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A lattice path approach to counting partitions with minimum rank t

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Abstract

In this paper, we give a combinatorial proof via lattice paths of the following result due to Andrews and Bressoud: for $t \leqslant 1$, the number of partitions of n with all successive ranks at least t is equal to the number of partitions of n with no part of size $2-t$. The identity is a special case of a more general theorem proved by Andrews and Bressoud using a sieve.

Keywords: Integer partitions; Lattice paths

1. Introduction

In this paper, we show how to use a lattice path counting technique to establish a relationship between partitions defined by rank conditions and partitions with forbidden part sizes. We begin with some background on identities of this form.

A *partition* λ of a non-negative integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of integers satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We regard the *Ferrers diagram* of λ as an array of unit squares, left justified, in which the number of squares in row i is λ_i. The largest square subarray in this diagram is the *Durfee square* and $d(\lambda)$ refers to the length of a side. The *conjugate* of λ, denoted λ', is the partition whose ith part is the number of squares in the ith column of the Ferrers diagram of λ.

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The *successive ranks* of λ are the entries of the sequence $(\lambda_1 - \lambda'_1, \ldots, \lambda_d - \lambda'_d)$, where $d = d(\lambda)$ [1,7].

In [2] Andrews proved Theorem 1 below, showing a relationship between partitions defined by a constraint on the successive ranks and partitions defined by a congruence condition on the parts. Theorem 1 is a significant generalization of the Rogers–Ramanujan identities [12] which can be interpreted in this framework. Andrews' original result was for odd moduli M, but Bressoud proved in [3] that the result holds for even moduli as well.

Theorem 1. *For integers M, r, satisfying $0 < r < M/2$, the number of partitions of n whose successive ranks lie in the interval $[-r+2, M-r-2]$ is equal to the number of partitions of n with no part congruent to 0, r, or $-r$ modulo M.*

Andrews used a sieve technique to prove Theorem 1. No bijective proof is known. For the special case of the Rogers–Ramanujan identities ($(r,M) = (1,5)$ and $(r,M) = (2,5)$) Garsia and Milne used their involution principle to produce a bijection [9], which, though far from simple, was the first bijective proof of these identities.

Recently, Theorem 1 attracted the attention of the graph theory community when Erdös and Richmond made use of it to establish a lower bound on the number of graphical partitions of an integer n [8]. A partition is *graphical* if it is the degree sequence of some simple graph. It was observed in [8] that the conjugate of a partition with all successive ranks positive is always graphical and that setting $r = 1$ and $M = n + 2$ in Theorem 1 gives:

Corollary 1. *The number of partitions of n with all successive ranks positive is equal to the number of partitions of n with no part* '1'.

Rousseau and Ali felt that since Corollary 1 is such a special case of Theorem 1, it should have a simple proof. In [13] they give a generating function proof which makes use of a generating function of MacMahon for plane partitions and an identity due to Cauchy. Venkatraman and Wilf, using a generating function for plane partitions due to Bender and Knuth, with the help of q-Ekhad, verified that Corollary 1 remains true when the number of parts is fixed [14]. In [6] a simple bijective proof of Corollary 1 was given, inspired by a result of Cheema and Gordon [5]. It turns out that a (different) bijection can be recovered from a result of Burge [4]. Setting $M = n + r + 1$ and $r = 2 - t$ in Theorem 1 gives the following generalization of Corollary 1. A bijective proof appears in [6].

Corollary 2. *For $t \leqslant 1$, The number of partitions of n with all successive ranks at least t is equal to the number of partitions of n with no part* '$2 - t$'.

So, for example, the number of partitions of 7 with all successive ranks at least -1:

$$\{(7),(6,1),(5,2),(5,1,1),(4,3),(4,2,1),(4,1,1,1),(3,3,1),(3,2,2),(3,2,1,1)\}$$

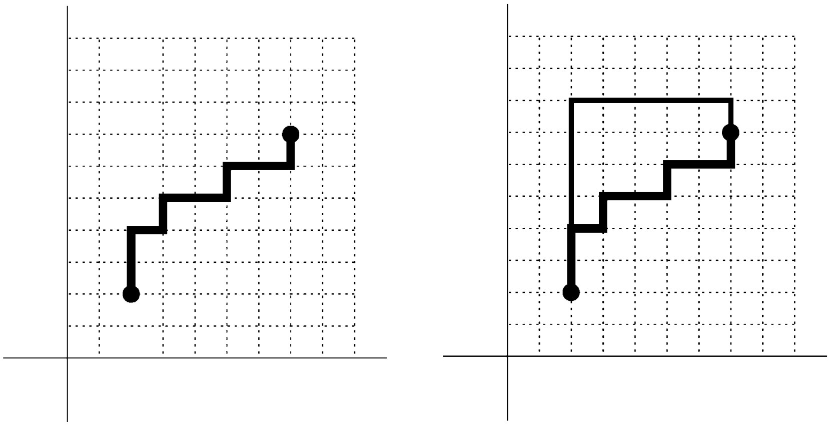

Fig. 1. A north–east lattice path p and the associated partition $\lambda(p) = (5,5,3,1)$.

is the same as the number of partitions of 7 with no part 3:

$$\{(7),(6,1),(5,2),(5,1,1),(4,2,1),(4,1,1,1),(2,2,2,1),$$

$$(2,2,1,1,1),(2,1,1,1,1,1),(1,1,1,1,1,1,1)\}.$$

In this paper, we show how to use a lattice path counting argument to give simple proofs of Corollaries 1 and 2 and several generalizations (all previously known).

In Section 2, we state and prove the 'lattice path identity' and then derive its consequences in Section 3.

2. Lattice paths

For integers $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, define a *north–east lattice path* $p[(x_1,y_1) \to (x_2,y_2)]$ to be a path in the plane from (x_1,y_1) to (x_2,y_2) consisting of unit steps north and east. The region enclosed by p and the lines $x = x_1$, x_2, and $y = y_2 + 1$ can be regarded as the Ferrers diagram of a partition λ_p (see Fig. 1). Let $a(\lambda_p)$ denote the *area* of this region. In this way, we get a bijection between north–east lattice paths $[(x_1,y_1) \to (x_2,y_2)]$ and partitions whose Ferrers diagram fits in an $(x_2 - x_1)$ by $(y_2 - y_1 + 1)$ box. Two lattice paths are called *non-crossing* if they have no point in common. Let $P(n,k)$ be the set of partitions of n with k parts and let $P(n,k,l)$, $P(n,k,>l)$, and $P(n,k,\leqslant l)$ be, respectively, those partitions in $P(n,k)$ with largest part l, those with largest part greater than l, and those with largest part at most l. Let $R_{\geqslant t}(n,k)$ be the set of partitions in $P(n,k)$ with all successive ranks at least t and similarly for $R_{\geqslant t}(n,k,l)$ and $R_{\geqslant t}(n,k,>l)$.

Theorem 2 (Lattice path identity). *For* $t \leqslant 1$,

$$|R_{\geqslant t}(n,k)| = |P(n,k,>k+t-1)| - |P(n-2+t,k-2+t,>k+1)|. \tag{1}$$

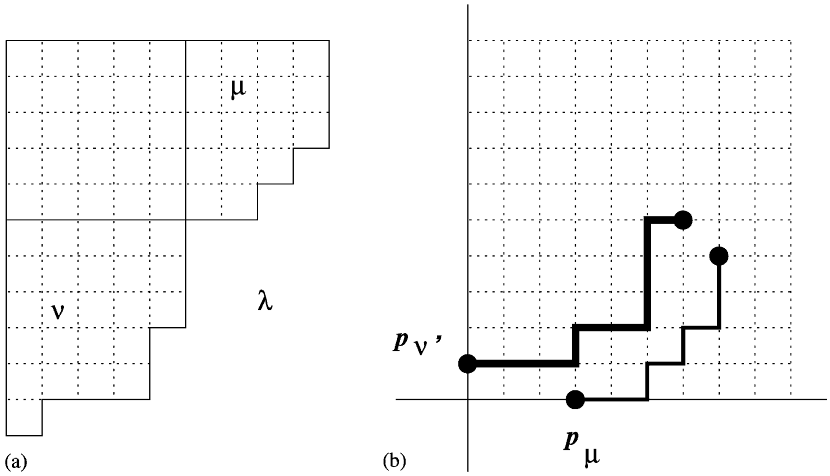

Fig. 2. For $t = -2$, (a) partition λ with $d = 5$, $k = 11$, $l = 9$, and all ranks at least t and (b) the associated lattice paths $p_{\nu'}$ and p_μ.

Proof. Let λ be a partition in $P(n,k,l)$ with Durfee square size d and define partitions μ and ν by $\mu = (\lambda_1 - d, \lambda_2 - d, \ldots, \lambda_d - d)$ (allowing entries to be 0 if necessary) and $\nu = (\lambda_{d+1}, \lambda_{d+2}, \ldots, \lambda_k)$. Note that for $t \leqslant 1$ and $l - k \geqslant t$ the lattice paths

$$p_\mu[(1-t,0) \to (1-t+l-d, d-1)]$$

and

$$p_{\nu'}[(0,1) \to (k-d, d)],$$

associated with μ and ν', are non-crossing if and only if $\mu_i - \nu'_i \geqslant t$ for $i = 1, \ldots, d$, that is, $\lambda \in R_{\geqslant t}(m,k,l)$ (see Fig. 2).

Now, assume $l - k \geqslant t$. (Otherwise, no partition in $P(n,k,l)$ is in $R_{\geqslant t}(n,k,l)$.) Then to count the partitions in $R_{\geqslant t}(n,k,l)$ with Durfee square size d we subtract from $|P(n,k,l)|$ the count of those partitions in $P(n,k,l)$ with Durfee square size d whose corresponding pairs (μ, ν') give rise to a pair of *crossing* lattice paths. We count them using the method of Gessel and Viennot [10,11].

Let $p_\mu[(1-t,0) \to (1-t+l-d, d-1)]$ and $p_{\nu'}[(0,1) \to (k-d,d)]$ be crossing lattice paths associated with the pair (μ, ν') corresponding to a partition $\lambda \in P(n,k,l)$ with Durfee square size d. We find the first point of intersection of these paths (moving north–east) and exchange the parts of those paths before the first intersection to obtain a pair of paths $q[(1-t,0) \to (k-d,d)]$ and $r[(0,1) \to (1-t+l-d, d-1)]$ (see Fig. 3).

Let the partitions $\tilde{\mu}$ and $\tilde{\nu}$ be such that $(\tilde{\mu}, \tilde{\nu}')$ is the pair of partitions associated with the lattice paths r and q, respectively. Then

$$a(\tilde{\mu}) + a(\tilde{\nu}) = a(\mu) + a(\nu) - (1-t), \tag{2}$$

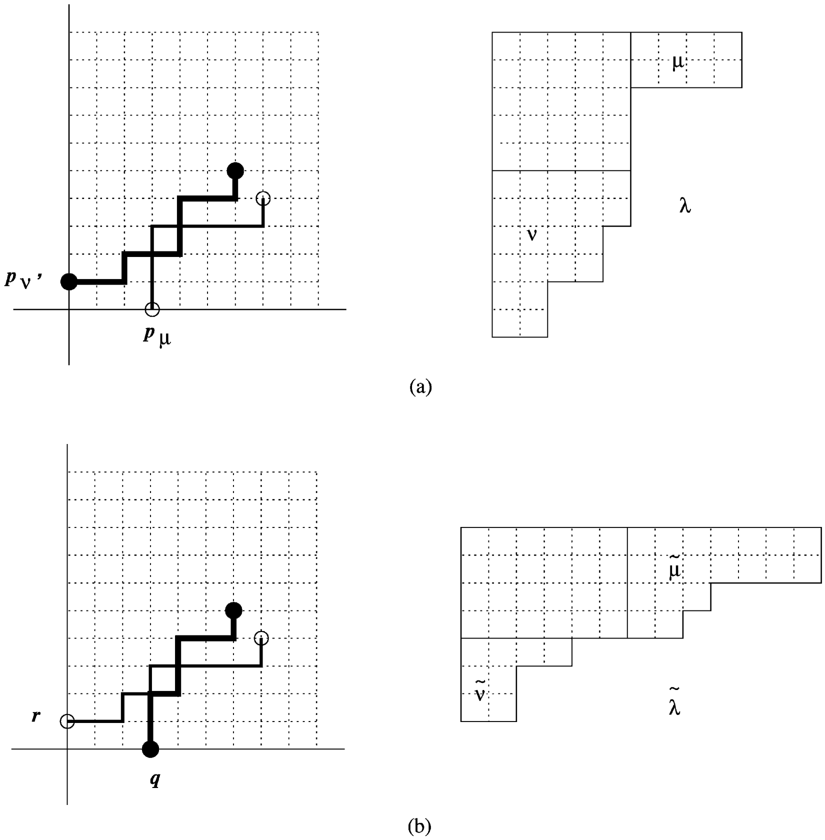

Fig. 3. For $t=-2$, (a) crossing paths corresponding to λ and (b) the paths after swapping, together with their corresponding $\tilde{\lambda}$.

since all the unit squares in both sums of the associated areas are counted with the same multiplicities except for the area defined by $0 \leqslant x \leqslant 1-t,\ d \leqslant y \leqslant d+1$. This area is counted once by the right-hand sum of areas (as part of $a(v)$) and not counted by the left-hand sum. Hence,

$$a(\tilde{\mu}) + a(\tilde{v}) = n - d^2 - (1-t).$$

Now, associate to the pair $(\tilde{\mu}, \tilde{v}')$ the partition $\tilde{\lambda}$ obtained by taking a $(d-1)\times(d+1)$ rectangle (of area d^2-1) and adjoining $\tilde{\mu}$ to the east and $\tilde{v}$ (the conjugate of $\tilde{v}'$) to the south. Then $\tilde{\lambda}$ has the largest part

$$(l-d) + (1-t) + (d+1) = l + 2 - t;$$

the total area of $\tilde{\lambda}$ is

$$(d^2-1) + (n - d^2 - (1-t)) = n - (2-t);$$

and the number of parts of $\tilde{\lambda}$ is

$$((k-d)-(1-t))+(d-1)=k-(2-t).$$

So, $\tilde{\lambda}$ is a partition in $P(n-2+t, k-2+t, l+2-t)$ in which d is the largest integer for which the Ferrers diagram of $\tilde{\lambda}$ contains a $(d-1)\times(d+1)$ subarray. Conversely, any such partition in $P(n-2+t, k-2+t, l+2-t)$, by removing the partitions to the east and south of the $(d-1)\times(d+1)$ rectangle, corresponds to a pair of lattice paths $[(1-t,0)\to(k-d,d)]$ and $[(0,1)\to(1-t+l-d,d-1)]$, which must necessarily cross since, because $l-k\geqslant t$,

$$1-t+l-d\geqslant 1+k-d>k-d.$$

Summing over all values of d gives, for $t\leqslant 1$ and $l-k\geqslant t$,

$$|R_{\geqslant t}(n,k,l)|=|P(n,k,l)|-|P(n-2+t,k-2+t,l+2-t)|.$$

Finally, summing over all $l\geqslant k+t$ gives exactly (1). □

3. Consequences

Let $R_{\geqslant t}(n)$ denote the set of partitions of n with all successive ranks at least t and, as in the previous section, let $R_{\geqslant t}(n,k)$ denote those with exactly k parts. Similarly, let $R_{=t}(n)$ denote the set of partitions of n with minimum rank *equal to t* and $R_{=t}(n,k)$ denote those with k parts. $P(n)$ is the set of all partitions of n. Let $P_s(n)$ denote the set of partitions of n with no part 's' and $P_s(n,k)$ those with k parts. The partitions of n which *do* contain a part 's' are counted by $|P(n-s)|$. So, by splitting $P(n)$ into those partitions which do not contain a part 's' and those which do, we get

$$|P_s(n)|=|P(n)|-|P(n-s)|. \tag{3}$$

If a partition in $P(n,k)$ has no part '1', we can decrease every part by 1 and still have k parts, so

$$|P_1(n,k)|=|P(n-k,k)|. \tag{4}$$

For $P(n,k,l)$, note that, by taking the conjugate,

$$|P(n,k,l)|=|P(n,l,k)|. \tag{5}$$

Also, by partitioning into those partitions which do have a part of size 1 and those which do not,

$$|P(n,k,>l)|=|P(n-1,k-1,>l)|+|P(n-k,k,>l-1)| \tag{6}$$

and

$$|P(n,k,l)|=|P(n-1,k-1,l)|+|P(n-k,k,l-1)|. \tag{7}$$

Therefore, we can write for $t < 1$, applying the lattice path identity (1) for the second equality,

$$\begin{aligned}|R_{=t}(n,k)| &= |R_{\geqslant t}(n,k)| - |R_{\geqslant t+1}(n,k)| \\ &= |P(n,k,>k+t-1)| - |P(n,k,>k+t)| \\ &\quad +|P(n+t-1,k-1+t,>k+1)| \\ &\quad -|P(n+t-2,k-2+t,>k+1)|.\end{aligned}$$

The first two terms on the right-hand side of the last equality give $|P(n,k,k+t)|$ and applying (6) to the last two terms gives

$$|R_{=t}(n,k)| = |P(n,k,k+t)| + |P(n-k,k-1+t,>k)|. \quad (8)$$

Theorem 3. $|R_{\geqslant 1}(n,k)| = |P(n-k,k)| = |P_1(n,k)|$.

Proof.

$$\begin{aligned}|R_{\geqslant 1}(n,k)| &= |P(n,k,>k)| - |P(n-1,k-1,>k+1)| && \text{(from (1))} \\ &= |P(n,k,k+1)| + |P(n,k,>k+1)| \\ &\quad -|P(n-1,k-1,>k+1)| \\ &= |P(n,k,k+1)| + |P(n-k,k,>k)| && \text{(applying (6) to second two terms)} \\ &= |P(n,k+1,k)| + |P(n-k,k,>k)| && \text{(from (5))} \\ &= |P(n-k,k,\leqslant k)| + |P(n-k,k,>k)| && \text{(removing} k \text{ in first term)} \\ &= |P(n-k,k)|.\end{aligned}$$

The last equality in the theorem follows from (4). □

We can now prove the first corollary of the Andrews–Bressoud theorem.

Proof of Corollary 1. From Theorem 3, summing over k, and from (3) we get

$$|R_{\geqslant 1}(n)| = |P_1(n)| = |P(n)| - |P(n-1)|. \quad \square$$

We can also use the lattice path identity to prove the following four lemmas from [6] and the second corollary of the Andrews–Bressoud theorem.

Lemma 1. *For* $t < 0$, $|R_{=t}(n,k)| = |R_{=t+1}(n-1,k-1)|$.

Proof. From (8),

$$|R_{=t}(n,k)| = |P(n,k,k+t)| + |P(n-k,k-1+t,>k)|.$$

Thus, for $t<0$,

$$|R_{=t+1}(n-1,k-1)| = |P(n-1,k-1,k+t)| + |P(n-k,k-1+t,>k-1)|.$$

Use (7) on the first term of the right-hand side and split the second term into those that do and do not have largest part k to get

$$\begin{aligned}|R_{=t+1}(n-1,k-1)| &= |P(n,k,k+t)| - |P(n-k,k,k+t-1)| \\ &\quad + |P(n-k,k-1+t,k)| + |P(n-k,k-1+t,>k)| \\ &= |P(n,k,k+t)| - |P(n-k,k-1+t,k)| \quad \text{from (5)} \\ &\quad + |P(n-k,k-1+t,k)| + |P(n-k,k-1+t,>k)| \\ &= |P(n,k,k+t)| + |P(n-k,k-1+t,>k)| \\ &= |R_{=t}(n,k)|. \qquad \square\end{aligned}$$

Lemma 2. $|R_{=0}(n,k)| = |R_{\geqslant 1}(n-1,k-1)|$.

Proof. From (8),

$$\begin{aligned}|R_{=0}(n,k)| &= |P(n,k,k)| + |P(n-k,k-1,>k)| \\ &= |P(n-k,k-1,\leqslant k)| + |P(n-k,k-1,>k)| \\ &= |P(n-k,k-1)| \\ &= |R_{\geqslant 1}(n-1,k-1)| \quad \text{from Theorem 1.} \qquad \square\end{aligned}$$

Lemma 3. *For* $t<1$, $|R_{=t}(n,k)| = |R_{\geqslant 1}(n-1+t,k-1+t)| = |P(n-k,k-1+t)|$.

Proof. Repeated application of Lemma 1, followed by application of Lemma 2 gives the first equality. The second follows from Theorem 3. □

Summing over k in Lemma 3 gives the following.

Lemma 4. *For* $t \leqslant 0$, $|R_{=t}(n)| = |R_{\geqslant 1}(n-1+t)|$.

Proof of Corollary 2. $(|R_{\geqslant t}(n)| = |P_{2-t}(n)|)$.

$$\begin{aligned}
|R_{\geqslant t}(n)| &= \left(\sum_{j=t}^{0} |R_{=j}(n)|\right) + |R_{\geqslant 1}(n)| \\
&= \sum_{j=t}^{1} |R_{\geqslant 1}(n-1+j)| && \text{by Lemma 4} \\
&= \sum_{j=t}^{1} |P_1(n-1+j)| && \text{by Corollary 1} \\
&= \sum_{j=t}^{1} |(P(n-1+j)| - |P(n-2+j)|) && \text{by (3)} \\
&= |P(n)| - |P(n-2+t)| \\
&= |P_{2-t}(n)| && \text{by (3).} \quad \square
\end{aligned}$$

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