times [d]$. Let $V = \{(v_1, \ldots, v_d) \in \mathbb{R}^d : v_1 + \ldots + v_d = 0\} \simeq \mathbb{R}^{d-1}$. The graphic arrangement $\mathcal{C}_G$ is the arrangement in V given by the hyperplanes $v_i = v_j$ for $(i, j) \in E$. It is unimodular [20]. For each subset $S \subset [d]$ we get an *induced subgraph* $G|_S = \big(S,\, E \cap (S \times S)\big)$. For a partition π of $[d]$, we denote by G/π the graph obtained from G by contracting all edges whose vertices lie in the same part of π. The intersection lattice L_G of the graphic arrangement $\mathcal{C}_G$ has the following well-known description in terms of the *partition lattice* Π_d. See e.g. [21] for proofs and references.

Proposition 5.1. *The intersection lattice L_G is isomorphic to the sublattice of the partition lattice Π_d consisting of partitions π such that, for each part S of π, the subgraph $G|_S$ is connected. The element V_π of L_G corresponding to $\pi \in \Pi_d$ is the intersection of the hyperplanes $\{v_i = v_j\}$ for pairs i, j in the same part of π. The dimension of V_π is equal to the number of parts of π minus 1. The interval $[V_\pi, \hat{1}\,]$ of the intersection lattice L_G is isomorphic to the intersection lattice $L_{G/\pi}$.*

We write $\mu(G) = |\mu_{L_G}(\hat{0}, \hat{1})|$ for the Möbius invariant of the intersection lattice. Thus $\mu(G)$ equals the *Cohen-Macaulay type* (top Betti number) of the matroid ideal

$$M_G \quad = \quad \bigcap \big\{ \langle\, x_{ij} \,:\, (i,j) \in F \,\rangle \;\mid\; F \subseteq E \text{ is a spanning tree of } G \,\big\}.$$

From Proposition 5.1 and (8), we conclude that all the lower Betti numbers can be expressed in terms of the Möbius invariants of the contractions G/π of G.

Corollary 5.2. *The cocharacteristic polynomial of the graphic arrangement $\mathcal{C}_G$ is*

$$\psi_{\mathcal{C}_G}(q) \quad = \quad \sum_{\pi \in L_G} \mu(G/\pi) \cdot q^{|\pi|-1}.$$

This reduces our problem to computing the Möbius invariant $\mu(G)$ of a graph G. Green and Zaslavsky [10] found the following combinatorial formula. An *orientation* of the graph G is a choice, for each edge (i, j) of G, of one of the two possible directions: $i \to j$ or $j \to i$. An orientation is *acyclic* if there is no directed cycle.

Proposition 5.3. *Fix a vertex i of G. Then $\mu(G)$ equals the number of acyclic orientations of G such that, for any vertex j, there is a directed path from i to j.*

Proof: The regions of the graphic arrangement $\mathcal{C}_G$ are in one-to-one correspondence with the acyclic orientations of G: the region corresponding to an acyclic orientation o is given by the inequalities $x_i > x_j$ for any directed edge $i \to j$ in o.

The linear functional $w : (u_1, \ldots, u_d) \mapsto u_i$ is generic for the arrangement $\mathcal{C}_G$. The Möbius invariant $\mu(G)$ equals the number of regions of $\mathcal{C}_G$ which are bounded below with respect to w. We claim that the acyclic orientations corresponding to the w-bounded regions are precisely the ones given in our assertion.

Suppose that, for any vertex j in G, there is a directed path $i \to \cdots \to j$. For any point $(u_1, \ldots, u_d)$ of the corresponding region, this path implies $u_i > \cdots > u_j$. The condition $u_1 + \cdots + u_m = 0$ forces $w(u) = u_i > 0$. This implies that the region is w-positive. Conversely, consider any acyclic orientation which does not satisfy

the condition in Proposition 5.3. Then there exists a vertex $j \neq i$ which is a source of o. Then the vector $v = (-1, \ldots, -1, d-1, -1, \ldots, -1)$, where $d-1$ is in the j-th coordinate, belongs to the closure of the region associated with o. But $w(v) = -1$. Hence the region is not w-positive. □

The above discussion can be translated into a combinatorial recipe for writing the minimal free resolution of graphic ideals M_G, where each syzygy is indexed by a certain acyclic orientation of a graph G/π. For the case of the *complete graph* $G = K_d$, we recover the resolution in [2, Theorem 5.3]. Note that the intersection lattice L_{K_d} is isomorphic to the partition lattice Π_d. For any partition π of $\{1, \ldots, d\}$ with $i+1$ parts, K_d/π is isomorphic to K_{i+1}. The number of such partitions equals $S(d, i+1)$, the *Stirling number* of the second kind. The number of acyclic orientations of K_{i+1} with a unique fixed source equals $i\,!$. We conclude

Corollary 5.4. *The number of minimal i-th syzygies of M_{K_d} equals $i!\ S(d, i+1)$.*

Remark 5.5. Vic Reiner suggested to us the following combinatorial interpretation of $\mu(G)$. It can be derived from Proposition 5.3. For any graph G, the Möbius invariant $\mu(G)$ counts the number of equivalence classes of linear orderings of the vertices of G, under the equivalence relation generated by the following operations:

- commuting two adjacent vertices v, v' in the ordering if $\{v, v'\}$ is not an edge of G,
- cyclically shifting the entire order, i.e. $v_1 v_2 \ldots v_n \leftrightarrow v_2 \ldots v_n v_1$.

Invariance under the second operation makes this interpretation convenient for writing down the minimal free resolution of the graphic Lawrence ideals in [2, §5].

Another application arises when (W, S) is a Coxeter system and G its Coxeter graph (considered without its edge labels). Suppose $S = \{s_1, \ldots, s_n\}$. Then $\mu(G)$ counts the number of Coxeter elements $s_{i_1} \ldots s_{i_n}$ of G up to the equivalence relation $s_{i_1} s_{i_2} \ldots s_{i_n} \leftrightarrow s_{i_2} \ldots s_{i_n} s_{i_1}$.

We now come to the cographic arrangement $\mathcal{C}_G^\perp$, whose matroid is dual to that of $\mathcal{C}_G$. Fix a directed graph G on $[d]$ with edges E, where G is allowed to have loops and multiple edges. We associate with G the multiset of vectors $\{v_e \in \mathbb{Z}^d : e \in E\}$, where for an edge $e = (i \to j)$, the ith coordinate of v_e is 1, the jth coordinate is -1, and all other coordinates are 0. Set $v_e = 0$ for a loop $e = (i \to i)$ of G. Let $V_G = \{\lambda : E \to \mathbb{R} \mid \sum_{e \in E} \lambda(e) v_e = 0\}$. Note that V_G is a vector space of dimension $\#\{\text{edges}\} - \#\{\text{vertices}\} + \#\{\text{connected components}\}$. The *cographic arrangement* $\mathcal{C}_G^\perp$ is the arrangement in V_G given by hyperplanes $H_e = \{\lambda \in V_G : \lambda(e) = 0\}$ for $e \in E$. It is unimodular [20]. We write $\mu^\perp(G) = |\mu_{L_G^\perp}(\hat{0}, \hat{1})|$ for the Möbius invariant of the intersection lattice $L_G^\perp$ of $\mathcal{C}_G^\perp$, and we refer to this number as the *Möbius coinvariant* of G. Thus $\mu^\perp(G)$ is the Cohen-Macaulay type of the cographic ideal $J_{C_G^\perp}$ in [2, §5].

Remark 5.6. The characteristic polynomial of a matroid can be expressed via the Tutte dichromatic polynomial [19]. Thus Möbius invariant and coinvariant of a graph G are certain values of the Tutte polynomial: $\mu(G) = T_G(1, 0)$ and $\mu^\perp(G) = T_G(0, 1)$. We do not know, however, how to express the cocharacteristic polynomial $\psi(q)$ in terms of the Tutte polynomial.

A formula for the Tutte polynomial due to Gessel and Sagan [9, Theorem 2.1] implies:

Proposition 5.7. *The Möbius coinvariant of G is $\mu^{\perp}(G) = \sum_{F \subseteq G} (-1)^{d-|F|-1}$, where the sum is over all forests in G and $|F|$ denotes the number of edges in F.*

We shall derive explicit formulas for the Möbius coinvariant of complete and complete bipartite graphs. A subgraph M of a graph G is called a *partial matching* if it is a collection of pairwise disjoint edges of the graph. For a partial matching M, let $a(M)$ be the number of vertices of G that have degree 0 in M. The *Hermite polynomial* $H_n(x)$, $n \geq 0$, is the generating function of partial matchings in the complete graph K_n:

$$H_n(x) = \sum_M x^{a(M)},$$

where the sum is over all partial matchings in K_n. In particular, $H_0(x) = 1$. Set also $H_{-1}(x) = 0$. The main result of this section is the following formula:

Theorem 5.8. *The Möbius coinvariant of the complete graph K_m equals*

$$\mu^{\perp}(K_m) \quad = \quad (m-2)\, H_{m-3}(m-1)\,, \qquad m \geq 2\,. \tag{11}$$

A few initial numbers $\mu^{\perp}(K_m)$ are given below.

m	2	3	4	5	6	7	8	9	10	$\cdots$
$\mu^{\perp}(K_m)$	0	1	6	51	560	7575	122052	2285353	48803904	$\cdots$

The proof of Theorem 5.8 relies on several auxiliary results and will be given below. The next proposition summarizes well-known properties of Hermite polynomials.

Proposition 5.9. *The Hermite polynomial $H_n(x)$ satisfies the recurrence*

$$\begin{aligned} &H_{-1}(x) = 0, \quad H_0(x) = 1, \\ &H_{n+1}(x) = x\, H_n(x) + n\, H_{n-1}(x), \quad n \geq 0. \end{aligned} \tag{12}$$

It is given explicitly by the formula

$$H_n(x) \quad = \quad x^n + \sum_{k \geq 1}^{[n/2]} \binom{n}{2k} (2k-1)!!\, x^{n-2k},$$

where $(2k-1)!! = (2k-1)(2k-3)(2k-5)\cdots 3 \cdot 1$.

Proof: In a partial matching the first vertex has either degree 0 or 1. This gives two terms in the right-hand side of the recurrence (12). The formula for $H_n(x)$ follows from the fact that there are $(2k-1)!!$ matchings with k edges on $2k$ vertices. □

Returning to general cographic arrangements, recall that an edge e of the graph G is called an *isthmus* if $G \backslash e$ has more connected components than G; a graph is called *isthmus-free* if no edge of G is an isthmus. The minimal nonempty isthmus-free subgraphs of G are the *cycles* of G. For a subgraph H of G, denote by G/H the graph obtained from G by contracting the edges of H. Note that G/H may have loops and multiple edges even if G does not. The following result appears in [10].

Proposition 5.10. *The intersection lattice $L_G^{\perp}$ of the cographic arrangement is isomorphic to the lattice of isthmus-free subgraphs of G ordered by reverse inclusion. The element of the intersection lattice that corresponds to an isthmus-free subgraph H is $V_H \subset V_G$. The coatoms of the lattice $L_G^{\perp}$ are the cycles of G. For two isthmus-free subgraphs $H \supset K$ of G, the interval $[V_H, V_K]$ of the intersection lattice $L_G^{\perp}$ is isomorphic to the interval $[\hat{0}, \hat{1}]$ of the intersection lattice $L_{H/K}^{\perp}$.*

Proposition 4.2 implies the following recurrence for the cocharacteristic polynomial $\psi_{\mathcal{C}_G^\perp}(q)$ of the cographic arrangement $\mathcal{C}_G^\perp$.

Corollary 5.11. *Let e be an edge of the graph G. Then*

$$\psi_{\mathcal{C}_G^\perp}(q) = \psi_{\mathcal{C}_{G\setminus e}^\perp}(q) + q\sum_C \psi_{\mathcal{C}_{G/C}^\perp}(q), \tag{13}$$

where the sum is over all cycles C of G that contain e.

Considering terms of the highest degree in (13), we obtain

Corollary 5.12. *If e is any edge of G that is not an isthmus, then*

$$\mu^\perp(G) \quad = \quad \sum_C \mu^\perp(G/C), \tag{14}$$

where the sum is over all cycles C of G that contain e.

Note that $\mu^\perp(G)$ is equal to the Möbius coinvariant of the graph $\widetilde{G}$ obtained from G by removing all loops and isthmuses. Thus when we use relation (14) to calculate $\mu^\perp(G)$, we may remove all new loops obtained after contracting the cycle C.

We are ready to prove Theorem 5.8. For $n \geq 0$ and $k \geq 1$, define $K_n^{(k)}$ to be the complete graph K_n on the vertices $1, \ldots, n$, together with one additional vertex $n+1$ (root) connected to each vertex $1, \ldots, n$ by k edges. Let $\mu_n^{(k)} = \mu^\perp(K_n^{(k)})$ be the Möbius-coinvariant of the graph $K_n^{(k)}$. Note that $K_m = K_{m-1}^{(1)}$ and $\mu^\perp(K_m) = \mu_{m-1}^{(1)}$. Theorem 5.8 can be extended as follows:

Proposition 5.13. $\mu_n^{(k)} = H_n(n+k-1) - n\,H_{n-1}(n+k-1)$ *for $n, k \geq 1$.*

Proof: We utilize Corollary 5.12. Select an edge $e = (n, n+1)$ of the graph $K_n^{(k)}$. There are $k-1$ choices for a cycle C of length 2 that contains the edge e, and the graph $K_n^{(k)}/C$, after removing loops, is isomorphic to $K_{n-1}^{(k+1)}$. There are $(n-1)\,k$ choices for a cycle C of length 3 that contains the edge e, and the graph $K_n^{(k)}/C$, after removing loops, is isomorphic to $K_{n-2}^{(k+2)}$. In general, for cycles of length $l \geq 3$, there are $k\,(n-1)(n-2)\cdots(n-l+2)$ choices, and we obtain a graph that is isomorphic to $K_{n-l+1}^{(k+l-1)}$. Equation (14) implies the following recurrence for $\mu_n^{(k)}$:

$$\begin{aligned} \mu_n^{(k)} \quad &= \quad (k-1)\,\mu_{n-1}^{(k+1)} + k\,(n-1)\,\mu_{n-2}^{(k+2)} + \\ &+ k\,(n-1)(n-2)\,\mu_{n-3}^{(k+3)} + k\,(n-1)(n-2)(n-3)\,\mu_{n-4}^{(k+4)} + \cdots, \end{aligned} \tag{15}$$

which, together with the initial condition $\mu_0^{(k)} = 1$, defines the numbers $\mu_n^{(k)}$ uniquely. Set

$$b_n^{(k)} \quad = \quad \mu_n^{(k)} + n\,\mu_{n-1}^{(k+1)} + n(n-1)\,\mu_{n-2}^{(k+2)} + \cdots + n(n-1)\cdots 1\,\mu_0^{(k+n)}.$$

Then $\mu_n^{(k)} = b_n^{(k)} - nb_{n-1}^{(k+1)}$ and the relation (15) can be rewritten as

$$b_n^{(k)} - n\,b_{n-1}^{(k+1)} = (k-1)\left(b_{n-1}^{(k+1)} - (n-1)\,b_{n-2}^{(k+2)}\right) + k\,(n-1)\,b_{n-2}^{(k+2)},$$

or, simplifying, as

$$b_n^{(k)} = (n+k-1)b_{n-1}^{(k+1)} + (n-1)\,b_{n-2}^{(k+2)}. \tag{16}$$

We claim that $b_n^{(k)} = H_n(n+k-1)$. Indeed, $b_0^{(k)} = 1$, $b_1(k) = k$, and equation (16) is equivalent to the defining relation (12) for the Hermite polynomials. Hence $\mu_n^{(k)} = b_n^{(k)} - n b_{n-1}^{(k+1)} = H_n(n+k-1) - nH_{n-1}(n+k-1)$. □

Proof of Theorem 5.8: By Proposition 5.13 and equation (12),

$$\mu^{\perp}(K_m) = \mu_{m-1}^{(1)} = H_{m-1}(m-1) - (m-1)\, H_{m-2}(m-1) = (m-2)\, H_{m-3}(m-1).$$

□

We now discuss a bipartite analog of Hermite polynomials. For a partial matching M in the complete bipartite graph $K_{m,n}$, denote by $a(M)$ the number of vertices in the first part that have degree 0 in M, and by $b(M)$ the number of vertices in the second part that have degree 0. Define

$$H_{m,n}(x,y) \quad = \quad \sum_M x^{a(M)}\, y^{b(M)}\,,$$

where the sum is over all partial matchings in $K_{m,n}$. In particular $H_{m,0} = x^m$ and $H_{0,n} = y^n$. Set also $H_{m,-1} = H_{-1,n} = 0$. The following statement is a bipartite analogue of Theorem 5.8.

Theorem 5.14. *The Möbius coinvariant of the complete bipartite graph $K_{m,n}$ equals*

$$\mu^{\perp}(K_{m,n}) \quad = \quad (m-1)(n-1)H_{m-2,\,n-2}(n-1, m-1)\,, \qquad m,n \geq 1\,.$$

Analogously to Proposition 5.9 we have

Proposition 5.15. *The polynomial $H_{m,n}(x,y)$ is given by*

$$H_{m,n}(x,y) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, x^{m-k} y^{n-k}\,.$$

It satisfies the following recurrence relations:

$$\begin{aligned} &H_{m,n}(x,y) = x\, H_{m-1,n}(x,y) + n\, H_{m-1,n-1}(x,y), \\ &H_{m,n}(x,y) = y\, H_{m,n-1}(x,y) + m\, H_{m-1,n-1}(x,y), \\ &H_{m,0} = x^m, \qquad H_{0,n} = y^n. \end{aligned} \tag{17}$$

Proof: The first formula is obtained by counting the partial matchings in $K_{m,n}$. The recurrence relations (17) are obtained by distinguishing two cases when the first vertex in the first (second) part of $K_{m,n}$ has degree 0 or 1 in a partial matching. □

Let us define the graph $K_{m,n}^{(k,l)}$ as the complete bipartite graph $K_{m,n}$ with an additional vertex v such that v is connected by k edges with each vertex in the first part and by l edges with each vertex in the second part. Let $\mu_{m,n}^{(k,l)} = \mu^{\perp}(K_{m,n}^{(k,l)})$ be the Möbius coinvariant of this graph. Note that $K_{m,n} = K_{m,n-1}^{(1,0)}$ and, thus, $\mu^{\perp}(K_{m,n}) = \mu_{m,n-1}^{(1,0)}$. Theorem 5.14 can be extended as follows:

Proposition 5.16. *We have*

$$\mu_{m,n}^{(k,l)} = H_{m,n}(n+k-1, m+l-1) - m\,n\; H_{m-1,\,n-1}(n+k-1, m+l-1).$$

Proof: Our proof is similar to that of Proposition 5.13. We utilize Corollary 5.12. Select an edge e of the graph $K^{(k,l)}_{m,n}$ that joins the additional vertex v with a vertex from the first part. There are $k-1$ choices for a cycle C of length 2 that contains the edge e, and the graph $K^{(k,l)}_{m,n}/C$, after removing loops, is isomorphic to $K^{(k,l+1)}_{m-1,n}$. There are $n\,l$ choices for a cycle C of length 3 that contains the edge e, and the graph $K^{(k,l)}_{m,n}/C$, after removing loops, is isomorphic to $K^{(k+1,l+1)}_{m-1,n-1}$. For cycles of length 4, we have $n(m-1)\,k$ choices, and obtain a graph isomorphic to $K^{(k+1,l+2)}_{m-2,n-1}$, etc. In general, for cycles of odd length $2r+1 \geq 3$, we have $l\,n(m-1)(n-1)(m-2)\cdots(m-r+1)(n-r+1)$ choices, and we obtain a graph isomorphic to $K^{(k+r,\,l+r)}_{m-r,\,n-r}$. For cycles of even length $2r+2 \geq 4$, we have $k\,n(m-1)(n-1)(m-2)\cdots(n-r+1)(m-r)$ choices, and we obtain a graph isomorphic to $K^{(k+r,\,l+r+1)}_{m-r-1,\,n-r}$. The equation (14) implies the following recurrence for $\mu^{(k,l)}_{m,n}$:

$$\begin{aligned}\mu^{(k,l)}_{m,n} &= (k-1)\,\mu^{(k,l+1)}_{m-1,n} + l\,n\,\mu^{(k+1,l+1)}_{m-1,n-1} + k\,n(m-1)\,\mu^{(k+1,l+2)}_{m-2,n-1} + \\ &\quad l\,n(m-1)(n-1)\,\mu^{(k+2,l+2)}_{m-2,n-2} + k\,n(m-1)(n-1)(m-2)\,\mu^{(k+2,l+3)}_{m-3,n-2} + \cdots,\end{aligned} \tag{18}$$

which, together with the initial conditions $\mu^{(k,l)}_{0,n} = (l-1)^n$ and $\mu^{(k,l)}_{m,0} = (k-1)^m$, unambiguously defines the numbers $\mu^{(k,l)}_{m,n}$. Let us fix the numbers $p = k+n-1$ and $q = l+m-1$ and write $\mu_{m,n}$ for $\mu^{(p-n+1,q-m+1)}_{m,n}$. Set

$$b_{m,n} = \mu_{m,n} + n\,m\,\mu_{m-1,n-1} + n(n-1)\,m(m-1)\,\mu_{m-2,n-2} + \cdots.$$

Then $\mu_{m,n} = b_{m,n} - m\,n b_{m-1,n-1}$ and the relation (18) can be rewritten as

$$\begin{aligned}b_{m,n} - m\,n\,b_{m-1,n-1} &= -\,(b_{m-1,n} - (m-1)\,n\,b_{m-2,n-1}) + \\ &\quad +(p-n+1)\,b_{m-1,n} + (q-m+1)\,n\,b_{m-1,n-1}\end{aligned}$$

or, simplifying, as

$$b_{m,n} = (p-n)\,b_{m-1,n} + (q+1)n\,b_{m-1,n-1} + (m-1)n\,b_{m-2,n-1}. \tag{19}$$

This relation, together with the initial conditions $b_{0,n} = q^n$, $b_{m,0} = p^m$, $b_{-1,n} = b_{m,-1} = 0$, uniquely determines the numbers $b_{m,n}$.

We claim that $b_{m,n} = H_{m,n}(p,q)$. Indeed, the above initial conditions are satisfied by $H_{m,n}(p,q)$ and (19) follows from the defining relations (17) for the bipartite Hermite polynomials. In order to see this, we write by (17)

$$\begin{aligned}H_{m,n}(p,q) &= p\,H_{m-1,\,n}(p,q) + n\,H_{m-1,\,n-1}(p,q), \\ n\,H_{m-1,\,n}(p,q) &= n\,q\,H_{m-1,\,n-1}(p,q) + n(m-1)\,H_{m-2,\,n-1}(p,q)\,.\end{aligned}$$

The sum of these two equations is equivalent to the equation (19). Hence $\mu^{(k,l)}_{m,n} = b_{m,n} - m\,n\,b_{m-1,n-1} = H_{m,n}(p,q) - m\,n\,H_{m-1,\,n-1}(p,q)$. □

An alternative expression for $\mu^{(k,l)}_{m,n}$ can be deduced from Proposition 5.16:

$$\mu^{(k,l)}_{m,n} = \sum_{r=0}^{\min(m,n)} (1-r)\binom{m}{r}\binom{n}{r} r!\,(n+k-1)^{m-r}(m+l-1)^{n-r}\,. \tag{20}$$

Proof of Theorem 5.14: By Proposition 5.16 and the recurrence relations (17),

$$\begin{aligned}
&\mu^{\perp}(K_{m,n}) \\
&= \mu^{(1,0)}_{m,n-1} = H_{m,n-1}(n-1,m-1) - m(n-1)\, H_{m-1,\,n-2}(n-1,m-1) \\
&= (n-1)\, H_{m-1,\,n-1}(n-1,m-1) - (m-1)(n-1)\, H_{m-1,\,n-2}(n-1,m-1) \\
&= (m-1)(n-1)\, H_{m-2,\,n-2}(n-1,m-1)\,.
\end{aligned}$$

□

For a commutative algebra example illustrating Theorem 5.14, consider the Lawrence ideal $J_B \subset \mathbf{k}[x_{11},\ldots,x_{33},y_{11},\ldots,y_{33}]$ associated with the bipartite graph $K_{3,3}$. This is the Lawrence lifting of the ideal of 2×2-minors of a generic 3×3-matrix. It was discussed in the end of Section 4. Its Cohen-Macaulay type is

$$\mu^{\perp}(K_{3,3}) \quad = \quad (3-1)\cdot(3-1)\cdot H_{1,1}(2,2) \quad = \quad 2\cdot 2\cdot 5 \quad = \quad 20.$$

This is the leading coefficient of the cocharacteristic polynomial in equation (10).

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