FACES OF GENERALIZED PERMUTOHEDRA

ALEXANDER POSTNIKOV, VICTOR REINER, AND LAUREN WILLIAMS

Abstract. The aim of the paper is to calculate face numbers of simple generalized permutohedra, and study their f-, h- and γ-vectors. These polytopes include permutohedra, associahedra, graph-associahedra, simple graphic zonotopes, nestohedra, and other interesting polytopes.

We give several explicit formulas for h-vectors and γ-vectors involving descent statistics. This includes a combinatorial interpretation for γ-vectors of a large class of generalized permutohedra which are flag simple polytopes, and confirms for them Gal's conjecture on nonnegativity of γ-vectors.

We calculate explicit generating functions and formulae for h-polynomials of various families of graph-associahedra, including those corresponding to all Dynkin diagrams of finite and affine types. We also discuss relations with Narayana numbers and with Simon Newcomb's problem.

We give (and conjecture) upper and lower bounds for f-, h-, and γ-vectors within several classes of generalized permutohedra.

An appendix discusses the equivalence of various notions of deformations of simple polytopes.

Contents

Date: April 28, 2007.

A.P. was supported in part by NSF CAREER Award DMS-0504629. V.R. was supported in part by NSF Grant DMS-0601010. L.W. was supported in part by an NSF Postdoctoral Fellowship.

1. Introduction

Generalized permutohedra are a very well-behaved class of convex polytopes studied in [Post'05], as generalizations of the classical permutohedra, associahedra, cyclohedra, etc. That work explored their wonderful properties from the point of view of valuations such as volumes, mixed volumes, and number of lattice points. This paper focuses on their further good behavior with respect to face enumeration in the case when they are simple polytopes.

Simple generalized permutohedra include as an important subclass (the duals of) the *nested set complexes* considered by DeConcini and Procesi in their work on *wonderful compactifications* of hyperplane arrangements; see [DP'95, FS'05]. In particular, when the arrangement comes from a Coxeter system, one obtains interesting flag simple polytopes studied by Davis, Januszkiewicz, and Scott [DJS'03]. These polytopes can be combinatorially described in terms of the corresponding Coxeter graph. Carr and Devadoss [CD'06] studied these polytopes for arbitrary graphs and called them *graph-associahedra.*

We mention here two other recent papers in which generalized permutohedra have appeared. Morton, Pachter, Shiu, Sturmfels, and Wienand [M–W'06] considered generalized permutohedra from the point of view of *rank tests* on ordinal data in statistics. The normal fans of generalized permutohedra are what they called *submodular rank tests.* Agnarsson and Morris [AM'06] investigated closely the 1-skeleton (vertices and edges) in the special case where generalized permutohedra are Minkowski sums of standard simplices.

Let us formulate several results of the present paper. A few definitions are required. A *connected building set* $\mathcal{B}$ on $[n] := \{1, \ldots, n\}$ is a collection of nonempty subsets in $[n]$ such that

(1) if $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$,
(2) $\mathcal{B}$ contains all singletons $\{i\}$ and the whole set $[n]$,

see Definition 6.1. An interesting subclass of *graphical building sets* $\mathcal{B}(G)$ comes from connected graphs G on $[n]$. The building set $\mathcal{B}(G)$ contains all nonempty subsets of vertices $I \subseteq [n]$ such that the induced graph $G|_I$ is connected.

The *nestohedron* $P_{\mathcal{B}}$ is defined (see Definition 6.3) as the Minkowski sum

$$P_{\mathcal{B}} = \sum_{I \in \mathcal{B}} \Delta_I$$

of the coordinate simplices $\Delta_I := \mathrm{ConvexHull}(e_i \mid i \in I)$, where the e_i are the endpoints of the coordinate vectors in $\mathbb{R}^n$. According to [Post'05, Theorem 7.4] and [FS'05, Theorem 3.14] (see Theorem 6.5 below), the nestohedron $P_{\mathcal{B}}$ is a simple polytope which is dual to a simplicial nested set complex. For a graphical building set $\mathcal{B}(G)$, the nestohedron $P_{\mathcal{B}(G)}$ is called the *graph-associahedron.* In the case when G is the n-path, $P_{\mathcal{B}(G)}$ is the usual associahedron; and in the case when $G = K_n$ is the complete graph, $P_{\mathcal{B}(G)}$ is the usual permutohedron.

Recall that the f-vector and the h-vector of a simple d-dimensional polytope P are $(f_0, f_1, \ldots, f_d)$ and $(h_0, h_1, \ldots, h_d)$, where f_i is the number of i-dimensional faces of P and $\sum h_i\,(t+1)^i = \sum f_i\,t^i$. It is known that the h-vector of a simple polytope is positive and symmetric. Since the h-vector is symmetric, one can define another vector called the γ-vector $(\gamma_1, \gamma_2, \ldots, \gamma_{\lfloor d/2 \rfloor})$ by the relation

$$\sum_{i=0}^{d} h_i\,t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i\,t^i (1+t)^{d-2i}.$$

A simplicial complex Δ is called a *flag complex* (or a *clique complex*) if its simplices are cliques (i.e., subsets of vertices with complete induced subgraphs) of some graph (1-skeleton of Δ). Say that a simple polytope is *flag* if its dual simplicial complex is flag.

Gal conjectured [Gal'05] that the γ-vector has nonnegative entries for any flag simple polytope.

Let us that say a connected building set $\mathcal{B}$ is *chordal* if, for any of the sets $I = \{i_1 < \cdots < i_r\}$ in $\mathcal{B}$, all subsets $\{i_s, i_{s+1}, \ldots, i_r\}$ also belong to $\mathcal{B}$; see Definition 9.2. By Proposition 9.4, graphical chordal building sets $\mathcal{B}(G)$ are exactly building sets coming from chordal graphs. By Proposition 9.7, all nestohedra $P_{\mathcal{B}}$ for chordal building sets are flag simple polytopes. So Gal's conjecture applies to this class of *chordal nestohedra,* which include graph-associahedra for chordal graphs and, in particular, for trees.

For a building set $\mathcal{B}$ on $[n]$, define (see Definition 8.7) the set $\mathfrak{S}_n(\mathcal{B})$ of $\mathcal{B}$-*permutations* as the set of permutations w of size n such that, for any $i = 1, \ldots, n$, there exists $I \in \mathcal{B}$ such that $I \subseteq \{w(1), \ldots, w(i)\}$, and I contains both $w(i)$ and

$\max\{w(1), w(2), \ldots, w(i)\}$. It turns out that $\mathcal{B}$-permutations are in bijection with vertices of the nestohedron $P_{\mathcal{B}}$; see Proposition 8.10.

Let $\mathrm{des}(w) = \#\{i \mid w(i) > w(i+1)\}$ denote the number of descents in a permutation w. Let $\widehat{\mathfrak{S}}_n$ be the subset of permutations w of size n without two consecutive descents and without final descent, i.e., there is no $i \in [n-1]$ such that $w(i) > w(i+1) > w(i+2)$, assuming that $w(n+1) = 0$.

Theorem 1.1. (Corollary 9.6 and Theorem 11.6) *Let $\mathcal{B}$ be a connected chordal building set on $[n]$. Then the h-vector of the nestohedron $P_{\mathcal{B}}$ is given by*

$$\sum_i h_i\, t^i = \sum_{w \in \mathfrak{S}_n(\mathcal{B})} t^{\mathrm{des}(w)},$$

and the γ-vector of the nestohedron $P_{\mathcal{B}}$ is given by

$$\sum_i \gamma_i\, t^i = \sum_{w \in \mathfrak{S}_n(\mathcal{B}) \cap \widehat{\mathfrak{S}}_n} t^{\mathrm{des}(w)}.$$

This result shows that Gal's conjecture is true for chordal nestohedra.

The paper is structured as follows.

Section 2 reviews polytopes, cones, fans, and gives basic terminology of face enumeration for polytopes (*f-vectors*), simple polytopes (*h-vectors*), and *flag* simple polytopes (*γ-vectors*).

Section 3 reviews the definition of generalized permutohedra, and recasts this definition equivalently in terms of their normal fans. It then sets up the dictionary between preposets, and cones and fans coming from the braid arrangement. In particular, one finds that each vertex in a generalized associahedron has associated to it a *poset* that describes its normal cone. This is used to characterize when the polytope is simple, namely when the associated posets have Hasse diagrams which are trees. In Section 4 this leads to a combinatorial formula formula for the h-vector in terms of descent statistics on these tree-posets.

The remainder of the paper deals with subclasses of simple generalized permutohedra. Section 5 dispenses quickly with the very restrictive class of simple zonotopal generalized permutohedra, namely the *simple graphic zonotopes*.

Section 6 then moves on to the interesting class of *nestohedra* $P_{\mathcal{B}}$ coming from a building set $\mathcal{B}$, where the posets associated to each vertex are rooted trees. These include graph-associahedra. Section 7 characterizes the flag nestohedra.

Section 8 discusses $\mathcal{B}$-trees and $\mathcal{B}$-permutations. These trees and permutations are in bijection with each other and with vertices of the nestohedron $P_{\mathcal{B}}$. The h-polynomial of a nestohedron is the descent-generating function for $\mathcal{B}$-trees. Then Section 9 introduces the class of chordal building sets and shows that h-polynomials of their nestohedra are descent-generating functions for $\mathcal{B}$-permutations.

Section 10 illustrates these formulas for h-polynomials by several examples: the classical permutohedron and associahedron, the cyclohedron, the stellohedron (the graph-associahedron for the star graph), and the Stanley-Pitman polytope.

Section 11 gives a combinatorial formula for the γ-vector of all chordal nestohedra as a descent-generating function (or peak-generating function) for a subset of $\mathcal{B}$-permutations. This result implies Gal's nonnegativity conjecture for this class of polytopes. The warm-up example here is the classical permutohedron, and the section concludes with the examples of the associahedron and cyclohedron.

Section 12 calculates the generating functions for f-polynomials of the graph-associahedra for all trees with one branching point and discuss a relation with Simon Newcomb's problem. Section 13 deals with graphs that are formed by a path with two small fixed graphs attached to the ends. It turns out that the h-vectors of graph-associahedra for such path-like graphs can be expressed in terms of h-vectors of classical associahedra. The section includes explicit formulas for graph-associahedra for the Dynkin diagrams of all finite and affine Coxeter groups.

Section 14 gives some bounds and monotonicity conjectures for face numbers of generalized permutohedra.

The paper ends with an Appendix which clarifies the equivalence between various kinds of deformations of a simple polytope.

ACKNOWLEDGMENTS: The authors thank Federico Ardila, Richard Ehrenborg, Ira Gessel, Sangwook Kim, Jason Morton, Margaret Readdy, Anne Shiu, Richard Stanley, John Stembridge, Bernd Sturmfels, Oliver Wienand, and Andrei Zelevinsky for helpful conversations.

2. Face numbers

This section recalls some standard definitions from the theory of convex polytopes and formulate Gal's extension of the Charney-Davis conjecture.

2.1. **Polytopes, cones, and fans.** A *convex polytope* P is the convex hull of a finite collection of points in $\mathbb{R}^n$. The *dimension* of a polytope (or any other subset in $\mathbb{R}^n$) is the dimension of its affine span.

A *polyhedral cone* in $\mathbb{R}^n$ is a subset defined by a conjunction of weak inequalities of the form $\lambda(x) \geq 0$ for linear forms $\lambda \in (\mathbb{R}^n)^*$. A *face* of a polyhedral cone is a subset of the cone given by replacing some of the inequalities $\lambda(x) \geq 0$ by the equalities $\lambda(x) = 0$.

Two polyhedral cones σ_1, σ_2 *intersect properly* if their intersection is a face of each. A *complete fan of cones* $\mathcal{F}$ in $\mathbb{R}^n$ is a collection of distinct nonempty polyhedral cones covering $\mathbb{R}^n$ such that (1) every nonempty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$, and (2) any two cones in $\mathcal{F}$ intersect properly. Cones in a fan $\mathcal{F}$ are also called *faces* of $\mathcal{F}$.

Note that fans can be alternatively defined only in terms of their top dimensional faces, as collections of distinct pairwise properly intersecting n-dimensional cones covering $\mathbb{R}^n$.

A *face* F of a convex polytope P is the set of points in P where some linear functional $\lambda \in (\mathbb{R}^n)^*$ achieves its maximum on P, i.e.,

$$F = \{x \in P \mid \lambda(x) = \max\{\lambda(y) \mid y \in P\}\}.$$

Faces that consist of a single point are called *vertices* and 1-dimensional faces are called *edges* of P.

Given any convex polytope P in $\mathbb{R}^n$ and a face F of P, the *normal cone to P at F*, denoted $\mathcal{N}_F(P)$, is the subset of linear functionals $\lambda \in (\mathbb{R}^n)^*$ whose maximum on P is achieved on all of the points in the face F, i.e.,

$$\mathcal{N}_F(P) := \{\lambda \in (\mathbb{R}^n)^* \mid \lambda(x) = \max\{\lambda(y) \mid y \in P\} \text{ for all } x \in F\}.$$

Then $\mathcal{N}_F(P)$ is a polyhedral cone in $(\mathbb{R}^n)^*$, and the collection of all such cones $\mathcal{N}_F(P)$ as one ranges through all faces F of P gives a complete fan in $(\mathbb{R}^n)^*$ called the *normal fan* $\mathcal{N}(P)$. A fan of the form $\mathcal{N}(P)$ for some polytope P is called a *polytopal fan.*

The combinatorial structure of faces of P can be encoded by the *lattice of faces* of P ordered via inclusion. This structure is also encoded by the normal fan $\mathcal{N}(P)$. Indeed, the map $F \mapsto \mathcal{N}_F(P)$ is an inclusion-reversing bijection between the faces of P and the faces of $\mathcal{N}(P)$.

A cone is called *pointed* if it contains no lines (1-dimensional linear subspaces), or equivalently, if it can be defined by a conjunction of inequalities $\lambda_i(x) \geq 0$ in which the λ_i span $(\mathbb{R}^n)^*$. A fan is called *pointed* if all its faces are pointed.

If the polytope $P \subset \mathbb{R}^n$ is full-dimensional, that is $\dim P = n$, then the normal fan $\mathcal{N}(P)$ is pointed. For polytopes P of lower dimension d, define the $(n-d)$-dimensional subspace $P^\perp \subset (\mathbb{R}^n)^*$ of linear functionals which are constant on P. Then all cones in the normal fan $\mathcal{N}(P)$ contain the subspace $P^\perp$. Thus $\mathcal{N}(P)$ can be reduced to a pointed fan in the space $(\mathbb{R}^n)^*/P^\perp$.

A polytope P is called *simple* if any vertex of P is incident to exactly $d = \dim P$ edges. A cone is called *simplicial* if it can be given by a conjunctions of linear inequalities $\lambda_i(x) \geq 0$ and linear equations $\mu_j(x) = 0$ where the covectors λ_i and μ_j form a basis in $(\mathbb{R}^n)^*$. A fan is called *simplicial* if all its faces are simplicial. Clearly, simplicial cones and fans are pointed. A convex polytope $P \subset \mathbb{R}^n$ is simple if and only if its (reduced) normal fan $\mathcal{N}(P)/P^\perp$ is simplicial.

The *dual simplicial complex* Δ_P of a simple polytope P is the simplicial complex obtained by intersecting the (reduced) normal fan $\mathcal{N}(P)/P^\perp$ with the unit sphere. Note that i-simplices of Δ_P correspond to faces of P of codimension $i+1$.

2.2. **f-vectors and h-vectors.** For a d-dimensional polytope P, the *face number* $f_i(P)$ is the number of i-dimensional faces of P. The vector $(f_0(P), \ldots, f_d(P))$ is called the *f-vector*, and the polynomial $f_P(t) = \sum_{i=0}^d f_i(P)\, t^i$ is called the *f-polynomial* of P.

Similarly, for a d-dimensional fan $\mathcal{F}$, $f_i(\mathcal{F})$ is the number of i-dimensional faces of $\mathcal{F}$, and $f_\mathcal{F}(t) = \sum_{i=0}^d f_i(\mathcal{F})\, t^i$. Note that face numbers of a polytope P and its (reduced) normal cone $\mathcal{F} = \mathcal{N}(P)/P^\perp$ are related as $f_i(P) = f_{d-i}(\mathcal{F})$, or equivalently, $f_P(t) = t^d\, f_\mathcal{F}(t^{-1})$.

We will most often deal with the case where P is a simple polytope, or equivalently, when $\mathcal{F}$ is a simplicial fan. In these situations, there is a more compact encoding of the face numbers $f_i(P)$ or $f_i(\mathcal{F})$ by smaller nonnegative integers. One defines the *h-vector* $(h_0(P), \ldots, h_d(P))$ and *h-polynomial* $h_P(t) = \sum_{i=0}^d h_i(P)\, t^i$ uniquely by the relation

$$(2.1)\quad f_P(t) = h_P(t+1), \quad \text{or equivalently,} \quad f_j(P) = \sum_i \binom{i}{j} h_i(P),\ j = 0, \ldots, d.$$

For a simplicial fan $\mathcal{F}$, the h-vector $(h_0(\mathcal{F}), \ldots, h_d(\mathcal{F})$ and the h-polynomial $h_\mathcal{F}(t) = \sum_{i=0}^d h_i(\mathcal{F})\, t^i$ are defined by the relation $t^d\, f_\mathcal{F}(t^{-1}) = h_\mathcal{F}(t+1)$, or equivalently, $f_j(\mathcal{F}) = \sum_i \binom{i}{d-j} h_i(\mathcal{F})$, for $j = 0, \ldots, d$. Thus the h-vector of a simple polytope coincides with the h-vector of its normal fan.

The nonnegativity of $h_i(P)$ for a simple polytope P comes from its well-known combinatorial interpretation [Zieg'94, §8.2] in terms of the 1-skeleton of the simple polytope P. Let us extend this interpretation to arbitrary complete simplicial fans.

For a simplicial fan $\mathcal{F}$ in $\mathbb{R}^d$, construct the graph $G_\mathcal{F}$ with vertices corresponding to d-dimensional cones and edges corresponding to $(d-1)$-dimensional cones of $\mathcal{F}$, where two vertices of $G_\mathcal{F}$ are connected by an edge whenever the corresponding cones share a $(d-1)$-dimensional face. Pick a vector $g \in \mathbb{R}^d$ that does not belong to any $(d-1)$-dimensional face of $\mathcal{F}$ and orient edges of $G_\mathcal{F}$, as follows. Orient an edge $\{\sigma_1, \sigma_2\}$ corresponding to two cones σ_1 and σ_2 in $\mathcal{F}$ as (σ_1, σ_2) if the vector g points from σ_1 to σ_2 (in a small neighborhood of the common face of these cones).

Proposition 2.1. *For a simplicial fan $\mathcal{F}$, the ith entry $h_i(\mathcal{F})$ of its h-vector equals the number of vertices with outdegree i in the oriented graph $G_\mathcal{F}$. These numbers satisfy the Dehn-Sommerville symmetry: $h_i(\mathcal{F}) = h_{d-i}(\mathcal{F})$.*

Corollary 2.2. (cf. [Zieg'94, §8.2]) *Let $P \in \mathbb{R}^n$ be a simple polytope. Pick a generic linear form $\lambda \in (\mathbb{R}^n)^*$. Let G_P be the 1-skeleton of P with edges directed so that λ increases on each edge. Then $h_i(P)$ is the number of vertices in G_P of outdegree i.*

Proof of Proposition 2.1. The graph $G_\mathcal{F}$ has a unique vertex of outdegree 0. Indeed, this is the vertex corresponding to the cone in $\mathcal{F}$ containing the vector g. For any face F of $\mathcal{F}$ (of an arbitrary dimension), let $G_\mathcal{F}(F)$ be the induced subgraph on the set of d-dimensional cones of $\mathcal{F}$ containing F as a face. Then $G_\mathcal{F}(F) \simeq G_{\mathcal{F}'}$, where $\mathcal{F}'$ is the link of the face F in the fan $\mathcal{F}$, which is also a simplicial fan of smaller dimension. Thus the subgraph $G_\mathcal{F}(F)$ also contains a unique vertex of outdegree 0 (in this subgraph).

There is a surjective map $\phi : F \mapsto \sigma$ from all faces of $\mathcal{F}$ to vertices of $G_\mathcal{F}$ (i.e., d-dimensional faces of $\mathcal{F}$) that sends a face F to the vertex σ of outdegree 0 in the subgraph $G_\mathcal{F}(F)$. Now, for a vertex σ of $G_\mathcal{F}$ of outdegree i, the preimage $\phi^{-1}(\sigma)$ contains exactly $\binom{d-i}{d-j}$ faces of dimension j. Indeed, $\phi^{-1}(\sigma)$ is formed by taking all possible intersections of σ with some subset of its $(d-1)$-dimensional faces $\{F_1, \dots, F_{d-i}\}$ on which the vector g is directed towards the interior of σ; there are exactly $d-i$ such faces because σ has indegree i in $G_\mathcal{F}$. Thus a face of dimension j in $\phi^{-1}(\sigma)$ has the form $F_{i_1} \cap \cdots \cap F_{i_{d-j}}$ for a $(d-j)$-element subset $\{i_1, \dots, i_{d-j}\} \subseteq [d-i]$.

Let $\tilde{h}_i$ be the number of vertices of $G_\mathcal{F}$ of outdegree i. Counting j-dimensional faces in preimages $\phi^{-1}(\sigma)$ one obtains the relation $f_j(\mathcal{F}) = \sum_i \binom{d-i}{d-j} \tilde{h}_i$. Comparing this with the definition of $h_i(\mathcal{F})$, one deduces that $h_i(\mathcal{F}) = \tilde{h}_{d-i}$.

Note that the numbers $h_i(\mathcal{F})$ do not depend upon the choice of the vector g. It follows that the numbers $\tilde{h}_i$ of vertices with given outdegrees also do not depend on g. Replacing the vector g with $-g$ reverses the orientation of all edges in the d-regular graph $G_\mathcal{F}$, implying the the symmetry $\tilde{h}_i = \tilde{h}_{d-i}$. ☐

The Dehn-Sommerville symmetry means that h-polynomials are palindromic polynomials: $t^d\, h_\mathcal{F}(\frac{1}{t}) = h_\mathcal{F}(t)$. In this sense the h-vector encoding is more compact, since it is determined by roughly half of its entries, namely $h_0, h_1, \dots, h_{\lfloor \frac{d}{2} \rfloor}$.

Whenever possible, we will try to either give further explicit combinatorial interpretations or generating functions for the f- and h-polynomials of simple generalized permutohedra.

2.3. **Flag simple polytopes and γ-vectors.** A simplicial complex Δ is called a *flag simplicial complex* or *clique complex* if it has the following property: a collection

C of vertices of Δ forms a simplex in Δ if and only if there is an edge in the 1-skeleton of Δ between any two vertices in C. Thus flag simplicial complexes can be uniquely recovered from their 1-skeleta.

Let us say that a simple polytope P is a *flag polytope* if its dual simplicial complex Δ_P is a flag simplicial complex.

We next discuss γ-vectors of flag simple polytopes, as introduced by Gal [Gal'05] and independently in a slightly different context by Brânden [Brä'04, Brä'06]; see also the discussion in [Stem'07, §1D]. A conjecture of Charney and Davis [ChD'95] led Gal [Gal'05] to define the following equivalent encoding of the f-vector or h-vector of a simple polytope P, in terms of smaller integers, which are conjecturally nonnegative when P is *flag*. Every palindromic polynomial $h(t)$ of degree d has a unique expansion in terms of centered binomials $t^i(1+t)^{d-2i}$ for $0 \leq i \leq d/2$, and so one can define the entries $\gamma_i = \gamma_i(P)$ of the γ-*vector* $(\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor \frac{d}{2} \rfloor})$ and the γ-polynomial $\gamma_P(t) := \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i$ uniquely by

$$h_P(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i \, t^i (1+t)^{d-2i} = (1+t)^d \gamma_P\left(\frac{t}{(1+t)^2}\right).$$

Conjecture 2.3. Gal [Gal'05] *The γ-vector has nonnegative entries for any flag simple polytope. More generally, the nonnegativity of the γ-vector holds for every flag simplicial homology sphere.*

Thus we will try to give explicit combinatorial interpretations, where possible, for the γ-vectors of flag simple generalized permutohedra. As will be seen in Section 7.1, any graph-associahedron is a flag simple polytope.

Remark 2.4. Section 11 later will employ a a certain combinatorial approach to Gal's conjecture and γ-vector nonnegativity that goes back to work of Shapiro, Woan, and Getu [SWG'83], also used by Foata and Strehl, and more recently by Brânden; see [Brä'06] for a thorough discussion.

Suppose P is a simple polytope and one has a combinatorial formula for the h-polynomial $h_P(t) = \sum_{a \in A} t^{f(a)}$, where $f(a)$ is some statistic on the set A. Suppose further that one has a partition of A into *f-symmetric Boolean classes*, i.e. such that the f-generating function for each class is $t^r(1+t)^{2n-r}$ for some r. Let $\widehat{A} \subset A$ be the set of representatives of the classes where $f(a)$ takes its minimal value. Then the γ-polynomial equals $\gamma_P(t) = \sum_{a \in \widehat{A}} t^{f(a)}$.

Call $f(a)$ a "generalized descent-statistic." Additionally, define

$$\text{peak}(a) = \min\{f(b) \mid a \text{ and } b \text{ in the same class}\} + 1$$

and call it a "generalized peak statistic." The reason for this terminology will become apparent in Section 11.

3. Generalized permutohedra and the cone-preposet dictionary

This section reviews the definition of generalized permutohedra from [Post'05]. It then records some observations about the relation between cones and fans coming from the braid arrangement and preposets. (Normal fans of generalized permutohedra are examples of such fans.) This leads to a characterization for when generalized permutohedra are simple, an interpretation for their h-vector in this situation, and a corollary about when the associated toric variety is smooth.

The material in this section and in the Appendix (Section 15) has substantial overlap with recent work on *rank tests of non-parametric statistics* [M–W'06]. We have tried to indicate in places the corresponding terminology used in that paper.

3.1. **Generalized permutohedra.** Recall that a *usual permutohedron* in $\mathbb{R}^n$ is the convex hull of $n!$ points obtained by permuting the coordinates of any vector $(a_1, \ldots, a_n)$ with strictly increasing coordinates $a_1 < \cdots < a_n$. So the vertices of a usual permutohedron can be labelled $v_w = (a_{w^{-1}(1)}, \ldots, a_{w^{-1}(n)})$ by the permutations w in the symmetric group $\mathfrak{S}_n$. The edges of this permutohedron are $[v_w, v_{ws_i}]$, where $s_i = (i, i+1)$ is an adjacent transposition. Then, for any $w \in \mathfrak{S}_n$ and any s_i, one has

$$v_w - v_{ws_i} = k_{w,i}(e_{w(i)} - e_{w(i+1)}) \tag{3.1}$$

where the $k_{w,i}$ are some *strictly positive* real scalars, and $e_1, \ldots, e_n$ are the standard basis vectors in $\mathbb{R}^n$.

Note that a usual permutohedron in $\mathbb{R}^n$ has dimension $d = n - 1$, because it is contained in an affine hyperplane where the sum of coordinates $x_1 + \cdots + x_n$ is constant.

Definition 3.1. [Post'05, Definition 6.1] A *generalized permutohedron* P is the convex hull of $n!$ points v_w in $\mathbb{R}^n$ labelled by the permutations w in the symmetric group $\mathfrak{S}_n$, such that for any $w \in \mathfrak{S}_n$ and any adjacent transposition s_i, one still has equation (3.1), but with $k_{w,i}$ assumed only to be nonnegative, that is, $k_{w,i}$ can vanish.

The Appendix shows that all $n!$ points v_w in a generalized permutohedron P are actually vertices of P (possibly with repetitions); see Theorem 15.3. Thus a generalized permutohedron P comes naturally equipped with the surjective map $\Psi_P : \mathfrak{S}_n \to \text{Vertices}(P)$ given by $\Psi_P : w \mapsto v_w$, for $w \in \mathfrak{S}_m$.

Definition 3.1 says that a generalized permutohedron is obtained by moving the vertices of the usual permutohedron in such a way that directions of edges are preserved, but some edges (and higher dimensional faces) may degenerate. In the Appendix such deformations of a simple polytope are shown to be equivalent to various other notions of deformation; see Proposition 3.2 below and the more general Theorem 15.3.

3.2. **Braid arrangement.** Let $x_1, \ldots, x_n$ be the usual coordinates in $\mathbb{R}^n$. Let $\mathbb{R}^n/(1, \ldots, 1)\mathbb{R} \simeq \mathbb{R}^{n-1}$ denote the quotient space modulo the 1-dimensional subspace generated by the vector $(1, \ldots, 1)$. The *braid arrangement* is the arrangement of hyperplanes $\{x_i - x_j = 0\}_{1 \le i < j \le n}$ in the space $\mathbb{R}^n/(1, \ldots, 1)\mathbb{R}$. These hyperplanes subdivide the space into the polyhedral cones

$$C_w := \{x_{w(1)} \le x_{w(2)} \le \cdots \le x_{w(n)}\}$$

labelled by permutations $w \in \mathfrak{S}_n$, called *Weyl chambers* (of type A). The Weyl chambers and their lower dimensional faces form a complete simplicial fan, sometimes called the *braid arrangement fan.*

Note that a usual permutohedron P has dimension $d = n - 1$, so one can reduce its normal fan modulo the 1-dimensional subspace $P^\perp = (1, \ldots, 1)\mathbb{R}$. The braid arrangement fan is exactly the (reduced) normal fan $\mathcal{N}(P)/P^\perp$ for a usual permutohedron $P \subset \mathbb{R}^n$. Indeed, the (reduced) normal cone $\mathcal{N}_{v_w}(P)/P^\perp$ of P at vertex

v_w is exactly the Weyl chamber C_w. (Here one identifies $\mathbb{R}^n$ with $(\mathbb{R}^n)^*$ via the standard inner product.)

Recall that the *Minkowski sum* $P+Q$ of two polytopes $P, Q \subset \mathbb{R}^n$ is the polytope $P+Q := \{x+y \mid x \in P, y \in Q\}$. Say that P is a *Minkowski summand* of R, if there is a polytope Q such that $P+Q=R$. Say that a fan $\mathcal{F}$ is *refined* by a fan $\mathcal{F}'$ if any cone in $\mathcal{F}$ is a union of cones in $\mathcal{F}'$. The following proposition is a special case of Theorem 15.3.

Proposition 3.2. *A polytope P in $\mathbb{R}^n$ is a generalized permutohedron if and only if its normal fan (reduced by $(1,\dots,1)\mathbb{R}$) is refined by the braid arrangement fan.*

Also, generalized permutohedra are exactly the polytopes arising as Minkowski summands of usual permutohedra.

This proposition shows that generalized permutohedra lead to the study of cones given by some inequalities of the form $x_i - x_j \geq 0$ and fans formed by such cones. Such cones are naturally related to posets and preposets.

3.3. **Preposets, equivalence relations, and posets.** Recall that a *binary relation* R on a set X is a subset of $R \subseteq X \times X$. A *preposet* is a reflexive and transitive binary relation R, that is $(x,x) \in R$ for all $x \in X$, and whenever $(x,y),(y,z) \in R$ one has $(x,z) \in R$. In this case we will often use the notation $x \preceq_R y$ instead of $(x,y) \in R$. Let us also write $x \prec_R y$ whenever $x \preceq_R y$ and $x \neq y$.

An *equivalence relation* $\equiv$ is the special case of a preposet whose binary relation is also symmetric. Every preposet Q gives rise to an equivalence relation $\equiv_Q$ defined by $x \equiv_Q y$ if and only if both $x \preceq_Q y$ and $y \preceq_Q x$. A *poset* is the special case of a preposet Q whose associated equivalence relation $\equiv_Q$ is the trivial partition, having only singleton equivalence classes.

Every preposet Q gives rise to the poset $Q/{\equiv_Q}$ on the equivalence classes $X/{\equiv_Q}$. A preposet Q is uniquely determined by $\equiv_Q$ and $Q/{\equiv_Q}$, that is, a preposet is just an equivalence relation together with a poset structure on the equivalence classes.

A *preposet* Q on X is *connected* if the undirected graph having vertices X and edges $\{x,y\}$ for all $x \preceq_Q y$ is connected.

A *covering relation* $x \lessdot_Q y$ in a poset Q is a pair of elements $x \prec_Q y$ such that there is no z such that $x \prec_Q z \prec_Q y$. The *Hasse diagram* of a poset Q on X is the directed graph on X with edges (x,y) for covering relations $x \lessdot_Q y$.

Let us say that a poset Q is a *tree-poset* if its Hasse diagram is a spanning tree on X. Thus tree-posets correspond to directed trees on the vertex set X.

A *linear extension* of a poset Q on X is a linear ordering $(y_1,\dots,y_n)$ of all elements in X such that $y_1 \prec_Q y_2 \prec_Q \cdots \prec_Q y_n$. Let $\mathcal{L}(Q)$ denote the set of all linear extensions of Q.

The *union* $R_1 \cup R_2$ of two binary relations R_1, R_2 on X is just their union as two subsets of $X \times X$. Given any reflexive binary relation Q, denote by $\overline{Q}$ the preposet which is its transitive closure. Note that if Q_1 and Q_2 are two preposets on the same set X, then the binary relation $Q_1 \cup Q_2$ is not necessarily a preposet. However, we can obtain a preposet by taking its transitive closure $\overline{Q_1 \cup Q_2}$.

Let $R \subseteq Q$ denote containment of binary relations on the same set, meaning containment as subsets of $X \times X$. Also let R^{op} be the *opposite* binary relation, that is $(x,y) \in R^{op}$ if and only if $(y,x) \in R$.

For two preposets P and Q on the same set, let us say that Q is a *contraction* of P if there is a binary relation $R \subseteq P$ such that $Q = \overline{P \cup R^{op}}$. In other words, the equivalence classes of $\equiv_Q$ are obtained by merging some equivalence classes of $\equiv_P$ *along relations* in P and the poset structure on equivalence classes of $\equiv_Q$ is induced from the poset structure on classes of $\equiv_P$.

For example, the preposet $1 < \{2,3\} < 4$ (where $\{2,3\}$ is an equivalence class) is a contraction of the poset $P = (1 < 3,\, 2 < 3,\, 1 < 4,\, 2 < 4)$. However, the preposet $(\{1,2\} < 3,\ \{1,2\} < 4)$ is not a contraction of P because 1 and 2 are incomparable in P.

Definition 3.3. Let us say that two preposets Q_1 and Q_2 on the same set *intersect properly* if the preposet $\overline{Q_1 \cup Q_2}$ is both a contraction of Q_1 and of Q_2.

A *complete fan of posets*[1] *on* X is a collection of distinct posets on X which pairwise intersect properly, and whose linear extensions (disjointly) cover all linear orders on X.

Compare Definition 3.3 to the definitions of properly intersecting cones and complete fan of cones; see Section 2.1. This connection will be elucidated in Proposition 3.5.

Example 3.4. The two posets $P_1 := 1 < 2$ and $P_2 := 2 < 1$ on the set $\{1,2\}$ intersect properly. Here $\overline{P_1 \cup P_2}$ is equal to $\{1 < 2, 2 < 1\}$. These P_1 and P_2 form a complete fan of posets.

However, the two posets $Q_1 := 2 < 3$ and $Q_2 := 1 < 2 < 3$ on the set $\{1,2,3\}$ do not intersect properly. In this case $\overline{Q_1 \cup Q_2} = Q_2$, which is not a contraction of Q_1.

3.4. **The dictionary.** Let us say that a *braid cone* is a polyhedral cone in the space $\mathbb{R}^n/(1,\dots,1)\mathbb{R} \simeq \mathbb{R}^{n-1}$ given by a conjunction of inequalities of the form $x_i - x_j \geq 0$. In other words, braid cones are polyhedral cones formed by unions of Weyl chambers or their lower dimensional faces.

There is an obvious bijection between preposets and braid cones. For a preposet Q on the set $[n]$, let σ_Q be the braid cone in the space $\mathbb{R}^n/(1,\dots,1)\mathbb{R}$ defined by the conjunction of the inequalities $x_i \leq x_j$ for all $i \preceq_Q j$. Conversely, one can always reconstruct the preposet Q from the cone σ_Q by saying that $i \preceq_Q j$ whenever $x_i \leq x_j$ for all points in σ_Q.

Proposition 3.5. *Let the cones σ, σ' correspond to the preposets Q, Q' under the above bijection. Then*

(1) *The preposet $\overline{Q \cup Q'}$ corresponds to the cone $\sigma \cap \sigma'$.*
(2) *The preposet Q is a contraction of Q' if and only if the cone σ is a face σ'.*
(3) *The preposets Q, Q' intersect properly if and only if the cones σ, σ' do.*
(4) *Q is a poset if and only if σ is a full-dimensional cone, i.e.,* $\dim \sigma = n - 1$.
(5) *The equivalence relation $\equiv_Q$ corresponds to the linear span* $\mathrm{Span}(\sigma)$ *of σ.*
(6) *The poset $Q/{\equiv_Q}$ corresponds to a full-dimensional cone inside* $\mathrm{Span}(\sigma_Q)$.
(7) *The preposet Q is connected if and only if the cone σ is pointed.*
(8) *If Q is a poset, then the minimal set of inequalities describing the cone σ is $\{x_i \leq x_j \mid i \lessdot_Q j\}$. (These inequalities associated with covering relations in Q are exactly the facet inequalities for σ.)*
(9) *Q is a tree-poset if and only if σ is a full-dimensional simplicial cone.*
(10) *For $w \in \mathfrak{S}_n$, the cone σ contains the Weyl chamber C_w if and only if Q is a poset and w is its linear extension, that is $w(1) \prec_Q w(2) \prec_Q \cdots \prec_Q w(n)$.*

[1]In [M–W'06], this is called a *convex rank test*.

Proof. (1) The cone $\sigma \cap \sigma'$ is given by conjunction of all inequalities for σ and σ'. The corresponding preposet is obtained by adding all inequalities that follow from these, i.e., by taking the transitive closure of $Q \cup Q'$.

(2) Faces of σ' are obtained by replacing some inequalities $x_i \le x_j$ defining σ' with equalities $x_i = x_j$, or equivalently, by adding the opposite inequalities $x_i \ge x_j$.

(3) follows from (1) and (2).

(4) σ is full-dimensional if its defining relations do not include any equalities $x_i = x_j$, that is $\equiv_Q$ has only singleton equivalence classes.

(5) The cone associated with the equivalence relation $\equiv_Q$ is given by the equations $x_i = x_j$ for $i \equiv_Q j$, which is exactly $\mathrm{Span}(\sigma)$.

(6) Follows from (4) and (5).

(7) The maximal subspace contained in the half-space $\{x_i \le x_j\}$ is given by $x_i = x_j$. Thus the maximal subspace contained in the cone σ is given by the conjunction of equations $x_i = x_j$ for $i \le_Q j$. If Q is disconnected then this subspace has a positive dimension. If Q is connected then this subspace is given by $x_1 = \cdots = x_n$, which is just the origin in the space $\mathbb{R}^n/(1,\ldots,1)\mathbb{R}$.

(8) The inequalities for the covering relations $i \lessdot_Q j$ imply all other inequalities for σ and they cannot be reduced to a smaller set of inequalities.

(9) By (4) and (7) full-dimensional pointed cones correspond to connected posets. These cones will be simplicial if they are given by exactly $n-1$ inequalities. By (8) this means that the corresponding poset should have exactly $n-1$ covering relations, i.e., it is a tree-poset.

(10) Follows from (4) and definitions. □

According to Proposition 3.5, a full-dimensional braid cone σ associated with a poset Q can be described in three different ways (via all relations in Q, via covering relations in Q, and via linear extensions $\mathcal{L}(Q)$ of Q) as

$$\sigma = \{x_i \le x_j \mid i \preceq_Q j\} = \{x_i \le x_j \mid i \lessdot_Q j\} = \bigcup_{w \in \mathcal{L}(Q)} C_w.$$

Let $\mathcal{F}$ be a family of d-cones in $\mathbb{R}^d$ which intersect properly. Since they have disjoint interiors, they will correspond to a complete fan if and only if their closures cover $\mathbb{R}^d$, or equivalently, their spherical volumes sum to the volume of the full $(d-1)$-sphere.

A braid cone corresponding to a poset Q is the union of the Weyl chambers C_w for all linear extensions $w \in \mathcal{L}(Q)$, and every Weyl chamber has the same spherical volume ($\frac{1}{n!}$ of the sphere) due to the transitive Weyl group action. Therefore, a collection of properly intersecting posets $\{Q_1, \ldots, Q_t\}$ on $[n]$ correspond to a complete fan on braid cones if and only if

$$\bigcup_{i=1}^{t} \mathcal{L}(Q_i) = \mathfrak{S}_n \text{ (disjoint union), or equivalently, if and only if } \sum_{i=1}^{t} |\mathcal{L}(Q_i)| = n!,$$

cf. Definition 3.3.

Corollary 3.6. *A complete fan of braid cones (resp., pointed braid cones, simplicial braid cones) in $\mathbb{R}^n/(1,\ldots,1)\mathbb{R}$ corresponds to a complete fan of posets (resp., connected posets, tree-posets) on $[n]$.*

Using Proposition 3.2, we can relate Proposition 3.5 and Corollary 3.6 to generalized permutohedra. Indeed, normal cones of a generalized permutohedron (reduced modulo $(1,\ldots,1)\mathbb{R}$) are braid cones.

For a generalized permutohedron P, define the *vertex poset* Q_v at a vertex $v \in \mathrm{Vertices}(P)$ as the poset on $[n]$ associated with the normal cone $\mathcal{N}_v(P)/(1,\ldots,1)\mathbb{R}$ at the vertex v, as above.

Corollary 3.7. *For a generalized permutohedron (resp., simple generalized permutohedron) P, the collection of vertex posets $\{Q_v \mid v \in \mathrm{Vertices}(P)\}$ is a complete fan of posets (resp., tree-posets).*

Thus normal fans of generalized permutohedra correspond to certain complete fans of posets, which we call *polytopal.* In [M–W'06], the authors call such fans *submodular rank tests,* since they are in bijection with the faces of the cone of submodular functions. That cone is precisely the deformation cone we discuss in the Appendix.

Example 3.8. In [M–W'06], the authors modify an example of Studený [Stud'05] to exhibit a non-polytopal complete fan of posets. They also kindly provided us with the following further nonpolytopal example, having 16 posets Q_v, all of them tree-posets: $(1,2<3<4)$ (which means that $1<3$ and $2<3$), $(1,2<4<3)$, $(3,4<1<2)$, $(3,4<2<1)$, $(1<4<2,3)$, $(4<1<2,3)$, $(2<3<1,4)$, $(3<2<1,4)$, $(1<3<2<4)$, $(1<3<4<2)$, $(3<1<2<4)$, $(3<1<4<2)$, $(2<4<1<3)$, $(2<4<3<1)$, $(4<2<1<3)$, $(4<2<3<1)$. This gives a complete fan of simplicial cones, but does *not* correspond to a (simple) generalized permutohedron.

Recall that $\Psi_P : \mathfrak{S}_n \to \mathrm{Vertices}(P)$ is the surjective map $\Psi_P : w \mapsto v_w$; see Definition 3.1. The previous discussion immediately implies the following corollary.

Corollary 3.9. *Let P be a generalized permutohedron in $\mathbb{R}^n$, and $v \in \mathrm{Vertices}(P)$ be its vertex. For $w \in \mathfrak{S}_n$, one has $\Psi_P(w) = v$ whenever the normal cone $\mathcal{N}_v(P)$ contains the Weyl chamber C_w. The preimage $\Psi_P^{-1}(v) \subseteq \mathfrak{S}_n$ of a vertex $v \in \mathrm{Vertices}(P)$ is the set $\mathcal{L}(Q_v)$ of all linear extensions of the vertex poset Q_v.*

We remark on the significance of this cone-preposet dictionary for toric varieties associated to generalized permutohedra or their normal fans; see Fulton [Ful'93] for further background.

A complete fan $\mathcal{F}$ of polyhedral cones in $\mathbb{R}^d$ whose cones are rational with respect to $\mathbb{Z}^d$ gives rise to a toric variety $X_{\mathcal{F}}$, which is normal, complete and of complex dimension d.

This toric variety is *projective* if and only if $\mathcal{F}$ is the normal fan $\mathcal{N}(P)$ for some polytope P, in which case one also denotes $X_{\mathcal{F}}$ by X_P.

The toric variety $X_{\mathcal{F}}$ is *quasi-smooth* or *orbifold* if and only if $\mathcal{F}$ is a complete fan of simplicial cones; in the projective case, where $\mathcal{F} = \mathcal{N}(P)$, this corresponds to P being a simple polytope.

In this situation, the h-numbers of $\mathcal{F}$ (or of P) have the auxiliary geometric meaning as the (singular cohomology) Betti numbers $h_i = \dim H^i(X_{\mathcal{F}}, \mathbb{C})$. The symmetry $h_i = h_{d-i}$ reflects Poincaré duality for this quasi-smooth variety.

The toric variety $X_{\mathcal{F}}$ is *smooth* exactly when every top-dimensional cone of $\mathcal{F}$ is not only simplicial but *unimodular*, that is, the primitive vectors on its extreme rays form a $\mathbb{Z}$-basis for $\mathbb{Z}^d$. Equivalently, the facet inequalities $\ell_1,\ldots,\ell_d$ can be chosen to form a $\mathbb{Z}$-basis for $(\mathbb{Z}^d)^* = \mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z})$ inside $(\mathbb{R}^d)^*$. One has $X_\mathcal{F}$ both smooth and projective if and only if $\mathcal{F} = \mathcal{N}(P)$ for a *Delzant* polytope P, that is, one which is simple and has every vertex normal cone unimodular.

Corollary 3.10. (cf. [Zel'06, §5]) *A complete fan $\mathcal{F}$ of posets gives rise to a complete toric variety $X_\mathcal{F}$, which will be projective if and only if $\mathcal{F}$ is associated with the normal fan $\mathcal{N}(P)$ for a generalized permutohedron.*

A complete fan $\mathcal{F}$ of tree-posets gives rise to a (smooth, not just orbifold) toric variety $X_\mathcal{F}$, which will be projective if and only $\mathcal{F}$ is associated with the normal fan $\mathcal{N}(P)$ of a simple generalized permutohedron. In other words, simple generalized permutohedra are always Delzant.

Proof. All the assertions should be clear from the above discussion, except for the last one about simple generalized permutohedra being Delzant. However, a tree-poset Q corresponds to a set of functionals $x_i - x_j$ for the edges $\{i, j\}$ of a tree, which are well-known to give a $\mathbb{Z}$-basis for $(\mathbb{Z}^d)^*$, cf. [Post'05, Proposition 7.10]. $\square$

4. Simple generalized permutohedra

4.1. **Descents of tree-posets and h-vectors.** The goal of this section is to combinatorially interpret the h-vector of any simple generalized permutohedron.

Definition 4.1. Given a poset Q on $[n]$, define the *descent set* $\mathrm{Des}(Q)$ to be the set of ordered pairs (i, j) for which $i \lessdot_Q j$ is a covering relation in Q with $i >_\mathbb{Z} j$, and define the statistic *number of descents* $\mathrm{des}(Q) := |\mathrm{Des}(Q)|$.

Theorem 4.2. *Let P be a simple generalized permutohedron, with vertex posets $\{Q_v\}_{v \in \mathrm{Vertices}(P)}$. Then one has the following expression for its h-polynomial:*

$$h_P(t) = \sum_{v \in \mathrm{Vertices}(P)} t^{\mathrm{des}(Q_v)}. \tag{4.1}$$

More generally, for a complete fan $\mathcal{F} = \{Q_v\}$ of tree-posets (see Definition 3.3), one also has $h_\mathcal{F}(t) = \sum_v t^{\mathrm{des}(Q_v)}$.

Proof. (cf. proof of Proposition 7.10 in [Post'05]) Let us prove the more general claim about fans of tree-posets, that is, simplicial fans coarsening the braid arrangement fan.

Pick a generic vector $g = (g_1, \ldots, g_n) \in \mathbb{R}^n$ such that $g_1 < \cdots < g_n$ and construct the directed graph $G_\mathcal{F}$, as in Proposition 2.1. Let $\sigma = \{x_i \leq x_j \mid i \lessdot_{Q_v} j\}$ be the cone of $\mathcal{F}$ associated with poset Q_v, see Proposition 3.5(8). Let σ' be an adjacent cone separated from σ by the facet $x_i = x_j$, $i \lessdot_{Q_v} j$. The vector g points from σ to σ' if and only if $g_i >_\mathbb{R} g_j$, or equivalently, $i >_\mathbb{Z} j$. Thus the outdegree of σ in the graph $G_\mathcal{F}$ is exactly the descent number $\mathrm{des}(Q)$. The claim now follows from Proposition 2.1. $\square$

For a usual permutohedron P in $\mathbb{R}^n$, the vertex posets Q_v are just all linear orders on $[n]$. So its h-polynomial $h_P(t)$ is the classical *Eulerian polynomial*[2]

$$A_n(t) := \sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(w)}, \tag{4.2}$$

[2]Note that a more standard convention is to call $tA_n(t)$ the Eulerian polynomial.

where $\mathrm{des}(w) := \#\{i \mid w(i) > w(i+1)\}$ is the usual *descent number* of a permutation w.

Any element w in the Weyl group $\mathfrak{S}_n$ sends a complete fan $\mathcal{F} = \{Q_i\}$ of tree-posets to another such complete fan $w\mathcal{F} = \{wQ_i\}$, by relabelling all of the posets. Since $w\mathcal{F}$ is an isomorphic simplicial complex, with the same h-vector, this leads to a curious corollary.

Definition 4.3. Given a tree-poset Q on $[n]$, define its *generalized Eulerian polynomial*

$$A_Q(t) := \sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(wQ)}.$$

Note that A_Q depends upon Q only as an *unlabelled* poset.

When Q is a linear order, $A_Q(t)$ is the usual Eulerian polynomial $A_n(t)$.

Corollary 4.4. *The h-polynomial $h_P(t)$ of a simple generalized permutohedron P is the "average" of the generalized Eulerian polynomials of its vertex tree-posets Q_v:*

$$h_P(t) = \frac{1}{n!} \sum_{v \in \mathrm{Vertices}(P)} A_{Q_v}(t).$$

See Example 5.5 below for an illustration of Theorem 4.2 and Corollary 4.4.

4.2. **Bounds on the h-vector and monotonicity.** It is natural to ask for upper and lower bounds on the h-vectors of simple generalized permutohedra. Some of these follow immediately from an h-vector monotonicity result of Stanley that applies to complete simplicial fans. To state it, we recall a definition from that paper.

Definition 4.5. Say that a simplicial complex Δ' is a *geometric subdivision* of a simplicial complex Δ if they have geometric realizations which are topological spaces on the same underlying set, and every face of Δ' is contained in a single face of Δ.

Theorem 4.6. (see [Stan'92, Theorem 4.1]) *If Δ' is a geometric subdivision of a Cohen-Macaulay simplicial complex Δ, then the h-vector of Δ' is componentwise weakly larger than that of Δ.*

In particular this holds when Δ, Δ' come from two complete simplicial fans and Δ' refines Δ, e.g., the normal fans of two simple polytopes P, P' in which P is a Minkowski summand of P.

Corollary 4.7. *A simple generalized permutohedron P in $\mathbb{R}^n$ has h-polynomial coefficientwise smaller than that of the permutohedron, namely the Eulerian polynomial $A_n(t)$.*

Proof. Proposition 3.2 tells us that the normal fan of P is refined by that of the permutohedron, so the above theorem applies. $\square$

Remark 4.8. Does the permutohedron also provide an upper bound for the f-vectors, flag f- and flag h-vectors, generalized h-vectors, and *cd*-indices of generalized permutohedra also in the non-simple case? Is there also a monotonicity result for these other forms of face and flag number data when one has two generalized permutohedra P, P' in which P is a Minkowski summand of P'?

The answer is "Yes" for f-vectors and flag f-vectors, which clearly increase under subdivision. The answer is also "Yes" for generalized h-vectors, which Stanley also showed [Stan'92, Corollary 7.11] can only increase under geometric subdivisions of rational convex polytopes. But for flag h-vectors and cd-indices, this is not so clear.

Later on (Example 6.11, Section 7.2, and Section 14) there will be more to say about lower bounds for h-vectors of simple generalized permutohedra within various classes.

5. The case of zonotopes

This section illustrates some of the foregoing results in the case where the simple generalized permutohedron is a zonotope; see also [Post'05, §8.6]. Zonotopal generalized permutohedra are exactly the graphic zonotopes, and the simple zonotopes among them correspond to a very restrictive class of graphs that are easily dealt with.

A *zonotope* is a convex polytope Z which is the Minkowski sum of one-dimensional polytopes (line segments), or equivalently, a polytope whose normal fan $\mathcal{N}(Z)$ coincides with chambers and cones of a hyperplane arrangement. Under this equivalence, the line segments which are the Minkowski summands of Z lie in the directions of the normal vectors to the hyperplanes in the arrangement. Given a graph $G = (V, E)$ without loops or multiple edges, on node set $V = [n]$ and with edge set E, define the associated *graphic zonotope* Z_G to be the Minkowski sum of line segments in the directions $\{e_i - e_j\}_{ij \in E}$.

Proposition 3.2 then immediately implies the following.

Proposition 5.1. *The zonotopal generalized permutohedra are exactly the graphic zonotopes Z_G.*

Simple zonotopes are very special among all zonotopes, and simple graphic zonotopes have been observed [Kim'06, Remark 5.2] to correspond to a very restrictive class of graphic zonotopes, namely those whose biconnected components are all complete graphs.

Recall that for a graph $G = (V, E)$, there is an equivalence relation on E defined by saying $e \sim e'$ if there is some circuit (i.e., cycle which is minimal with respect to inclusion of edges) of G containing both e, e'. The $\sim$-equivalence classes are then called *biconnected components* of G.

Proposition 5.2. [Kim'06, Remark 5.2] *The graphic zonotope Z_G corresponding to a graph $G = (V, E)$ is a simple polytope if and only if every biconnected component of G is the set of edges of a complete subgraph some subset of the vertices V.*

In this case, if $V_1, \ldots, V_r \subseteq V$ are the node sets for these complete subgraphs, then Z_G is isomorphic to the Cartesian product of usual permutohedra of dimensions $|V_j| - 1$ for $j = 1, 2, \ldots, r$.

Let us give another description for this class of graphs. For a graph F with n edges $e_1, \ldots, e_n$, the *line graph* $\mathrm{Line}(F)$ of F is the graph on the vertex set $[n]$ where $\{i, j\}$ is an edge in $\mathrm{Line}(F)$ if and only if the edges e_i and e_j of F have a common vertex. The following claim is left an exercise for the reader.

Exercise 5.3. *For a graph G, all biconnected components of G are edge sets of complete graphs if and only if G is isomorphic to the line graph $\mathrm{Line}(F)$ of some forest F. Biconnected components of $\mathrm{Line}(F)$ correspond to non-leaf vertices of F.*

For the sake of completeness, included here is a proof of Proposition 5.2.

Proof of Proposition 5.2. If the biconnected components of G induce subgraphs isomorphic to graphs $G_1, \ldots, G_r$ then one can easily check that Z_G is the Cartesian product of the zonotopes Z_{G_i}. Since a Cartesian product of polytopes is simple if and only if each factor is simple, this reduces to the case where $r = 1$. Also note that when $r = 1$ and G is a complete graph, then Z_G is the permutohedron, which is well-known to be simple.

For the reverse implication, assume G is biconnected but not a complete graph, and it will suffice, by Proposition 3.5(9), to construct a vertex v of Z_G whose poset Q_v is not a tree-poset. One uses the fact [GZ'83] that a vertex v in the graphic zonotope Z_G corresponds to an acyclic orientation of G, and the associated poset Q_v on V is simply the transitive closure of this orientation. Thus it suffices to produce an acyclic orientation of G whose transitive closure has Hasse diagram which is not a tree.

Since G is biconnected but not complete, there must be two vertices $\{x, y\}$ that do not span an edge in E, but which lie in some circuit C. Traverse this circuit C in some cyclic order, starting at the node x, passing through some nonempty set of vertices V_1 before passing through y, and then through a nonempty set of vertices V_2 before returning to x. One can then choose arbitrarily a total order on the node set V so that these sets appear as segments in this order:

$$V_1, \quad x, \quad y, \quad V_2, \quad V - (V_1 \cup V_2 \cup \{x, y\}).$$

It is then easily checked that if one orients the edges of G consistently with this total order, then the associated poset has a non-tree Hasse diagram: for any $v_1 \in V_1$ and $v_2 \in V_2$, one has $v_1 < x, y < v_2$ with x, y incomparable. $\square$

Corollary 5.4. *Let Z_G be a simple graphic zonotope, with notation as in Proposition 5.2.*

Then Z_G is flag, and its f-polynomial, h-polynomials, γ-polynomials are all equal to products for $j = 1, 2, \ldots, r$ of the f-, h-, or γ-polynomials of $(|V_j| - 1)$-dimensional permutohedra.

Proof. Use Proposition 5.2 along with the fact that a Cartesian product of simple polytopes is flag if and only if each factor is flag, and has f-, h- and γ-polynomial equal to the product of the same polynomials for each factor. $\square$

Note that the h-polynomial for an $(n-1)$-dimensional permutohedron is the Eulerian polynomial $A_n(t)$ described in (4.2) above, and the γ-polynomial is given explicitly in Theorem 11.1 below.

Example 5.5. Consider the graph $G = (V, E)$ with $V = [4] := \{1, 2, 3, 4\}$ and $E = \{12, 13, 23, 14\}$, whose biconnected components are the triangle 123 and the edge 14, which are both complete subgraphs on node sets $V_1 = \{1, 2, 3\}$ and $V_2 = \{1, 4\}$. Hence the associated graphic zonotope Z_G is simple and flag, equal to the Cartesian product of a hexagon with a line segment, that is, Z_G is a hexagonal prism.

Its f-, h- and γ-polynomials are

$$\begin{array}{lll} f_{Z_G}(t) & = (2+t)(6+6t+t^2) & = 12 + 18t + 8t^2 + t^3 \\ h_{Z_G}(t) & = A_2(t)A_3(t) = (1+t)(1+4t+t^2) & = 1 + 5t + 5t^2 + t^3 \\ \gamma_{Z_G}(t) & = (1)(1+2t) & = 1 + 2t. \end{array}$$

One can arrive at the same h-polynomial using Theorem 4.2. One lists the tree-posets Q_v for each of the 12 vertices v of the hexagonal prism Z_G, coming in 5 isomorphism types, along with the number of descents for each:

type	poset Q_v	des
chain:	$2<3<1<4$	1
	$3<2<1<4$	2
	$4<1<2<3$	1
	$4<1<3<2$	2
vee:	$1<2<3$ and $1<4$	0
	$1<3<2$ and $1<4$	1

type	poset Q_v	des
wedge:	$2<3<1$ and $4<1$	2
	$3<2<1$ and $4<1$	3
wye:	$2<1<3$ and $1<4$	1
	$3<1<2$ and $1<4$	1
lambda:	$3<1<2$ and $4<1$	2
	$2<1<3$ and $4<1$	2

and finds that $\sum_v t^{\mathrm{des}(Q_v)} = 1+5t+5t^2+t^3$.

Lastly one can get this h-polynomial from Corollary 4.4, by calculating directly that

$$\begin{aligned}
A_{\mathrm{chain}}(t) &= 1+11t+11t^2+t^3 = A_4(t)\\
A_{\mathrm{vee}}(t) &= 3+10t+8t^2+3t^3\\
A_{\mathrm{wedge}}(t) &= 3+8t+10t^2+3t^3\\
A_{\mathrm{wye}}(t) = A_{\mathrm{lambda}}(t) &= 2+10t+10t^2+2t^3
\end{aligned}$$

and then the h-polynomial is

$$\frac{1}{4!}\left[4A_{\mathrm{chain}}(t)+2A_{\mathrm{vee}}(t)+2A_{\mathrm{wedge}}(t)+2A_{\mathrm{wye}}(t)+2A_{\mathrm{lambda}}(t)\right] = 1+5t+5t^2+t^3.$$

6. Building sets and nestohedra

This section reviews some results from [FS'05], [Post'05], and [Zel'06] regarding the important special case of generalized permutohedra that arise from building sets. These generalized permutohedra turn out always to be simple. Their dual simplicial complexes, the nested set complexes, are defined, and several tools are given for calculating their f- and h-vectors. The notion of nested sets goes back to work of Fulton and MacPherson [FM'94], and DeConcini and Procesi [DP'95] defined building sets and nested set complexes. However, our exposition mostly follows [Post'05] and [Zel'06].

6.1. **Building sets, nestohedra, and nested set complexes.**

Definition 6.1. [Post'05, Definition 7.1] Let us say that a collection $\mathcal{B}$ of nonempty subsets of a finite set S is a *building set* if it satisfies the conditions:

(B1) If $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$.
(B2) $\mathcal{B}$ contains all singletons $\{i\}$, for $i \in S$.

For a building set $\mathcal{B}$ on S and a subset $I \subseteq S$, define the *restriction* of $\mathcal{B}$ to I as $\mathcal{B}|_I := \{J \in \mathcal{B} \mid J \subseteq I\}$. Let $\mathcal{B}_{\max} \subset \mathcal{B}$ denote the inclusion-maximal subsets of a building $\mathcal{B}$. Then elements of $\mathcal{B}_{\max}$ are pairwise disjoint subsets that partition the set S. Call the restrictions $\mathcal{B}|_I$, for $I \in \mathcal{B}_{\max}$, the *connected components* of $\mathcal{B}$. Say that a building set is *connected* if $\mathcal{B}_{\max}$ has only one element: $\mathcal{B}_{\max} = \{S\}$.

Example 6.2. Let G be a graph (with no loops nor multiple edges) on the node set S. The *graphical building* $\mathcal{B}(G)$ is the set of nonempty subsets $J \subseteq S$ such that the induced graph $G|_J$ on node set J is connected. Then $\mathcal{B}(G)$ is indeed a building set.

The graphical building set $\mathcal{B}(G)$ is connected if and only if the graph G is connected. The connected components of the graphical $\mathcal{B}(G)$ building set correspond to connected components of the graph G. Also each restriction $\mathcal{B}(G)|_I$ is the graphical building set $\mathcal{B}(G|_I)$ for the induced subgraph $G|_I$.

Definition 6.3. Let $\mathcal{B}$ be a building set on $[n] := \{1, \ldots, n\}$. Faces of the standard coordinate simplex in $\mathbb{R}^n$ are the simplices $\Delta_I := \text{ConvexHull}(e_i \mid i \in I)$, for $I \subseteq [n]$, where the e_i are the endpoints of the coordinate vectors in $\mathbb{R}^n$.

Define the *nestohedron*[3] $P_\mathcal{B}$ as the Minkowski sum of these simplices

$$P_\mathcal{B} := \sum_{I \in \mathcal{B}} y_I \Delta_I, \tag{6.1}$$

where y_I are strictly positive real parameters; see [Post'05, Section 6].

Note that since each of the normal fans $\mathcal{N}(\Delta_I)$ is refined by the braid arrangement fan, the same holds for their Minkowski sum [Zieg'94, Prop. 7.12], and hence the nestohedra $P_\mathcal{B}$ are generalized permutohedra by Proposition 3.2.

It turns out that $P_\mathcal{B}$ is always a simple polytope, whose combinatorial structure (poset of faces) does not depend upon the choice of the positive parameters y_I. In describing this combinatorial structure, it is convenient to instead describe the dual simplicial complex of $P_\mathcal{B}$.

Definition 6.4. [Post'05, Definition 7.3] For a building set $\mathcal{B}$, let us say that a subset $N \subseteq \mathcal{B} \setminus \mathcal{B}_{\max}$ is a *nested set* if it satisfies the conditions:

(N1) For any $I, J \in N$ one has either $I \subseteq J$, $J \subseteq I$, or I and J are disjoint.
(N2) For any collection of $k \geq 2$ disjoint subsets $J_1, \ldots, J_k \in N$, their union $J_1 \cup \cdots \cup J_k$ is not in $\mathcal{B}$.

Define the *nested set complex* $\Delta_\mathcal{B}$ as the collection of all nested sets for $\mathcal{B}$.

It is immediate from the definition that the nested set complex $\Delta_\mathcal{B}$ is an abstract simplicial complex on node set $\mathcal{B}$. Note that this slightly modifies the definition of a nested set from [Post'05], following [Zel'06], in that one does not include elements of $\mathcal{B}_{\max}$ in nested sets.

Theorem 6.5. [Post'05, Theorem 7.4], [FS'05, Theorem 3.14] *Let $\mathcal{B}$ be a building set on $[n]$. The nestohedron $P_\mathcal{B}$ is a simple polytope of dimension $n - |\mathcal{B}_{\max}|$. Its dual simplicial complex is isomorphic to the nested set complex $\Delta_\mathcal{B}$.*

An explicit correspondence between faces of $P_\mathcal{B}$ and nested sets in $\Delta_\mathcal{B}$ is described in [Post'05, Proposition 7.5]. The dimension of the face of $P_\mathcal{B}$ associated with a nested set $N \in \Delta_\mathcal{B}$ equals $n - |N| - |\mathcal{B}_{\max}|$. Thus vertices of $P_\mathcal{B}$ correspond to inclusion-maximal nested sets in $\Delta_\mathcal{B}$, and all maximal nested sets contain exactly $n - |\mathcal{B}_{\max}|$ elements.

Remark 6.6. For a building set $\mathcal{B}$ on $[n]$, it is known [FY'04, Theorem 4] that one can obtain the nested set complex $\Delta_\mathcal{B}$ (resp., the nestohedron $P_\mathcal{B}$) via the following stellar subdivision (resp., shaving) construction, a common generalization of

- the barycentric subdivision of a simplex as the dual of the permutohedron,
- Lee's construction of the associahedron [Lee'89, §3].

[3]Called the *nested set polytope* in [Zel'06].

Start with an $(n-1)$-simplex whose vertices (resp., facets) have been labelled by the singletons i for $i \in [n]$, which are all in $\mathcal{B}$. Then proceed through each of the non-singleton sets I in $\mathcal{B}$, in any order that reverses inclusion (i.e., where larger sets come before smaller sets), performing a *stellar subdivision* on the face with vertices (resp., *shave off* the face which is the intersection of facets) indexed by the singletons in I.

Remark 6.7. Note that if $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are the connected components of a building set $\mathcal{B}$, then $P_{\mathcal{B}}$ is isomorphic to the direct product of polytopes $P_{\mathcal{B}_1} \times \cdots \times P_{\mathcal{B}_k}$. Thus it is enough to investigate generalized permutohedra $P_{\mathcal{B}}$ and nested set complexes $\Delta_{\mathcal{B}}$ only for connected buildings.

Remark 6.8. The definition (6.1) of the nestohedron $P_{\mathcal{B}}$ as a Minkowski sum should make it clear that whenever one has two building sets $\mathcal{B} \subseteq \mathcal{B}'$, then $P_{\mathcal{B}}$ is a Minkowski summand of $P_{\mathcal{B}'}$. Hence Theorem 4.6 implies the h-vector of $P_{\mathcal{B}'}$ is componentwise weakly larger than that of $P_{\mathcal{B}}$.

Remark 6.9. Nestohedra $P_{\mathcal{B}(G)}$ associated with graphical building sets $\mathcal{B}(G)$ are called *graph-associahedra*, and have been studied in [CD'06, Post'05, Tol'05, Zel'06]. In [CD'06], the sets in $\mathcal{B}(G)$ are called *tubes*, and the nested sets are called *tubings* of G.

In particular, the h-vector monotonicity discussed in Remark 6.8 applies to graph-associahedra $P_{\mathcal{B}(G)}, P_{\mathcal{B}(G')}$ associated to graphs G, G' where G is an edge-subgraph of G'.

Example 6.10. (*Upper bound for nestohedra: the permutohedron*) see [Post'05, Sect. 8.1] For the complete graph K_n, the building set $\mathcal{B}(K_n) = 2^{[n]} \setminus \{\emptyset\}$ consists of all nonempty subsets in $[n]$. Let us call it the *complete building set.* The corresponding nestohedron (the graph-associahedron of the complete graph) is the usual $(n-1)$-dimensional permutohedron in $\mathbb{R}^n$. The k-th component h_k of its h-vector is the Eulerian number, that is the number of permutations in $\mathfrak{S}_n$ with k descents; and its h-polynomial is the Eulerian polynomial $A_n(t)$; see (4.2).

This h-vector gives the componentwise upper bound on h-vectors for all $(d-1)$-dimensional nestohedra. This also implies that the f-vector of the permutohedron gives componentwise upper bound on f-vectors of nestohedra.

Example 6.11. (*Lower bound for nestohedra: the simplex*) The smallest possible connected building set $\mathcal{B} = \{\{1\}, \{2\}, \ldots, \{n\}, [n]\}$ gives rise to the nestohedron $P_{\mathcal{B}}$ which is the $(n-1)$-simplex in $\mathbb{R}^n$. In this case

$$f(t) = \sum_{i=1}^{n} \binom{n}{i} t^{i-1} = \frac{(1+t)^n - 1}{t} \quad \text{and} \quad h(t) = 1 + t + t^2 + \cdots + t^{n-1}$$

give trivial componentwise lower bounds on the f-, h-vectors of nestohedra.

6.2. **Two recurrences for f-polynomials of nestohedra.** It turns out that there are two useful recurrences for f-polynomials of nestohedra and nested set complexes.

Let $f_{\mathcal{B}}(t)$ be the f-polynomial of the nestohedron $P_{\mathcal{B}}$:

$$f_{\mathcal{B}}(t) := \sum f_i \, t^i = \sum_{N \in \Delta_{\mathcal{B}}} t^{|S| - |\mathcal{B}_{\max}| - |N|},$$

where $f_i = f_i(P_{\mathcal{B}})$ is the number of i-dimensional faces of $P_{\mathcal{B}}$. As usual, it is related to the h-polynomial as $f_{\mathcal{B}}(t) = h_{\mathcal{B}}(t+1)$.

Theorem 6.12. [Post'05, Theorem 7.11] *The f-polynomial $f_{\mathcal{B}}(t)$ is determined by the following recurrence relations:*

(1) *If $\mathcal{B}$ consists of a single singleton, then $f_{\mathcal{B}}(t) = 1$.*
(2) *If $\mathcal{B}$ has connected components $\mathcal{B}_1, \dots, \mathcal{B}_k$, then*

$$f_{\mathcal{B}}(t) = f_{\mathcal{B}_1}(t) \cdots f_{\mathcal{B}_k}(t).$$

(3) *If $\mathcal{B}$ is a connected building set on S, then*

$$f_{\mathcal{B}}(t) = \sum_{I \subsetneq S} t^{|S|-|I|-1} f_{\mathcal{B}|_I}(t).$$

Another recurrence relation for f-polynomials was derived in [Zel'06], and will be used in Section 12.4 below. It will be more convenient to work with the f-polynomial of nested set complexes

$$\tilde{f}_{\mathcal{B}}(t) := \sum_{N \in \Delta_{\mathcal{B}}} t^{|N|} = t^{|S|-|\mathcal{B}_{\max}|} f_{\mathcal{B}}(t^{-1}),$$

where $\mathcal{B}$ is a building set on S.

For a building set $\mathcal{B}$ on S and a subset $I \subset S$, recall that the restriction of $\mathcal{B}$ to I is defined as $\mathcal{B}|_I = \{J \in \mathcal{B} \mid J \subseteq I\}$. Also define the *contraction* of I from $\mathcal{B}$ as the building set on $S \setminus I$ given by

$$\mathcal{B}/I := \{J \in S \setminus I \mid J \in \mathcal{B} \text{ or } J \cup I \in \mathcal{B}\},$$

see [Zel'06, Definition 3.1]. A link decomposition of nested set complexes was constructed in [Zel'06]. It implies the following recurrence relation for the f-vector.

Theorem 6.13. [Zel'06, Proposition 4.7] *For a building set $\mathcal{B}$ on a nonempty set S, one has*

$$\frac{d}{dt}\tilde{f}_{\mathcal{B}}(t) = \sum_{I \in \mathcal{B}\setminus\mathcal{B}_{\max}} \tilde{f}_{\mathcal{B}|_I}(t) \cdot \tilde{f}_{\mathcal{B}/I}(t) \quad \text{and} \quad \tilde{f}_{\mathcal{B}}(0) = 1.$$

Let G be a simple graph on S and let $I \in \mathcal{B}(G)$, i.e., I is a connected subset of nodes of G. It has already been mentioned that $\mathcal{B}(G)|_I = \mathcal{B}(G|_I)$; see Example 6.2. Let G/I be the graph on the node set $S \setminus I$ such that two nodes $i, j \in S \setminus I$ are connected by an edge in G/I if and only if

(1) i and j are connected by an edge in G, or
(2) there are two edges (i, k) and (j, l) in G with $k, l \in I$.

Then the contraction of I from the graphical building set $\mathcal{B}(G)$ is the graphical building set associated with the graph G/I, that is $\mathcal{B}(G)/I = \mathcal{B}(G/I)$.

7. Flag nestohedra

This section characterizes the flag nested set complexes and nestohedra, and then identifies those which are "smallest".

7.1. When is the nested set complex flag?

For a graphical building set $\mathcal{B}(G)$ it has been observed ([Post'05, §8,4], [Zel'06, Corollary 7.4]) that one can replace condition (N2) in Definition 6.4 with a weaker condition:

(N2') For a disjoint pair of subsets $I, J \in N$, one has $I \cup J \notin \mathcal{B}$.

This implies that nested set complexes associated to graphical buildings are flag complexes. More generally, one has the following characterization of the nested set complexes which are flag.

Proposition 7.1. *For a building set $\mathcal{B}$, the following are equivalent.*

(i) *The nested set complex $\Delta_{\mathcal{B}}$ (or equivalently, the nestohedron $P_{\mathcal{B}}$) is flag.*
(ii) *The nested sets for $\mathcal{B}$ are the subsets $N \subseteq \mathcal{B} \setminus \mathcal{B}_{\max}$ which satisfy conditions* (N1) *and* (N2').
(iii) *If $J_1, \ldots, J_\ell \in \mathcal{B}$ with $\ell \geq 2$ are pairwise disjoint and their union $J_1 \cup \cdots \cup J_\ell$ is in $\mathcal{B}$, then one can reindex so that for some k with $1 \leq k \leq \ell - 1$ one has both $J_1 \cup \cdots \cup J_k$ and $J_{k+1} \cup \cdots \cup J_\ell$ in $\mathcal{B}$.*

Proof. The equivalence of (i) and (ii) essentially follows from the definitions. We will show here the equivalence of (i) and (iii).

Assume that (iii) fails, and let $J_1, \ldots, J_\ell$ provide such a failure with ℓ minimal. Note that this means $\ell \geq 3$, and minimality of ℓ forces $J_r \cup J_s \notin \mathcal{B}$ for each $r \neq s$; otherwise one could replace the two sets J_r, J_s on the list with the one set $J_r \cup J_s$ to obtain a counterexample of size $\ell - 1$. This means that all of the pairs $\{J_r, J_s\}$ index edges of $\Delta_{\mathcal{B}}$, although $\{J_1, \ldots, J_\ell\}$ does not. Hence $\Delta_{\mathcal{B}}$ is not flag, i.e., (i) fails.

Now assume (i) fails, i.e., $\Delta_{\mathcal{B}}$ is not flag. Let $J_1, \ldots, J_\ell$ be subsets in $\mathcal{B}$, for which each pair $\{J_r, J_s\}$ with $r \neq s$ is a nested set, but the whole collection $M := \{J_1, \ldots, J_\ell\}$ is not, and assume that this violation has ℓ minimal. Because $\{J_r, J_s\}$ are nested for $r \neq s$, it must be that M *does* satisfy condition (N1), and so M must fail condition (N2). By minimality of ℓ, it must be that the $J_1, \ldots, J_\ell$ are pairwise disjoint and their union $J_1 \cup \cdots \cup J_\ell$ is in $\mathcal{B}$. Bearing in mind that $J_r \cup J_s \notin \mathcal{B}$ for $r \neq s$, it must be that $\ell \geq 3$. But then M must give a violation of property (iii), else one could use property (iii) to produce a violation of (i) either of size k or of size $\ell - k$, which are both smaller than ℓ. ☐

Corollary 7.2. *For graphical buildings $\mathcal{B}(G)$, the graph-associahedron $P_{\mathcal{B}(G)}$ and nested set complex $\Delta_{\mathcal{B}(G)}$ are flag.*

7.2. **Stanley-Pitman polytopes and their relatives.** One can now use Proposition 7.1 to characterize the inclusion-minimal connected building sets $\mathcal{B}$ for which $\Delta_{\mathcal{B}}$ and $P_{\mathcal{B}}$ are flag.

For any building set $\mathcal{B}$ on $[n]$ with $\Delta_{\mathcal{B}}$ flag, one can apply Proposition 7.1(iii) with $\{J_1, \ldots, J_\ell\}$ equal to the collection of singletons $\{\{1\}, \ldots, \{n\}\}$, since they are disjoint and their union $[n]$ is also in $\mathcal{B}$. Thus after reindexing, some initial segment $[k]$ and some final segment $[n] \setminus [k]$ must also be in $\mathcal{B}$. Iterating this, one can assume after reindexing that there is a *plane binary tree* τ with these properties

- the singletons $\{\{1\}, \ldots, \{n\}\}$ label the leaves of τ,
- each internal node of τ is labelled by the set I which is the union of the singletons labelling the leaves of the subtree below it (so $[n]$ labels the root node), and
- the building set $\mathcal{B}$ contains of all of the sets labelling nodes in this tree.

It is not hard to see that these sets labelling the nodes of τ already comprise a building set $\mathcal{B}_\tau$ which satisfies Proposition 7.1(iii), and therefore give rise to a nested set complex $\Delta_{\mathcal{B}_\tau}$ and nestohedron $P_{\mathcal{B}_\tau}$ which are flag. See Figure 7.1 for an example.

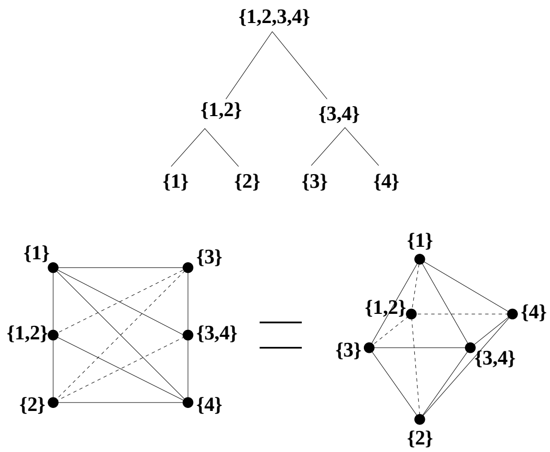

FIGURE 7.1. A binary tree τ and building set $\mathcal{B}_\tau$, along with its complex of nested sets $\Delta_{\mathcal{B}_\tau}$, drawn first as in the construction of Remark 6.6, and then redrawn as the boundary of an octahedron.

The previous discussion shows the following.

Proposition 7.3. *The building sets $\mathcal{B}_\tau$ parametrized by plane binary trees τ are exactly the inclusion-minimal building sets among those which are connected and have the nested set complex and nestohedron flag.*

As a special case, when τ is the plane binary tree having leaves labelled by the singletons and internal nodes labelled by all initial segments $[k]$, one obtains the building set $\mathcal{B}_\tau$ whose nestohedron $P_{\mathcal{B}_\tau}$ is *Stanley-Pitman polytope* from [StPi'02]; see [Post'05, §8.5]. The Stanley-Pitman polytope is shown there to be combinatorially (but not affinely) isomorphic to an $(n-1)$-cube; the argument given there generalizes to prove the following.

Proposition 7.4. *For any plane binary tree τ with n leaves, the nested set complex $\Delta_{\mathcal{B}_\tau}$ is isomorphic to the boundary of a $(n-1)$-dimensional cross-polytope (hyperoctahedron), and $P_{\mathcal{B}_\tau}$ is combinatorially isomorphic to an $(n-1)$-cube.*

Proof. Note that the sets labelling the non-root nodes of τ can be grouped into $n-1$ pairs $\{I_1, J_1\}, \ldots, \{I_{n-1}, J_{n-1}\}$ of siblings, meaning that I_k, J_k are nodes with a common parent in τ. One then checks that the nested sets for $\mathcal{B}_\tau$ are exactly the collections N containing *at most one set from each pair* $\{I_k, J_k\}$. As a simplicial complex, this is the boundary complex of an $(n-1)$-dimensional cross-polytope in which each pair $\{I_k, J_k\}$ indexes an antipodal pair of vertices. □

Note that in this case,

$$f_{\mathcal{B}_\tau}(t) = (2+t)^{n-1}, \quad h_{\mathcal{B}_\tau}(t) = (1+t)^{n-1}, \quad \gamma_{\mathcal{B}_\tau}(t) = 1 = 1 + 0 \cdot t + 0 \cdot t^2 + \cdots .$$

which gives a lower bound for the f- and h-vectors of flag nestohedra by Remark 6.8. If one assumes Conjecture 2.3, then it would also give a lower bound for γ-vectors of flag nestohedra (and for flag simplicial polytopes in general).

Note that the permutohedron is a graph-associahedron (and hence a flag nestohedra). Therefore, Corollary 4.7 implies that the permutohedron provides the upper bound on the f- and h-vectors among the flag nestohedra.

8. $\mathcal{B}$-trees and $\mathcal{B}$-permutations

This section discusses $\mathcal{B}$-trees and $\mathcal{B}$-permutations, which are two types of combinatorial objects associated with vertices of the nestohedron $P_{\mathcal{B}}$. The h-polynomial of $P_{\mathcal{B}}$ equals the descent-generating function for $\mathcal{B}$-trees.

8.1. $\mathcal{B}$-trees and h-polynomials. This section gives a combinatorial interpretation of the h-polynomials of nestohedra. Since nestohedra $P_{\mathcal{B}}$ are always simple, one should expect some description of their vertex tree-posets Q_v (see Corollaries 3.7 and 3.9) in terms of the building set $\mathcal{B}$.

Recall that a *rooted tree* is a tree with a distinguished node, called its *root.* One can view a rooted tree T as a partial order on its nodes in which $i <_T j$ if j lies on the unique path from i to the root. One can also view it as a directed graph in which all edges are directed towards the root; both viewpoints will be employed here.

For a node i in a rooted tree T, let $T_{\leq i}$ denote the set of all descendants of i, that is $j \in T_{\leq i}$ if there is a directed path from the node j to the node i. Note that $i \in T_{\leq i}$. Nodes i and j in a rooted tree are called *incomparable* if neither i is a descendant of j, nor j is a descendant of i.

Definition 8.1. [Post'05, Definition 7.7], cf. [FS'05] For a connected building set $\mathcal{B}$ on $[n]$, let us define a *$\mathcal{B}$-tree* as a rooted tree T on the node set $[n]$ such that

(T1) For any $i \in [n]$, one has $T_{\leq i} \in \mathcal{B}$.
(T2) For $k \geq 2$ incomparable nodes $i_1, \ldots, i_k \in [n]$, one has $\bigcup_{j=1}^{k} T_{\leq i_j} \notin \mathcal{B}$.

Note that, when the nested set complex $\Delta_{\mathcal{B}}$ is flag, that is when $\mathcal{B}$ satisfies any of the conditions of Proposition 7.1, one can define a $\mathcal{B}$-tree by requiring condition (T2) only for $k = 2$.

Proposition 8.2. [Post'05, Proposition 7.8], [FS'05, Proposition 3.17] *For a connected building set $\mathcal{B}$, the map sending a rooted tree T to the collection of sets $\{T_{\leq i} \mid i \text{ is a nonroot vertex}\} \subset \mathcal{B}$ gives a bijection between $\mathcal{B}$-trees and maximal nested sets. (Recall that maximal nested sets correspond to the facets of the nested set complex $\Delta_{\mathcal{B}}$ and to the vertices of the nestohedron $P_{\mathcal{B}}$.)*

Furthermore, if the $\mathcal{B}$-tree T corresponds to the vertex v of $P_{\mathcal{B}}$ then $T = Q_v$, that is, T is the vertex tree-poset for v in the notation of Corollary 3.7.

Question 8.3. Does a simple (indecomposable) generalized permutohedron P come from a (connected) building set if and only if every poset Q_v is a rooted tree, i.e. has a unique maximal element?

Proposition 8.2 and Theorem 4.2 yield the following corollary.

Corollary 8.4. *For a connected building set $\mathcal{B}$ on $[n]$, the h-polynomial of the generalized permutohedron $P_{\mathcal{B}}$ is given by*

$$h_{\mathcal{B}}(t) = \sum_{T} t^{\mathrm{des}(T)},$$

where the sum is over $\mathcal{B}$-trees T.

The following recursive description of $\mathcal{B}$-trees is straightforward from the definition.

Proposition 8.5. [Post'05, Section 7] *Let $\mathcal{B}$ be a connected building set on S and let $i \in S$. Let $\mathcal{B}_1, \dots, \mathcal{B}_r$ be the connected components of the restriction $\mathcal{B}|_{S \setminus \{i\}}$. Then all $\mathcal{B}$-trees with the root at i are obtained by picking a $\mathcal{B}_j$-tree T_j, for each component $\mathcal{B}_j$, $j = 1, \dots, r$, and connecting the roots of $T_1, \dots, T_r$ with the node i by edges.*

In other words, each $\mathcal{B}$-tree is obtained by picking a root $i \in S$, splitting the restriction $\mathcal{B}|_{S \setminus \{i\}}$ into connected components, then picking nodes in all connected components, splitting corresponding restrictions into components, etc.

Recall Definition 3.1 of the surjection $\Psi_\mathcal{B} := \Psi_{P_\mathcal{B}}$

$$\Psi_\mathcal{B} : \mathfrak{S}_n \longrightarrow \mathrm{Vertices}(P_\mathcal{B}) = \{\mathcal{B}\text{-trees}\},$$

Here and below one identifies vertices of the nestohedron $P_\mathcal{B}$ with $\mathcal{B}$-trees via Proposition 8.2. By Corollary 3.9, for a $\mathcal{B}$-tree T, one has $\Psi_\mathcal{B}(w) = T$ if and only if w is a linear extension of T.

Proposition 8.5 leads to an explicit recursive description of the surjection $\Psi_\mathcal{B}$.

Proposition 8.6. *Let $\mathcal{B}$ be a connected building set on $[n]$. Given a permutation $w = (w(1), \dots, w(n)) \in \mathfrak{S}_n$, one recursively constructs a $\mathcal{B}$-tree $T = T(w)$, as follows.*

The root of T is the node $w(n)$. Let $\mathcal{B}_1, \dots, \mathcal{B}_r$ be the connected components of the restriction $\mathcal{B}|_{\{w(1), \dots, w(n-1)\}}$. Restricting w to each of the sets $\mathcal{B}_i$ gives a subword of w, to which one can recursively apply the construction and obtain a $\mathcal{B}_i$-tree T_i. Then attach these $T_1, \dots, T_r$ as subtrees of the root node $w(n)$ in T. This association $w \mapsto T(w)$ is the map $\Psi_\mathcal{B}$.

8.2. **$\mathcal{B}$-permutations.** It is natural to ask for a nice section of the surjection $\Psi_\mathcal{B}$; these are the $\mathcal{B}$-permutations defined next.

Definition 8.7. Let $\mathcal{B}$ be a building set on $[n]$. Define the set $\mathfrak{S}_n(\mathcal{B}) \subset \mathfrak{S}_n$ of *$\mathcal{B}$-permutations* as the set of permutations $w \in \mathfrak{S}_n$ such that for any $i \in [n]$, the elements $w(i)$ and $\max\{w(1), w(2), \dots, w(i)\}$ lie in the same connected component of the restricted building set $\mathcal{B}|_{\{w(1), \dots, w(i)\}}$.

The following recursive construction of $\mathcal{B}$-permutations is immediate from the definition.

Lemma 8.8. *A permutation $w \in \mathfrak{S}_n$ is a $\mathcal{B}$-permutation if and only if it can be constructed via the following procedure.*

Pick $w(n)$ from the connected component of $\mathcal{B}$ that contains n; then pick $w(n-1)$ from the connected component of $\mathcal{B}|_{[n] \setminus \{w(n)\}}$ that contains the maximal element of $[n] \setminus \{w(n)\}$; then pick $w(n-2)$ from the connected component of $\mathcal{B}|_{[n] \setminus \{w(n), w(n-1)\}}$ that contains the maximal element of $[n] \setminus \{w(n), w(n-1)\}$, etc. Continue in this manner until $w(1)$ has been chosen.

Let T be a rooted tree on $[n]$ viewed as a tree-poset where the root is the unique maximal element. The *lexicographically minimal linear extension* of T is the permutation $w \in \mathfrak{S}_n$ such that $w(1)$ is the minimal leaf of T (in the usual order on $\mathbb{Z}$), $w(2)$ is the minimal leaf of $T - \{w(1)\}$ (the tree T with the vertex $w(1)$ removed), $w(3)$ is the minimal leaf of $T - \{w(1), w(2)\}$, etc. There is the following alternative "backward" construction for the lexicographically minimal linear extension of T.

Lemma 8.9. *Let w be the lexicographically minimal linear extension of a rooted tree T on $[n]$. Then the permutation w can be constructed from T, as follows: $w(n)$ is the root of T; $w(n-1)$ is the root of the connected component of $T - \{w(n)\}$ that contains the maximal vertex of this forest (in the usual order on $\mathbb{Z}$); $w(n-2)$ is the root of the connected component of $T - \{w(n), w(n-1)\}$ that contains the maximal vertex of this forest, etc.*

In general, $w(i)$ is the root of the connected component of the forest

$$T - \{w(n), \ldots, w(i+1)\}$$

that contains the vertex $\max(w(1), \ldots, w(i))$.

Proof. The proof is by induction on the number of vertices in T. Let T' be the rooted tree obtained from T by removing the minimal leaf l. Then the lexicographically minimal linear extension w of T is $w = (l, w')$, where w' is the lexicographically minimal linear extension of T', and both w and w' are written in list notation. By induction, w' can be constructed from T' backwards. When one performs the backward construction for T, the vertex l can never be the root of the connected component of $T - \{w(n), \ldots, w(i+1)\}$ containing the maximal vertex, for $i > 1$. So the backward procedure for T produces the same permutation $w = (l, w')$. □

The next claim gives a correspondence between $\mathcal{B}$-trees and $\mathcal{B}$-permutations.

Proposition 8.10. *Let $\mathcal{B}$ be a connected building set on $[n]$. The set $\mathfrak{S}_n(\mathcal{B})$ of $\mathcal{B}$-permutations is exactly the set of lexicographically minimal linear extensions of the $\mathcal{B}$-trees. (Equivalently, $\mathfrak{S}_n(\mathcal{B})$ is the set of lexicographically minimal representatives of fibers of the map $\Psi_{\mathcal{B}}$.)*

In particular, the map $\Psi_{\mathcal{B}}$ induces a bijection between $\mathcal{B}$-permutations and $\mathcal{B}$-trees, and $\mathfrak{S}_n(\mathcal{B})$ is a section of the map $\Psi_{\mathcal{B}}$.

Proof. Let $w \in \mathfrak{S}_n$ be a permutation and let $T = T(w)$ be the corresponding $\mathcal{B}$-tree constructed as in Proposition 8.6. Note that, for $i = n-1, n-2, \ldots, 1$, the connected components of the forest $T|_{\{w(1),\ldots,w(i)\}} = T - \{w(n), \ldots, w(i+1)\}$ correspond to the connected components of the building set $\mathcal{B}|_{\{w(1),\ldots,w(i)\}}$, and corresponding components have the same vertex sets. According to Lemma 8.9, the permutation w is the lexicographically minimal linear extension of T if and only if w is a $\mathcal{B}$-permutation as described in Lemma 8.8. □

9. Chordal building sets and their nestohedra

This section describes an important class of building sets $\mathcal{B}$, for which the descent numbers of $\mathcal{B}$-trees are equal to the descent numbers of $\mathcal{B}$-permutations. In this case, the h-polynomial of the nestohedron $P_{\mathcal{B}}$ equals the descent-generating function of the corresponding $\mathcal{B}$-permutations.

9.1. **Descents in posets vs. descents in permutations.** Let us say that a *descent* of a permutation $w \in \mathfrak{S}_n$ is a pair[4] $(w(i), w(i+1))$ such that $w(i) > w(i+1)$. Let $\mathrm{Des}(w)$ be the set of all descents in w. Also recall that the descent set $\mathrm{Des}(Q)$ of a poset Q is the set of pairs (a, b) such that $a <_Q b$ and $a >_{\mathbb{Z}} b$; see Definition 4.1.

Lemma 9.1. *Let Q be any poset on $[n]$, and let $w = w(Q)$ be the lexicographically minimal linear extension of Q. Then one has* $\mathrm{Des}(w) \subseteq \mathrm{Des}(Q)$.

[4] A more standard convention is say that a descent is an *index* i such that $w(i) > w(i+1)$.

Proof. One must show that any descent (a,b) (with $a >_{\mathbb{Z}} b$) in w must come from a covering relation $a \lessdot_Q b$ in the poset Q. Indeed, if a and b are incomparable in Q, then the permutation obtained from w by transposing a and b would be a linear extension of P which is lexicographically smaller than w. On the other hand, if a and b are comparable but not adjacent elements in Q, then they can never be adjacent elements in a linear extension of Q. $\square$

In particular, this lemma implies that, for a $\mathcal{B}$-tree T and the corresponding $\mathcal{B}$-permutation w (i.e., w is the lexicographically minimal linear extension of T), one has $\mathrm{Des}(w) \subseteq \mathrm{Des}(T)$. The rest of this section discusses a special class of building sets for which one always has $\mathrm{Des}(w) = \mathrm{Des}(T)$.

9.2. **Chordal building sets.**

Definition 9.2. Let us say that a building set $\mathcal{B}$ on $[n]$ is *chordal* if it satisfies the following condition: for any $I = \{i_1 < \cdots < i_r\} \in \mathcal{B}$ and $s = 1, \dots, r$, the subset $\{i_s, i_{s+1}, \dots, i_r\}$ also belongs to $\mathcal{B}$.

Recall that a graph is called *chordal* if it has no induced k-cycles for $k \geq 4$. It is well known [FG'65] that chordal graphs are exactly the graphs that admit a *perfect elimination ordering,* which is an ordering of vertices such that, for each vertex v, the neighbors of v that occur later than v in the order form a clique. Equivalently, a graph G is chordal if its vertices can be labelled by numbers in $[n]$ so that G has no induced subgraph $G|_{\{i<j<k\}}$ with the edges (i,j), (i,k) but without the edge (j,k). Let us call such graphs on $[n]$ *perfectly labelled chordal graphs.*[5]

Example 9.3. Let us say that a tree on $[n]$ is *decreasing* if the labels decrease in the shortest path from the vertex n (the root) to another vertex. It is easy to see that decreasing trees are exactly the trees which are perfectly labelled chordal graphs. Clearly, any unlabelled tree has such a decreasing labelling of vertices.

The following claim justifies the name "chordal building set."

Proposition 9.4. *A graphical building set $\mathcal{B}(G)$ is chordal if and only if G is a perfectly labelled chordal graph.*

Proof. Suppose that G contains an induced subgraph $G|_{\{i<j<k\}}$ with exactly two edges (i,j), (i,k). Then $\{i,j,k\} \in \mathcal{B}(G)$ but $\{j,k\} \notin \mathcal{B}(G)$. Thus $\mathcal{B}(G)$ is not a chordal building set.

On the other hand, suppose that $\mathcal{B}(G)$ is not chordal. Then one can find a connected subset $I = \{i_1 < \cdots < i_r\}$ of vertices in G such that $\{i_s, i_{s+1}, \dots, i_r\} \notin \mathcal{B}(G)$, for some s. In other words, the induced graph $G' = G|_{\{i_s,\dots,i_k\}}$ is disconnected. Let us pick a shortest path P in $G|_{\{i_1,\dots,i_r\}}$ that connects two different components of G'. Let i be the minimal vertex in P and let j and k be the two vertices adjacent of i in the path P. Clearly, $j > i$ and $k > i$. It is also clear that (i,j) is not an edge of G. Otherwise there is a shorter path obtained from P by replacing the edges (i,j) and (i,k) with the edge (j,k). So one has found a forbidden induced subgraph $G|_{\{i,j,k\}}$. Thus G is not a perfectly labelled chordal graph. $\square$

---

[5]We can also call them 312-*avoiding graphs* because they are exactly the graphs that have no induced 3-path a—b—c with the relative order of the vertices a, b, c as in the permutation 312. Note that, unlike the pattern avoidance in permutations, a 312-avoiding graph is the same thing as a 213-avoiding graph.

Proposition 9.5. *Let $\mathcal{B}$ be a connected chordal building set. Then, for any $\mathcal{B}$-tree T and the corresponding $\mathcal{B}$-permutation w, one has* $\mathrm{Des}(w) = \mathrm{Des}(T)$.

Proof. Let T be a $\mathcal{B}$-tree and let w be the corresponding $\mathcal{B}$-permutation, which can be constructed backward from T as described in Lemma 8.9. Let us fix $i \in \{n-1, n-2, \dots, 1\}$. Let $T_1, \dots, T_r, T_1', \dots, T_s'$ be the connected components of the forest $T - \{w(n), w(n-1), \dots, w(i+1)\}$, where $T_1, \dots, T_r$ are the subtrees whose roots are the children of the vertex $w(i+1)$, and $T_1', \dots, T_s'$ are the remaining subtrees. Let $I = T_{\leq w(i+1)} \subset [n]$ be the set of all descendants of $w(i+1)$ in T. By Definition 8.1(T1), one has $I \in \mathcal{B}$.

Suppose that the vertex $m = \max(w(1), \dots, w(i))$ appears in one of the subtrees $T_1, \dots, T_r$, say, in the tree T_1. Then, by Lemma 8.9, $w(i)$ should be the root of T_1. We claim that all vertices in $T_2, \dots, T_r$ are less than $w(i+1)$. Indeed, this is clear if $w(i+1)$ is the maximal element in I. Otherwise, the set $I' = I \cap \{w(i+1)+1, \dots, n-1, n\}$ is nonempty, $I' \in \mathcal{B}$ because $\mathcal{B}$ is chordal, and I' contains the maximal vertex m. Since the vertex set J of T_1 should be an element of $\mathcal{B}$, it follows that $I' \subseteq J$. So all vertices of $T_2, \dots, T_r$ are less than $w(i+1)$.

Thus none of the edges of T joining the vertex $w(i+1)$ with the roots of $T_2, T_3, \dots, T_r$ can be a descent edge. The only potential descent edge is the edge $(w(i), w(i+1))$ that attaches the subtree T_1 to $w(i+1)$. This edge will be a descent edge in T if and only if $w(i) > w(i+1)$, i.e., exactly when $(w(i), w(i+1))$ is a descent in the permutation w.

Now suppose that the maximal vertex $m = \max(w(1), \dots, w(i))$ appears in one of the remaining subtrees $T_1', \dots, T_s'$, which are not attached to the vertex $w(i+1)$, say, in T_1'. In this case $w(i+1)$ should be greater than all $w(1), \dots, w(i)$. (Otherwise, if $w(i+1) < m$, then at the previous step of the backward construction for w, T_1' is the connected component of $T - \{w(n), \dots, w(i+1)\}$ that contains the vertex $\max(w(1), \dots, w(i+1)) = m$. So $w(i+1)$ should have been the root of T_1'.) In this case, none of the edges joining the vertex $w(i+1)$ with the components $T_1, \dots, T_r$ can be a descent edge and $(w(i), w(i+1))$ cannot be a descent in w.

This proves that descent edges of T are in bijection with descents in w. ☐

Corollary 8.4 and Proposition 9.5 imply the following formula.

Corollary 9.6. *For a connected chordal building set $\mathcal{B}$, the h-polynomial of the nestohedron $P_\mathcal{B}$ equals*

$$h_\mathcal{B}(t) = \sum_{w \in \mathfrak{S}_n(\mathcal{B})} t^{\mathrm{des}(w)},$$

where $\mathrm{des}(w)$ *is the usual descent number of a permutation $w \in \mathfrak{S}_n(\mathcal{B})$.*

Let us give an additional nice property of nestohedra for chordal building sets.

Proposition 9.7. *For a chordal building set $\mathcal{B}$, the nestohedron $P_\mathcal{B}$ is a flag simple polytope.*

Proof. Let us check that a chordal building set $\mathcal{B}$ satisfies the condition in Proposition 7.1(iii). Using the notation of that proposition, let $J_1 \cup \cdots \cup J_\ell = \{i_1 < \cdots < i_r\}$. Let U_s be the union of those subsets $J_1, ..., J_\ell$ that have a nonempty intersection with $\{i_s, i_{s+1}, \dots, i_r\}$. Since $\{i_s, i_{s+1}, \dots, i_n\}$ is in $\mathcal{B}$ (because $\mathcal{B}$ is chordal), the subset U_s should also be in $\mathcal{B}$ (by Definition 6.1(B1)). Clearly, U_1 is the union of all J_i's and U_r consists of a single J_i. It is also clear that U_{j+1} either equals U_j

or is obtained from U_j by removing a single subset J_i. It follows that there exists an index s such that $U_s = (J_1 \cup \cdots \cup J_\ell) \setminus J_i$. This gives an index i such that $(J_1 \cup \cdots \cup J_\ell) \setminus J_i$ and J_i are both in $\mathcal{B}$, as needed. $\square$

10. Examples of nestohedra

Let us give several examples which illustrate Corollary 8.4 and Corollary 9.6. The f- and h-numbers for the permutohedron and associahedron are well-known.

10.1. **The permutohedron.** For the complete building set $\mathcal{B} = \mathcal{B}(K_n)$ the nestohedron $P_\mathcal{B}$ is the usual permutohedron; see Example 6.10 and [Post'05, Sect. 8.1]. In this case $\mathcal{B}$-trees are linear orders on $[n]$ and $\mathcal{B}$-permutations are all permutations $\mathfrak{S}_n(\mathcal{B}) = \mathfrak{S}_n$. Thus, as noted before in Example 6.10, the h-polynomial is the usual Eulerian polynomial $A_n(t)$, and the h-numbers are the Eulerian numbers $h_k(P_\mathcal{B}) = A(n,k) := \#\{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = k\}$.

10.2. **The associahedron.** Let $G = \mathrm{Path}_n$ denote the graph which is a path having n nodes labelled consecutively $1, \ldots, n$. The graphical building set $\mathcal{B} = \mathcal{B}(\mathrm{Path}_n)$ consists of all intervals $[i,j]$, for $1 \le i \le j \le n$. The corresponding nestohedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ is the usual Stasheff associahedron; see [CD'06, Post'05].

In this case, the $\mathcal{B}$-trees correspond to unlabelled plane binary trees on n nodes, as follows; see [Post'05, Sect. 8.2] for more details. A *plane binary tree* is a rooted tree with two types of edges (left and right) such that every node has at most one left and at most one right edge descending from it. From Proposition 8.5, one can see that a $\mathcal{B}$-tree is a binary tree with n nodes labelled $1, 2, \ldots, n$ so that, for any node, all nodes in its left (resp., right) branch have smaller (resp., bigger) labels. Conversely, given an unlabelled plane binary tree, there is a unique way to label its nodes $1, 2, \ldots, n$ to create a $\mathcal{B}$-tree, namely in the order of traversal of a *depth-first search*. Furthermore, note that descent edges correspond to right edges.

It is well-known that the number of unlabelled binary trees on n nodes is equal to the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$, and the number of binary trees on n nodes with $k-1$ right edges is the *Narayana number* $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$; see [Stan'99, Exer. 6.19c and Exer. 6.36]. Therefore, the h-numbers of the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ are the Narayana numbers: $h_k(P_{\mathcal{B}(\mathrm{Path}_n)}) = N(n, k+1)$, for $k = 0, \ldots, n-1$.

It is also well-known that the f-numbers of the associahedron are $f_k(P_{\mathcal{B}(\mathrm{Path}_n)}) = \frac{1}{n+1}\binom{n-1}{k}\binom{2n-k}{n}$. This follows from a classical Kirkman-Cayley formula [Cay'1890] for the number of ways to draw k noncrossing diagonals in an n-gon.

In this case, the $\mathcal{B}$-permutations are exactly 312-avoiding permutations $w \in \mathfrak{S}_n$. Recall that a permutation w is 312-*avoiding* if there is no triple of indices $i < j < k$ such that $w(j) < w(k) < w(i)$. Thus Corollary 9.6 says that the h-polynomial of the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ is $\sum_w t^{\mathrm{des}(w)}$ where the sum runs over all 312-avoiding permutations in $\mathfrak{S}_n$. This is consistent with the known fact that the Narayana numbers count 312-avoiding permutations according to their number of descents; see Simion [Sim'94, Theorem 5.4] for a stronger statement.

10.3. **The cyclohedron.** If $G = \mathrm{Cycle}_n$ is the n-cycle, then the nestohedron $P_{\mathcal{B}(\mathrm{Cycle}_n)}$ is the *cyclohedron* also introduced by Stasheff; see [CD'06, Post'05]. The

h-polynomial of the cyclohedron was computed by Simion [Sim'03, Corollary 1]:

$$h_{\mathcal{B}(\mathrm{Cycle}_n)}(t) = \sum_{k=0}^{n} \binom{n}{k}^2 t^k. \tag{10.1}$$

Note that the n-cycle (for $n > 3$) is not a chordal graph, so Corollary 9.6 does not apply to this case.

10.4. **The stellohedron.** Let $m = n-1$. Let $G = K_{1,m}$ be the m-star graph with the central node $m+1$ connected to the nodes $1,\dots,m$. Let us call the associated polytope $P_{\mathcal{B}(K_{1,m})}$ the *stellohedron.*

From Proposition 8.5 one sees that $\mathcal{B}(K_{1,m})$-trees are in bijection with *partial permutations* of $[m]$, which are ordered sequences $u = (u_1,\dots,u_r)$ of distinct numbers in $[m]$, where $r = 0,\dots,m$. The tree T associated to a partial permutation $u = (u_1,\dots,u_r)$ has the edges

$$(u_r,u_{r-1}),\dots,(u_2,u_1),(u_1,m+1),(m+1,i_1),\dots,(m+1,i_{m-r})$$

where $i_1,\dots,i_{m-r}$ are the elements of $[m] \setminus \{u_1,\dots,u_r\}$. The root of T is u_r if $r \geq 1$, or $m+1$ if $r=0$. For $r \geq 1$, one has $\mathrm{des}(T) = \mathrm{des}(u)+1$, where the descent number of a partial permutation is

$$\mathrm{des}(u) := \#\{i = 1,\dots,r-1 \mid u_i > u_{i+1}\}.$$

Also for the tree T associated with the empty partial permutation (for $r=0$) one has $\mathrm{des}(T) = 0$. Corollary 8.4 then says that

$$h_{\mathcal{B}(K_{1,m})}(t) = 1 + \sum_u t^{\mathrm{des}(u)+1} = 1 + \sum_{r=1}^{m} \binom{m}{r} \sum_{k=1}^{r} A(r,k)\, t^k, \tag{10.2}$$

where the first sum is over nonempty partial permutations w of $[m]$. In particular, the total number of vertices of the stellohedron $P_{\mathcal{B}(K_{1,m})}$ equals

$$f_0(P_{\mathcal{B}(K_{1,m})}) = \sum_{r=0}^{m} \binom{m}{r} \cdot r! = \sum_{r=0}^{m} \frac{m!}{r!}.$$

This sequence appears in Sloan's On-Line Encyclopedia of Integer Sequences[6] as A000522.

In this case, $\mathcal{B}(K_{1,m})$-permutations are permutations $w \in \mathfrak{S}_{m+1}$ such that $m+1$ appears before the first descent. Such permutations w are in bijection with partial permutations u of $[m]$. Indeed, u is the part of w after the entry $m+1$. Since our labelling of $K_{1,m}$ (with the central node labelled $m+1$) is decreasing (see Example 9.3), Corollary 9.6 implies that the h-polynomial of the stellohedron $P_{\mathcal{B}(K_{1,m})}$ is $h(t) = \sum_w t^{\mathrm{des}(w)}$, where the sum runs over all such permutations $w \in \mathfrak{S}_{m+1}$. This agrees with the above expression in terms of partial permutations.

10.5. **The Stanley-Pitman polytope.** Let $\mathcal{B}_{\mathrm{PS}} = \{[i,n],\{i\} \mid i = 1,\dots,n\}$ (the collection of all intervals $[i,n]$ and singletons $\{i\}$). This (non-graphical) building set is chordal. According to [Post'05, §8.5], the corresponding nestohedron $P_{\mathcal{B}_{\mathrm{PS}}}$ is the Stanley-Pitman polytope from [StPi'02].

By Proposition 8.5, $\mathcal{B}_{\mathrm{PS}}$-trees have the following form $T(I)$. For an increasing sequence I of positive integers $i_1 < i_2 < \cdots < i_k = n$, construct the tree $T(I)$ on $[n]$ with the root at i_1 and the chain of edges $(i_1,i_2),(i_2,i_3),\dots,(i_{k-1},i_k)$; also, for

[6] `http://akpublic.research.att.com/~njas/sequences/`

each $j \in [n] \setminus I$, one has the edge (i_l, j) where i_l is the minimal element of I such that $i_l > j$.

In this case, $\mathcal{B}_{\mathrm{PS}}$-permutations are permutations $w \in \mathfrak{S}_n$ such that $w(1) < w(2) < \cdots < w(k) > w(k+1) > \cdots > w(n)$, for some $k = 1, \ldots, n$.

Using $\mathcal{B}_{\mathrm{PS}}$-trees or $\mathcal{B}_{\mathrm{PS}}$-permutations one can easily deduce that the h-polynomial of the Stanley-Pitman polytope is $h_{\mathcal{B}_{\mathrm{PS}}}(t) = (1+t)^{n-1}$. This is not surprising since $P_{\mathcal{B}_{\mathrm{PS}}}$ is combinatorially isomorphic to the $(n-1)$-dimensional cube.

11. γ-vectors of nestohedra

Recall that the γ-vector $(\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor})$ of a d-dimensional simple polytope is defined via its h-polynomial as $h(t) = \sum \gamma_i\, t^i (1+t)^{d-2i}$; and the γ-polynomial is $\gamma(t) = \sum \gamma_i\, t^i$; see Section 2.3.

The main result of this section is a formula for the γ-polynomial of a chordal nestohedron as a descent-generating function (or peak-generating function) for some set of permutations. This implies that Gal's conjecture (Conjecture 2.3) holds for this class of flag simple polytopes.

11.1. A warm up: γ-vector for the permutohedron.

We review here the beautiful construction of Shapiro, Woan, and Getu [SWG'83] that leads to a non-negative formula for the γ-vector of the usual permutohedron. This subsection also serves as a warm-up for a more general construction in the following subsection.

Some notation is necessary. Recall that a *descent* in a permutation $w \in \mathfrak{S}_n$ is a pair $(w(i), w(i+1))$ such that $w(i) > w(i+1)$, where $i \in [n-1]$. A *final descent* is when $w(n-1) > w(n)$, and a *double descent* is a pair of consecutive descents, i.e. a triple $w(i) > w(i+1) > w(i+2)$.

Additionally, define a *peak* of w to be an entry $w(i)$ for $1 \leq i \leq n$ such that $w(i-1) < w(i) > w(i+1)$. Here (and below) set $w(0) = w(n+1) = 0$ and so a peak can occur in positions 1 or n. On the other hand, a *valley* of w is an entry $w(i)$ for $1 < i < n$ such that $w(i-1) > w(i) < w(i+1)$. The *peak-valley sequence* of w is the subsequence in w formed by all peaks and valleys.

Let $\widehat{\mathfrak{S}}_n$ denote the set of permutations in $\mathfrak{S}_n$ which do not contain any final descents or double descents. Let $\mathrm{peak}(w)$ denote the number of peaks in a permutation w. It is clear that $\mathrm{peak}(w) - 1 = \mathrm{des}(w)$, for permutations $w \in \widehat{\mathfrak{S}}_n$ (and only for these permutations).

Theorem 11.1. (cf. [SWG'83, Proposition 4]) The γ-polynomial of the usual permutohedron $P_{\mathcal{B}(K_n)}$ is

$$\sum_{w \in \widehat{\mathfrak{S}}_n} t^{\mathrm{peak}(w)-1} = \sum_{w \in \widehat{\mathfrak{S}}_n} t^{\mathrm{des}(w)}.$$

Example 11.2. Let us calculate the γ-polynomial of the two dimensional permutohedron $P_{\mathcal{B}(K_3)}$. One has $\widehat{\mathfrak{S}}_3 = \{(1,2,3), (2,1,3), (3,1,2)\}$. Of these, $(1,2,3)$ has one peak (and no descents), and $(2,1,3)$ and $(3,1,2)$ have two peaks (and one descent). Therefore, the γ-polynomial is $1 + 2t$.

Say that an entry $w(i)$ of w is an *intermediary* entry if $w(i)$ is not a peak or a valley. Say that $w(i)$ is an *ascent-intermediary entry* if $w(i-1) < w(i) < w(i+1)$ and that it is a *descent-intermediary entry* if $w(i-1) > w(i) > w(i+1)$. (Here again one should assume that $w(0) = w(n+1) = 0$.) Note that the set $\widehat{\mathfrak{S}}_n$ is exactly the set of permutations in $\mathfrak{S}_n$ without descent-intermediary entries.

It is convenient to graphically represent a permutation $w \in \mathfrak{S}_n$ by a piecewise linear "mountain range" M_w obtained by connecting the points $(x_0, 0)$, $(x_1, w(1))$, $(x_2, w(2))$, $\dots$, $(x_n, w(n))$, $(x_{n+1}, 0)$ on $\mathbb{R}^2$ by straight line intervals, for some $x_0 < x_1 < \cdots < x_{n+1}$; see Figure 11.1. Then peaks in w correspond to local maxima of M_w, valleys correspond to local minima of M_w, ascent-intermediary entries correspond to nodes on ascending slopes of M_w, and descent-intermediary entries correspond to nodes on descending slopes of M_w. For example, the permutation $w = (6, 5, 4, 10, 8, 2, 1, 7, 9, 3)$ shown in Figure 11.1 has three peaks $6, 10, 9$, two valleys $4, 1$, one ascent-intermediary entry 7, and four descent-intermediary entries $5, 8, 2, 3$. Its peak-valley sequence is $(6, 4, 10, 1, 9)$.

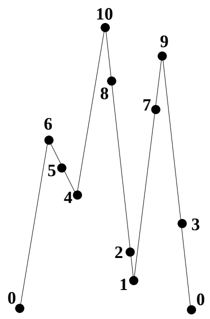

FIGURE 11.1. Mountain range M_w for $w = (6, 5, 4, 10, 8, 2, 1, 7, 9, 3)$

As noted in Section 4.1, the h-polynomial of the permutohedron is the descent-generating function for permutations in $\mathfrak{S}_n$ (the Eulerian polynomial). In order to prove Theorem 11.1, one constructs an appropriate partitioning of $\mathfrak{S}_n$ into equivalence classes (cf. Remark 2.4), where each equivalence classes has exactly one element from $\widehat{\mathfrak{S}}_n$. To describe the equivalence classes of permutations, one must introduce some operations on permutations.

Definition 11.3. Let us define the *leap* operations L_a and L_a^{-1} that act on permutations. Informally, the permutation $L_a(w)$ is obtained from w by moving an intermediary node a on the mountain range M_w directly to the right until it hits the next slope of M_w. The permutation $L_a^{-1}(w)$ is obtained from w by moving a directly to the left until it hits the next slope of M_w.

More formally, for an intermediary entry $a = w(i)$ in w, the permutation $L_a(w)$ is obtained from w by removing a from the i-th position and inserting a in the position between $w(j)$ and $w(j+1)$, where j is the minimal index such that $j > i$ and a is between $w(j)$ and $w(j+1)$, i.e., $w(j) < a < w(j+1)$ or $w(j) > a > w(j+1)$. The leap operation L_a is not defined if all entries following a in w are less than a.

Similarly, the inverse operation $L_a^{-1}(w)$ is given by removing a from the i-th position in w and inserting a between $w(k)$ and $w(k+1)$, where k is the maximum index such that $k < i$ and a is between $w(k)$ and $w(k+1)$. The operation L_a^{-1} is is not defined if all entries preceding a in w are less than a.

For example, for the permutation w shown on Figure 11.1, one has $L_2(w) = (6, 5, 4, 10, 8, 1, 2, 7, 9, 3)$ and $L_2^{-1}(w) = (2, 6, 5, 4, 10, 8, 1, 7, 9, 3)$.

Clearly, if a is an ascent-intermediary entry in w then a is a descent-intermediary entry in $L_a^{\pm 1}(w)$, and vise versa. Note that if a is an ascent-intermediary entry in w, then $L_a(w)$ is always defined, and if a is a decent-intermediary entry, then $L_a^{-1}(w)$ is always defined.

Definition 11.4. Let us also define the *hop* operations H_a on permutations. For an ascent-intermediary entry a in w, define $H_a(w) = L_a(w)$; and, for a descent-intermediary entry a in w, define $H_a(w) = L_a^{-1}(w)$.

For example, for the permutation w shown on Figure 11.1, the permutation $H_2(w) = (2, 6, 5, 4, 10, 8, 1, 7, 9, 3)$ is obtained by moving the descent-intermediary entry 2 to the left to the first ascending slope, and $H_7(w) = (6, 5, 4, 10, 8, 2, 1, 9, 7, 3)$ is obtained by moving the ascent-intermediary entry 7 to the right to the last descending slope.

Note that leaps and hops never change the shape of the mountain range M_w, that is, they never change the peak-valley sequence of w. They just move intermediary nodes from one slope of M_w to another. It is quite clear from the definition that all leap and hop operations pairwise commute with each other. It is also clear that two hops H_a get us back to the original permutation.

Lemma 11.5. *For intermediary entries a and b in w, one has $(H_a)^2(w) = w$ and $H_a(H_b(w)) = H_b(H_a(w))$.*

Thus the hop operations H_a generate the action of the group $(\mathbb{Z}/2\mathbb{Z})^m$ on the set of permutations with a given peak-valley sequence, where m is the number of intermediary entries in such permutations.

Let us say that two permutations are *hop-equivalent* if they can be obtained from each other by the hop operations H_a for various a's. The partitioning of $\mathfrak{S}_n$ into hop-equivalence classes allows us to prove Theorem 11.1.

Proof of Theorem 11.1. The number $\mathrm{des}(w)$ of descents in w equals the number of peaks in w plus the number of descent-intermediary entries in w minus 1 (because the last entry is either a peak or a descent-intermediary entry, but it does not contribute a descent). Notice that if a is an ascent-intermediary (resp., descent-intermediary) entry in w then the number of descent-intermediary entries in $H_a(w)$ increases (resp., decreases) by 1 and the number of peaks does not change.

If $w \in \mathfrak{S}_n$ has $p = \mathrm{peak}(w)$ peaks then it has $p-1$ valleys and $n - 2p + 1$ intermediary entries. Lemma 11.5 implies that the hop-equivalence class C of w involves 2^{n-2p+1} permutations. Moreover, the descent-generating function for these permutations is $\sum_{u\in C} t^{\mathrm{des}(u)} = t^p(t+1)^{n-2p+1}$. Each hop-equivalence class has exactly one representative u without descent-intermediary entries, that is $u \in \widehat{\mathfrak{S}}_n$. Thus, summing the contributions of hop-equivalence classes, one can write the h-polynomial of the permutohedron as

$$h(t) = \sum_{w\in\mathfrak{S}_n} t^{\mathrm{des}(w)} = \sum_{w\in\widehat{\mathfrak{S}}_n} t^{\mathrm{peak}(w)-1}(t+1)^{n+1-2\,\mathrm{peak}(w)}.$$

Comparing this to the definition of the γ-polynomial, one derives the theorem. ☐

11.2. **γ-vectors of chordal nestohedra.** According to Proposition 9.7, nestohedra for chordal building sets are flag simple polytopes. Thus Gal's conjecture

(Conjecture 2.3) applies. This section proves this conjecture and present a nonnegative combinatorial formula for γ-polynomials of such nestohedra as peak-generating functions for some subsets of permutations.

Let $\mathcal{B}$ be a connected chordal building set on $[n]$. Recall that $\mathfrak{S}_n(\mathcal{B})$ is the set of $\mathcal{B}$-permutations; see Definition 8.7. Let $\widehat{\mathfrak{S}}_n(\mathcal{B}) := \mathfrak{S}_n(\mathcal{B}) \cap \widehat{\mathfrak{S}}_n$ be the subset of $\mathcal{B}$-permutations which have no final descent or double descent.

The following theorem is the main result of this section.

Theorem 11.6. *For a connected chordal building $\mathcal{B}$ on $[n]$, the γ-polynomial of the nestohedron $P_{\mathcal{B}}$ is the peak-generating function for the permutations in $\widehat{\mathfrak{S}}_n(\mathcal{B})$:*

$$\gamma_{\mathcal{B}}(t) = \sum_{w \in \widehat{\mathfrak{S}}_n(\mathcal{B})} t^{\mathrm{peak}(w)-1} = \sum_{w \in \widehat{\mathfrak{S}}_n(\mathcal{B})} t^{\mathrm{des}(w)}.$$

As noted earlier, $\mathrm{peak}(w) - 1 = \mathrm{des}(w)$ for $w \in \widehat{\mathfrak{S}}_n$.

The proof of Theorem 11.6 will be an extension of the proof given for the γ-vector of the permutohedron in Section 11.1. Recall that Corollary 9.6 interprets the h-polynomial of $P_{\mathcal{B}}$ as the descent-generating function for $\mathcal{B}$-permutations $w \in \mathfrak{S}_n(\mathcal{B})$. Theorem 11.6 will be proven by constructing an appropriate partitioning of the set $\mathfrak{S}_n(\mathcal{B})$ into equivalence classes, where each equivalence class has exactly one representative from $\widehat{\mathfrak{S}}_n(\mathcal{B})$. As before, one uses (suitably generalized) hop operations to describe equivalence classes of elements of $\mathfrak{S}_n(\mathcal{B})$.

One needs powers of the leap operations $L_a^r := (L_a)^r$, for $r \geq 0$, and $L_a^r := (L_a^{-1})^{-r}$, for $r \leq 0$; see Definition 11.3. In other words, for $r > 0$, $L_a^r(w)$ is obtained from w by moving the intermediary entry a to the right until it hits the r-th slope from its original location; and, for $r < 0$, by moving a to the left until it hits $(-r)$-th slope from its original location. Clearly, $L_a^r(w)$ is defined whenever r is in a certain integer interval $r \in [r_{\min}, r_{\max}]$. It is also clear that, if a is an ascent-intermediary entry in w, then a is ascent-intermediary in $L_a^r(w)$ for even r and a is descent-intermediary in $L_a^r(w)$ for odd r, and vice versa if a is descent-intermediary in w.

Note that for a $\mathcal{B}$-permutation $w \in \mathfrak{S}_n(\mathcal{B})$, the permutations $L_a^r(w)$ may no longer be $\mathcal{B}$-permutations. The next lemma ensures that at least some of them will be $\mathcal{B}$-permutations.

Lemma 11.7. *Let $\mathcal{B}$ be a chordal building on $[n]$. Suppose that $w \in \mathfrak{S}_n(\mathcal{B})$ is a $\mathcal{B}$-permutation.*

(1) *If a is an ascent-intermediary letter in w, then there exists an odd positive integer $r > 0$ such that $L_a^r(w) \in \mathfrak{S}_n(\mathcal{B})$ and $L_a^s(w) \notin \mathfrak{S}_n(\mathcal{B})$, for all $0 < s < r$.*

(2) *If a is a descent-intermediary letter in w, then there exists an odd negative integer $r < 0$ such that $L_a^r(w) \in \mathfrak{S}_n(\mathcal{B})$ and $L_a^s(w) \notin \mathfrak{S}_n(\mathcal{B})$, for all $0 > s > r$.*

The proof of Lemma 11.7 will require some preparatory notation and observations.

For a permutation $w \in \mathfrak{S}_n$ and $a \in [n]$ such that $w(i) = a$, let

$$\{w \nwarrow a\} := \{w(j) \mid j \leq i,\, w(j) \geq a\}$$

be the set of all entries in w which are located to the left of a and are greater than or equal to a (including the entry a itself). The arrow in this notation refers to our graphical representation of a permutation as a mountain range M_w: the set $\{w \nwarrow a\}$ is the set of entries in w located to the North-West of the entry a.

According to Definition 8.7, the set $\mathfrak{S}_n(\mathcal{B})$ is the set of permutations w such that, for $i = 1, \dots, n$, there exists $I \in \mathcal{B}$ such that both $w(i)$ and $\max(w(1), \dots, w(i))$ are in I and $I \subset \{w(1), \dots, w(i)\}$. If $\mathcal{B}$ is chordal, then $I' := I \cap [w(i), \infty]$ also belongs to $\mathcal{B}$ (see Definition 9.2) and satisfies the same properties. Clearly $\max(w(1), \dots, w(i)) = \max\{w \nwarrow w(i)\}$. Thus, for a chordal building set, one can reformulate Definition 8.7 of $\mathcal{B}$-permutations as follows.

Lemma 11.8. *Let $\mathcal{B}$ be a chordal building set. Then $\mathfrak{S}_n(\mathcal{B})$ is the set of permutations $w \in \mathfrak{S}_n$ such that for any $a \in [n]$, the elements a and $\max\{w \nwarrow a\}$ are in the same connected component of $\mathcal{B}|_{\{w \nwarrow a\}}$. Equivalently, there exists $I \in \mathcal{B}$ such that $a \in I$, $\max\{w \nwarrow a\} \in I$, and $I \subset \{w \nwarrow a\}$.*

Let us now return to the setup of Lemma 11.7. There are 2 possible reasons why the permutation $u = L_a^r(w)$ may no longer be a $\mathcal{B}$-permutation, that is, fail to satisfy the conditions in Lemma 11.8:

(A) It is possible that the entry a and the entry $\max\{u \nwarrow a\}$ are in different connected components of $\mathcal{B}|_{\{u \nwarrow a\}}$.

(B) It is also possible that another entry $b \neq a$ in u and $\max\{u \nwarrow b\}$ are in different connected components of $\mathcal{B}|_{\{u \nwarrow b\}}$.

Let us call these two types of failure *A-failure* and *B-failure.* The following auxiliary result is needed.

Lemma 11.9. *Let us use the notation of Lemma 11.7.*

(1) *For left leaps $u = L_a^r(w)$, $r < 0$, one can never have a B-failure.*

(2) *For the maximal left leap $u = L_a^{r_{\min}}(w)$, where the entry a goes all the way to the left, one cannot have an A-failure.*

(3) *For the maximal right leap $u = L_a^{r_{\max}}(w)$, where the entry a goes all the way to the right, one cannot have an A-failure.*

(4) *Let $u = L_a^r(w)$ and $u' = L_a^{r+1}(w)$, for $r \in \mathbb{Z}$, be two adjacent leaps such that a is descent-intermediary in u (and, thus, a is ascent-intermediary in u'). Then there is an A-failure in u if and and only if there is an A-failure in u'.*

Proof. (1) Since $w \in \mathfrak{S}_n(\mathcal{B})$, there is a subset $I \in \mathcal{B}$ that contains both b and $\max\{w \nwarrow b\}$ and such that $I \subset \{w \nwarrow b\}$. The same subset I works for u because $\{u \nwarrow b\} = \{w \nwarrow b\}$ or $\{u \nwarrow b\} = \{w \nwarrow b\} \cup \{a\}$.

(2) In this case, a is greater than all preceding entries in u, so $a = \max\{u \nwarrow a\}$.

(3) In this case, a is greater than all following entries in u. The interval $I = [a, n]$ contains both a and $\max\{u \nwarrow a\}$, $I \subset \{u \nwarrow a\}$, and $I \in \mathcal{B}$ because $\mathcal{B}$ is chordal.

(4) In this case, all entries between the position of a in u and the position of a in u' are less than a. Thus $\{u \nwarrow a\} = \{u' \nwarrow a\}$. So u has an A-failure if and only if u' has an A-failure. □

Proof of Lemma 11.7. It is easier to prove the second part of the lemma.

(2) By parts (1) and (2) of Lemma 11.9, there exists a negative r such that $L_a^r(w) \in \mathfrak{S}_n(\mathcal{B})$. Let us pick such an r with minimal possible absolute value. Then r should be odd, by part (4) of Lemma 11.9, which proves (2).

(1) Suppose that there is an entry $b \neq a$ in the permutation w such that b and $m = \max\{w \nwarrow b\}$ are in different connected components of $\mathcal{B}|_{\{w \nwarrow b\} \setminus \{a\}}$. In this case, $a \in \{w \nwarrow b\}$, that is $b < a$ and b is located to the right of a in w. (Otherwise, b and m are in different connected components of $\mathcal{B}|_{\{w \nwarrow b\}}$, which is impossible because w is a $\mathcal{B}$-permutation.) Let us pick the leftmost entry b in w that satisfies

this condition. Then the permutation $u = L_a^r(w)$ has a B-failure if the letter a moves to the right of this entry b; and u has no B-failure if a stays to the left of b. By our assumptions, a stays to the left of b in $L_a^1(w)$, so such a u exists.

Let $u = L_a^r(w)$ be the maximal right leap (i.e., with maximal $r > 0$) such that the entry a stays to the left of b. Then all entries in u between the positions of a and b should be less than a. Thus $m = \max\{u \nwarrow a\} = \max\{w \nwarrow b\}$. Since $w \in \mathfrak{S}_n(\mathcal{B})$, there is an $I \in \mathcal{B}$ such that $b, m \in I$ and $I \subset \{w \nwarrow b\}$. This subset I should also contain the entry a. (Otherwise, b and m would be in the same connected component $\mathcal{B}|_{\{w \nwarrow b\} \setminus \{a\}}$, contrary to our choice of b.) Thus $I' := I \cap [a, +\infty] \in \mathcal{B}$ contains both a and m and $I' \subset \{u \nwarrow a\}$. This means that there is no A-failure in u. Thus $u \in \mathfrak{S}_n(\mathcal{B})$.

If there is no entry b in w as above, then none of the permutations $L_a^r(w)$ has a B-failure. In this case $L_a^{r_{\max}}(w) \in \mathfrak{S}_n(\mathcal{B})$ by part (3) of Lemma 11.9.

In all cases, there exists a positive r such that $L_a^r(w) \in \mathfrak{S}_n(\mathcal{B})$ and only A-failures are possible in $L_a^s(w)$, for $0 < s < r$. Let us pick the minimal such r. Then r should be odd by part (4) of Lemma 11.9, as needed. ☐

Definition 11.10. Let us define the $\mathcal{B}$*-hop* operations $\mathcal{B}H_a$. For a $\mathcal{B}$-permutation w with an ascent-intermediary (resp., descent-intermediary) entry a, the permutation $\mathcal{B}H_a(w)$ is the right leap $u = L_a^r(w)$, $r > 0$ (resp., the left leap $u = L_a^r(w)$, $r < 0$) with minimal possible $|r|$ such that u is a $\mathcal{B}$-permutation.

Informally, $\mathcal{B}H_a(w)$ is obtained from w by moving the node a on its mountain range M_w directly to the right if a is ascent-intermediary in w, or directly left if a is descent-intermediary in w (possibly passing through several slopes) until one hits a slope and obtain a $\mathcal{B}$-permutation.

Lemma 11.7 says that the $\mathcal{B}$-hop $\mathcal{B}H_a(w)$ is well-defined for any intermediary entry a in w. It also says that if a is ascent-intermediary in w then a is descent-intermediary in $\mathcal{B}H_a(w)$, and vice versa. Moreover, according to that lemma, $(\mathcal{B}H_a)^2(w) = w$.

Example 11.11. Let G be the decreasing tree shown on Figure 11.2. Then the graphical building $\mathcal{B} = \mathcal{B}(G)$ is chordal; see Example 9.3. Figure 11.2 shows several $\mathcal{B}$-hops of the $\mathcal{B}$-permutation $w = (1, 10, 8, 3, 6, 9, 7, 4, 12, 11, 5, 2)$:

$$\begin{aligned}
\mathcal{B}H_1(w) &= L_1(w) = (10, 8, 3, 6, 9, 7, 4, 12, 11, 5, 2, 1), \\
\mathcal{B}H_5(w) &= (L_5)^{-5}(w) = (1, 5, 10, 8, 3, 6, 9, 7, 4, 12, 11, 2), \\
\mathcal{B}H_6(w) &= L_6(w) = (1, 10, 8, 3, 9, 7, 6, 4, 12, 11, 5, 2).
\end{aligned}$$

Let us now show that the $\mathcal{B}$-hop operations pairwise commute with each other.

Lemma 11.12. *Let a and b be two intermediary entries in a $\mathcal{B}$-permutation w. Then $\mathcal{B}H_a(\mathcal{B}H_b(w)) = \mathcal{B}H_b(\mathcal{B}H_a(w))$.*

Proof. Let us first assume that both a and b are descent-intermediary entries in w. Without loss of generality assume that $a > b$. In this case $\mathcal{B}H_a(w) = L_a^r(w)$ and $\mathcal{B}H_b(w) = L_b^s(w)$ for some *negative* odd r and s, that is the entries a and b of w are moved to the left. According to Lemma 11.9(1), in this case one does not need to worry about B-failures. In other words, $\mathcal{B}H_a(w)$ is the first left leap $L_a^r(w)$ (i.e., with minimal $-r > 0$) that has no A-failure. Similarly, $\mathcal{B}H_b(w)$ is the first left leap $L_b^s(w)$ without A-failures (where A-failures concern the entry b).

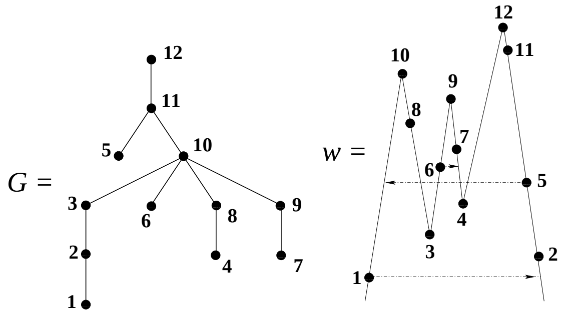

FIGURE 11.2. A $\mathcal{B}(G)$-permutation w and some $\mathcal{B}$-hops

Since A-failures for permutations $u = L_a^t(w)$, $t < 0$, are described in terms of the set $\{u \nwarrow a\} \subset [a, \infty]$, moving the entry $b < a$ in w will have no effect on these A-failures. Thus, for the permutation $w' = \mathcal{B}H_b(w)$, one has $\mathcal{B}H_a(w') = L_a^r(w')$ with exactly the same r as in $\mathcal{B}H_a(w) = L_a^r(w)$.

However, for permutations $u = L_b^t(w)$, $t < 0$, the sets $\{u \nwarrow b\}$ might change if one first performs the operation $\mathcal{B}H_a$ to w. Namely, let $\tilde{w} = \mathcal{B}H_a(w)$ and $\tilde{u} = L_b^t(\tilde{w}) = L_b^t(L_a^r(w))$. Then $\{\tilde{u} \nwarrow b\} = \{u \nwarrow b\} \cup \{a\}$ if a is located to the left of b in $\tilde{u}$ and a is located to the right of b in u (and $\{\tilde{u} \nwarrow b\} = \{u \nwarrow b\}$ otherwise). Notice that one always has $m = \max\{u \nwarrow b\} = \max\{\tilde{u} \nwarrow b\}$, since this maximum is the maximal peak preceding b in u (or in $\tilde{u}$), and leaps and hops have no affect on the peaks.

If b and m are in the same connected component of $\mathcal{B}|_{\{u \nwarrow b\}}$ then they are also in the same connected component of $\mathcal{B}|_{\{\tilde{u} \nwarrow b\}}$, that is if there is no A-failure for u then there is no A-failure for $\tilde{u}$.

Suppose that there is no A-failure for $\tilde{u}$ but there is an A-failure for u. Then the sets $\{u \nwarrow b\}$ and $\{\tilde{u} \nwarrow b\}$ have to be different. That means that a is located to the left of b in $\tilde{u}$ and a is located to the right of b in u. Let I be the element $I \in \mathcal{B}$ such that $b, m \in I$ and $I \subset \{\tilde{u} \nwarrow b\}$. Then I should contain the entry a. (Otherwise, $I \subset \{u \nwarrow b\}$ and there would be no A-failure for u.)

Let $\hat{w} = L_a^{\hat{t}}(w)$ be the left leap with maximal possible $-\hat{t} \geq 0$ such that the position of a in $\hat{w}$ is located to the right of the position of b in $\tilde{u}$. Since $\tilde{u} = L_b^t(L_a^r(w))$, it follows that $|\hat{t}| < |r|$. In other words, if one starts moving to the right from the node b along the mountain range $M_{\tilde{u}}$, the (ascending) slope that first crosses the level a is the place where the entry a is located in $\hat{w}$. Note that $\hat{t}$ is odd because a should be an ascent-intermediary entry in $\hat{w}$; in particular $\hat{t} < 0$.

Since all entries in $\hat{w}$ located between the position of b in $\tilde{u}$ and the position of a in $\hat{w}$ are less than a, one deduces that $\{\tilde{u} \nwarrow b\} \cap [a, \infty] = \{\hat{w} \nwarrow a\}$. Thus the subset $\hat{I} = I \cap [a, \infty]$ has three important properties: it lies in $\mathcal{B}$ (because $\mathcal{B}$ is chordal); it contains both a and $m = \max\{\hat{w} \nwarrow a\}$; and it is a subset of $\{\hat{w} \nwarrow a\}$. It follows that there is no A-failure in $\hat{w}$. This contradicts the fact that $L_a^r(w) \neq L_a^{\hat{t}}(w)$ is the first left leap that has no A-failure.

Thus u has an A-failure if and only if $\tilde{u}$ has an A-failure. It follows that $\mathcal{B}H_b(\tilde{w}) = L_b^s(\tilde{w})$ with exactly the same s as in $\mathcal{B}H_b(w) = L_b^s(w)$.

This proves that $\mathcal{B}H_a(\mathcal{B}H_b(w)) = L_a^r(L_b^s(w)) = L_b^s(L_a^r(w)) = \mathcal{B}H_b(\mathcal{B}H_a(w))$, in the case when both a and b are descent-intermediary in w.

Let us now show that the general case easily follows. Suppose that, say, a is ascent-intermediary and b is descent-intermediary in w. Then, for $w'' = \mathcal{B}H_a(w)$ both a and b are descent-intermediary. One has $\mathcal{B}H_a(\mathcal{B}H_b(w'')) = \mathcal{B}H_b(\mathcal{B}H_a(w''))$. Thus $\mathcal{B}H_a(\mathcal{B}H_b(\mathcal{B}H_a(w))) = \mathcal{B}H_b(\mathcal{B}H_a(\mathcal{B}H_a(w))) = \mathcal{B}H_b(w)$. Applying $\mathcal{B}H_a$ to both sides, one deduces $\mathcal{B}H_b(\mathcal{B}H_a(w)) = \mathcal{B}H_a(\mathcal{B}H_b(w))$. The other cases are similar. $\square$

Thus the $\mathcal{B}$-hop operations $\mathcal{B}H_a$ generate the action of the group $(\mathbb{Z}/2\mathbb{Z})^m$ on the set of $\mathcal{B}$-permutations with a given peak-valley sequence, where m is the number of intermediary entries in such permutations.

Let us say that two $\mathcal{B}$-permutation are *$\mathcal{B}$-hop-equivalent* if they can be obtained from each other by the $\mathcal{B}$-hop operations $\mathcal{B}H_a$ for various a's. This gives the partitioning of the set of $\mathcal{B}$-permutations into $\mathcal{B}$-hop-equivalence classes.

One can now prove Theorem 11.6 by literally repeating the argument in the proof of Theorem 11.1.

Proof of Theorem 11.6. For a $\mathcal{B}$-permutation $w \in \mathfrak{S}_n(\mathcal{B})$ with $p = \mathrm{peak}(w)$, the descent-generating function of the $\mathcal{B}$-hop-equivalence class C of w is $\sum_{u\in C} t^{\mathrm{des}(u)} = t^p(t+1)^{n-2p+1}$. Each $\mathcal{B}$-hop-equivalence class has exactly one representative without descent-intermediary entries, that is, in the set $\widehat{\mathfrak{S}}_n(\mathcal{B})$. Thus the h-polynomial of the nestohedron $P_\mathcal{B}$ (see Corollary 9.6) is

$$h_{P_\mathcal{B}}(t) = \sum_{w\in\mathfrak{S}_n(\mathcal{B})} t^{\mathrm{des}(w)} = \sum_{w\in\widehat{\mathfrak{S}}_n(\mathcal{B})} t^{\mathrm{peak}(w)-1}(t+1)^{n+1-2\,\mathrm{peak}(w)}.$$

Comparing this to the definition of the γ-polynomial, one derives the theorem. $\square$

Corollary 11.13. *Gal's conjecture holds for all graph-associahedra corresponding to chordal graphs.*

11.3. **γ-vectors for the associahedron and cyclohedron.**

Proposition 11.14. *The γ-polynomial of the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ is*

$$\gamma(t) = \sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor} C_r \binom{n-1}{2r} t^r,$$

where $C_r = \frac{1}{r+1}\binom{2r}{r}$ is the r-th Catalan number.

Proposition 11.15. *The γ-polynomial of the cyclohedron $P_{\mathcal{B}(\mathrm{Cycle}_n)}$ is*

$$\gamma(t) = \sum_{r=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n}{r, r, n-2r} t^r,$$

These two formulas can be derived from the expressions for the corresponding h-polynomials (see Sections 10.2 and 10.3) using standard quadratic transformations of hypergeometric series; e.g., see [RSW'03, Lemma 4.1].

On the other hand, let us mention the following three combinatorial interpretations of the γ-vector for the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$.

First proof of Proposition 11.14. It is known that the Narayana polynomial which is the h-polynomial of $P_{\mathcal{B}(\mathrm{Path}_n)}$ is also the rank generating function for the well-studied *lattice of noncrossing partitions* $NC(n)$. An explicit *symmetric chain decomposition* for $NC(n)$ was given by Simion and Ullman [SU'91], who actually produced a much stronger decomposition of $NC(n)$ into disjoint Boolean intervals placed symmetrically about the middle rank(s) of $NC(n)$. Their decomposition contains exactly $C_r\binom{n-1}{2r}$ such Boolean intervals of rank $n-(2r+1)$ for each $r = 0, 1, \ldots, \frac{n-1}{2}$, which immediately implies the formula for the γ-polynomial; see [SU'91, Corollary 3.2]. $\square$

Second proof of Proposition 11.14. By Section 10.2, the h-polynomial of $P_{\mathcal{B}(\mathrm{Path}_n)}$ counts plane binary trees on n nodes according to their number of right edges. There is a natural map from binary trees to *full* binary trees, i.e., those in which each node has zero or two children: if a node has a unique child, contract this edge from the node to its child. If the original binary tree T has n nodes, then the resulting full binary tree T' will have $2r+1$ nodes, $2r$ edges and r right edges for some $r = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor$. There are C_r such full binary trees for each r. Given such a full binary tree T', one can produce all of the binary trees in its preimage by inserting $n-(2r+1)$ more nodes and deciding if they create left or right edges. One chooses the locations of these nodes from $2r+1$ choices, either an edge of the full binary tree they will subdivide or located above the root, giving $\binom{n-(2r+1)+(2r+1)-1}{n-(2r+1)} = \binom{n-1}{2r}$ possible locations. Thus the generating function with respect to the number of right edges for the preimage of T' is $\binom{n-1}{2r} t^r (t+1)^{n-(2r+1)}$, where the term $t^r(t+1)^{n-(2r+1)}$ comes from choosing whether each of the new nodes creates a left or a right edge. It follows that the generating function for all binary trees on n nodes is $h_{\mathrm{Path}_n}(t) = \sum_r C_r \binom{n-1}{2r} t^r (t+1)^{n-(2r+1)}$, where C_r counts full binary trees. This implies the needed expression for the γ-vector of the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$.

Equivalently, one can describe the subdivision of all binary trees into classes where two binary trees are in the same class if they can be obtained from each other by switches of left and right edges coming from single child nodes. Then one gets exactly $C_r\binom{n-1}{2r}$ classes having $t^r(t+1)^{n-(2r+1)}$ as its generating function counting number of right edges, for each $r = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor$. $\square$

Third proof of Proposition 11.14. This proof is based on our general approach to γ-vectors of chordal nestohedra. According to Section 10.2, $\mathcal{B}$-permutations for the associahedron are 312-avoiding permutations and h-polynomial is equal to the sum $h_{P_{\mathcal{B}(\mathrm{Path}_n)}}(t) = \sum_w q^{\mathrm{peak}(w)-1}$ over all 312-avoiding permutations $w \in \mathfrak{S}_n$. By Theorem 11.6, $\gamma_r(P_{\mathcal{B}(\mathrm{Path}_n)})$ equals the number of 312-avoiding permutations with no descent-intermediary elements and $r+1$ peaks. The (flattenings of) peak-valley sequences of such permutations are exactly 312-avoiding *alternating permutations* in $\mathfrak{S}_{2r+1}$, that is 312-avoiding permutations w' such that $w'_1 > w'_2 < w'_3 > \cdots < w'_{2r+1}$. It is known that the number of such permutations equals the Catalan number C_r; see [Man'02, Theorem 2.2]. Then there are $\binom{n-1}{2r}$ ways to insert the remaining $n-(2r+1)$ descent-intermediary elements. $\square$

12. Graph-associahedra for single branched trees

Our goal in this section is to compute a generating function that computes the h-polynomials of all graph-associahedra in which the graph is a tree having at most one *branched* vertex (i.e., a vertex of valence 3 or more).

12.1. **Associahedra and Narayana polynomials.** First recall (see Section 10.2) that the h-numbers of the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ are the Narayana numbers

$h_k(P_{\mathcal{B}(\mathrm{Path}_n)}) = N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, and the h-polynomial of the associahedron is the *Narayana polynomial*:

$$h_{\mathcal{B}(\mathrm{Path}_n)}(t) = C_n(t) := \sum_{k=1}^{n} N(n,k)\, t^{k-1}. \tag{12.1}$$

Recall the well-known recurrence relation and the generating function for the Narayana polynomials $C_n(t)$. The recurrence for the f-polynomials $f_{\mathcal{B}(\mathrm{Path}_n)}(t) = h_{\mathcal{B}(\mathrm{Path}_n)}(t+1) = C_n(t+1)$ given by Theorem 6.12 can be written as follows. When one removes k vertices from the n-path, it splits into $k+1$ (possibly empty) paths. Thus one obtains

$$C_n(t) = \sum_{k\geq 1}(t-1)^{k-1} \sum_{m_1+\cdots+m_{k+1}=n-k} C_{m_1}(t)\cdots C_{m_{k+1}}(t), \quad \text{for } n\geq 1, \tag{12.2}$$

where the sum is over $m_1,\dots,m_{k+1}\geq 0$ such that $\sum m_i = n-k$. Here one assumes that $C_0(t)=1$.

Let $C(t,x)$ be the generating function for the Narayana polynomials:

$$\begin{aligned} C(t,x) &:= \sum_{n\geq 1} C_n(t)\, x^n = x + (1+t)x^2 + (1+3t+t^2)x^3 + \cdots \\ &= \frac{1-x-tx-\sqrt{(1-x-tx)^2-4tx^2}}{2tx}. \end{aligned} \tag{12.3}$$

The recurrence relation (12.2) is equivalent to the following well-known functional equation:

$$C = tx\, C^2 + (1+t)\, x\, C + x, \tag{12.4}$$

see [Stan'99, Exer. 6.36b].

12.2. **Generating function for single branched trees.** Trees with at most one branched vertex have the following form. For $a_1,\dots,a_k\geq 0$, let $T_{a_1,\dots,a_k}$ be the graph obtained by attaching k chains of lengths $a_1,\dots,a_k$ to one central node. For example, $T_{0,\dots,0}$ is the graph with a single node and $T_{1,\dots,1}$ is the k-star graph $K_{1,k}$.

Theorem 12.1. *One has the following generating function for the h-polynomials of graph-associahedra $P_{\mathcal{B}(T_{a_1,\dots,a_k})}$ for the graphs $T_{a_1,\dots,a_k}$:*

$$\begin{aligned} T(t,x_1,\dots,x_k) &:= \sum_{a_1,\dots,a_k\geq 0} h_{T_{a_1,\dots,a_k}}(t)\, x_1^{a_1+1}\cdots x_k^{a_k+1} \\ &= \frac{(t-1)\,\phi_1\cdots\phi_k}{t-\prod_{i=1}^{k}(1+(t-1)\,\phi_i)} \end{aligned}$$

where $\phi_i = x_i(1+t\,C(t,x_i))$, and $C(t,x)$ is the generating function for the Narayana polynomials from (12.3).

This theorem immediately implies the following formula from [Post'05].

Corollary 12.2. [Post'05, Proposition 8.7] *The generating function for the number of vertices in the graph-associahedron $P_{\mathcal{B}(T_{a_1,\dots,a_k})}$ is*

$$\sum_{a_1,\dots,a_k} f_0(P_{\mathcal{B}(T_{a_1,\dots,a_k})})\, x_1^{a_1}\cdots x_k^{a_k} = \frac{\bar{C}(x_1)\cdots\bar{C}(x_k)}{1-x_1\,\bar{C}(x_1)-\cdots-x_k\,\bar{C}(x_k)},$$

where $\bar{C}(x) = \sum_{n\geq 0} C_n\, x^n = \frac{1-\sqrt{1-4x}}{2x}$ *is the generating function for the Catalan numbers.*

Proof. The claim is obtained from Theorem 12.1 in the limit $t \to 1$. Note however that one needs to use l'Hôpital's rule before plugging in $t = 1$. □

The first proof of Theorem 12.1 is fairly direct, using Corollary 8.4 and the solution to Simon Newcomb's problem. The second uses Theorem 6.13 to set up a system of PDE's and solve them; it has the advantage of producing a generating function for the h-polynomials of one further family of graph-associahedra.

12.3. **Theorem 12.1 via Simon Newcomb's problem.** Let us first review Simon Newcomb's problem and its solution.

Let $w = (w(1), \ldots, w_m)$ be a permutation of the multiset $\{1^{c_1}, \ldots, k^{c_k}\}$, that is, each i appears in w exactly c_i times, for $i = 1, \ldots, k$. A *descent* in w is an index i such that $w(i) > w(i+1)$. Let $\mathrm{des}(w)$ denote the number of descents in w. *Simon Newcomb's Problem* is the problem of counting permutations of a multiset with a given number of descents, see [Mac'17, Sec. IV, Ch. IV] and [GJ'83, Sec. 4.2.13]. Let us define the *multiset Eulerian polynomial* as

$$A_{c_1,\ldots,c_k}(t) := \sum_{w} t^{\mathrm{des}(w)},$$

where the sum is over all permutations w of the multiset $\{1^{c_1}, \ldots, k^{c_k}\}$. By convention, set $A_{0,\ldots,0}(t) = 1$.

In particular, the polynomial $A_{1,\ldots,1}(t)$ is the usual Eulerian polynomial. It is clear that $A_{c_1,\ldots,c_k}(1) = \binom{m}{c_1,\ldots,c_k}$, the total number of multiset permutations. A solution to Simon Newcomb's problem can be expressed by the following generating function for the $A_{c_1,\ldots,c_k}(t)$.

Proposition 12.3. [GJ'83, Sec. 4.2.13] *One has*

$$\sum_{c_1,\ldots,c_k\geq 0} A_{c_1,\ldots,c_k}(t)\, y_1^{c_1}\cdots y_k^{c_k} = \frac{t-1}{t - \prod_{i=1}^{k}(1+(t-1)\,y_i)}.$$

Theorem 12.1 then immediately follows from Proposition 12.3 and the following proposition.

Proposition 12.4. *The generating function for the h-polynomials of the polytopes $P_{\mathcal{B}(T_{a_1,\ldots,a_k})}$ equals*

$$T(t, x_1, \ldots, x_k) = \sum_{c_1,\ldots,c_k\geq 0} A_{c_1,\ldots,c_k}(t)\, \phi_1^{c_1+1}\cdots\phi_k^{c_k+1}.$$

Proof. Let us label nodes of the graph $T_{a_1,\ldots,a_k}$ by integers in $[n]$, where $n = a_1 + \cdots + a_k + 1$, so that the first chain is labelled by $1, \ldots, a_1$, the second chain is labelled by $a_1 + 1, \ldots, a_1 + a_2$, etc., with all labels increasing towards the central node, and finally the central node has the maximal label n.

Let T be a $T_{a_1,\ldots,a_k}$-tree. Suppose that the root r of T belongs to the $w(1)$-st chain of the graph $T_{a_1,\ldots,a_k}$. If one removes the node r from the graph $T_{a_1,\ldots,a_k}$, then the graph decomposes into 2 connected components, one of which is a chain Path_{b_1} and the other is $T_{a_1,\ldots,a'_{w(1)},\ldots,a_k}$, where $a'_{w(1)} = a_{w(1)} - b_1 - 1$ and all other indices are the same as before. (The first component is empty if $b_1 = 0$.) According to Proposition 8.5, the tree T is obtained by attaching a Path_{b_1}-tree T_1 and a

$T_{a_1,\dots,a'_{w(1)},\dots,a_k}$-tree T' to the root r. (Here one assumes that there is one empty Path_0-tree T_1, for $b_1 = 0$.) Let us repeat the same procedure with the tree T'. Assume that its root belongs to the $w(2)$-nd chain and split it into a Path_{b_2}-tree T_2 and a tree T''. Then repeat this procedure with T'', etc. Keep on doing this until one gets a tree $T^{'\cdots'}$ with the root at the central node n. Finally, if one removes the central node n from $T^{'\cdots'}$, then it splits into k trees $\tilde{T}_1,\dots,\tilde{T}_k$ such that $\tilde{T}_j$ is a Path_{d_j}-tree, for $j = 1,\dots,k$.

So each $T_{a_1,\dots,a_k}$-tree T gives us the following data:

(1) a sequence $(w(1),\dots,w_m) \in [k]^m$;
(2) a Path_{b_i}-tree T_i, for $i = 1,\dots,m$;
(3) a Path_{d_j}-tree $\tilde{T}_j$, for $j = 1,\dots,k$.

This data satisfies the following conditions:

$$m, b_1,\dots,b_m, d_1,\dots,d_k \geq 0, \text{ and}$$
$$(b_1+1)e_{w(1)} + \cdots + (b_m+1)e_{w_m} + (d_1,\dots,d_k) = (a_1,\dots,a_k),$$

where $e_1,\dots,e_k$ are the standard basis vectors in $\mathbb{R}^k$. Conversely, data of this form gives us a unique $T_{a_1,\dots,a_k}$-tree T. The number of descents in the tree T is

$$\mathrm{des}(T) = \sum_{i=1}^{m} \mathrm{des}(T_i) + \sum_{j=1}^{k} \mathrm{des}(\tilde{T}_j) + l + \mathrm{des}(w),$$

where l is the number of nonempty trees among $T_1, \dots, T_m, \tilde{T}_1, \dots, \tilde{T}_k$. Indeed, all descents in trees T_i and $\tilde{T}_j$ correspond to descents in T, each nonempty tree T_i or $\tilde{T}_j$ gives an additional descent for the edge that attaches this tree, and descents in w correspond to descent edges that attach trees $T', T'', \dots$.

Let us fix a sequence $w = w(1),\dots,w(m)$. For $i \in [k]$, let c_i be the number of times the integer i appears in w. In other words, w is a permutation of the multiset $\{1^{c_1},\dots,k^{c_k}\}$. Then the total contribution to the generating function $T(t,x_1,\dots,x_k)$ of trees T whose data involve w is equal to $t^{\mathrm{des}(w)}\,\phi_1^{c_1+1}\cdots\phi_k^{c_k+1}$. Indeed, the term 1 in $\phi_i = x_i(1+t\cdot C(t,x_i))$ corresponds to an empty tree, and the term $t\cdot C(t,x_i)$ corresponds to nonempty trees, which contribute one additional descent. The term $\phi_i^{c_i}$ comes from the c_i trees $T_{j_1},\dots,T_{j_{c_i}}$, where $w_{j_1},\dots,w_{j_{c_i}}$ are all occurrences of i in w. Finally, additional 1's in the exponents of ϕ_i's come from the trees $\tilde{T}_1,\dots,\tilde{T}_k$. Summing this expression over all permutations w of the multiset $\{1^{c_1},\dots,k^{c_k}\}$ and then over all $c_1,\dots,c_k \geq 0$, one obtains the needed expression for the generating function $T(t,x_1,\dots,x_k)$. □

Remark 12.5. One can dualize all definitions, statements, and arguments in this section, as follows. An equivalent dual formulation to Theorem 12.1 says

$$T(t,x_1,\dots,x_k) = \frac{(1-t)\,\psi_1\cdots\psi_k}{1 - t\,\prod_{i=1}^{k}(1+(1-t)\,\psi_i)}$$

where $\psi_i = x_i(1+C(t,x_i))$. The equivalence to Theorem 12.1 follows from the relation $\phi_i\cdot\psi_i = (t-1)(\phi_i - \psi_i)$, which is a reformulation of the functional equation (12.4).

The *dual multiset Eulerian polynomial* is $\bar{A}_{c_1,\dots,c_k}(t) := \sum_w t^{\mathrm{wdes}(w)+1}$, where the sum is over permutations w of the multiset $M = \{1^{c_1},\dots,k^{c_k}\}$, $m = c_1+\cdots+c_k$, and $\mathrm{wdes}(w)$ is the number of *weak descents* in the multiset permutation w, that is, the number of indices i for which $w(i) \geq w(i+1)$. The bijection which reverses the word w shows that $\bar{A}_{c_1,\dots,c_k}(t) = t^m\, A_{c_1,\dots,c_k}(t^{-1})$ and consequently one has an equivalent formulation of the solution to Simon Newcomb's problem:

$$\sum_{c_1,\dots,c_k\geq 0} \bar{A}_{c_1,\dots,c_k}(t)\, y_1^{c_1}\cdots y_k^{c_k} = \frac{1-t}{1-t\prod_{i=1}^k (1+(1-t)\,y_i)}.$$

Then one can modify the proof of Proposition 12.4, by switching the labels $i \leftrightarrow n+1-i$ in the graph $T_{a_1,\dots,a_k}$, and applying a similar argument to show

$$T(t,x_1,\dots,x_k) = \sum_{c_1,\dots,c_k\geq 0} \bar{A}_{c_1,\dots,c_k}(t)\, \psi_1^{c_1+1}\cdots\psi_k^{c_k+1}.$$

12.4. **Proof of Theorem 12.1 via PDE.** This section rederives Theorem 12.1 using Theorem 6.13. It also calculates the generating function for f-polynomials of graph-associahedra corresponding to another class of graphs, the *hedgehog graphs* defined below.

Recall that Path_n is the path with n nodes, and $T_{a_1,\dots,a_k}$ is the graph obtained by attaching the paths $\mathrm{Path}_{a_1}, \dots, \mathrm{Path}_{a_k}$ to a central node. Let us also define the *hedgehog graph* $H_{a_1,\dots,a_k}$ as the graph obtained from the disjoint union of the chains $\mathrm{Path}_{a_1}, \dots, \mathrm{Path}_{a_k}$ by adding edges of the complete graph between the first vertices of all chains. For example, $H_{0,\dots,0}$ is the empty graph, $H_{1,\dots,1} = K_k$, and $H_{2,\dots,2}$ is a graph with $2k$ vertices obtained from the complete graph K_k by adding a "leaf" edge hanging from each of the k original nodes. By convention, for the empty graph, one has $\tilde{f}_{H_{0,\dots,0}}(t) = 0$.

Theorem 6.13 gives the following recurrence relation for f-polynomials of path graphs:

$$\frac{d}{dt}\tilde{f}_{\mathrm{Path}_n}(t) = \sum_{r=1}^{n-1}(n-r+1)\cdot \tilde{f}_{\mathrm{Path}_r}(t)\cdot \tilde{f}_{\mathrm{Path}_{n-r}}(t).$$

Indeed, there are $n-r+1$ connected r-element subsets I of nodes of Path_n, the deletion $\mathrm{Path}_n|_I$ is isomorphic to Path_r, and the contraction Path_n/I is isomorphic to Path_{n-r}.

For graphs $T_{a_1,\dots,a_k}$, Theorem 6.13 gives the following recurrence relation

$$\frac{d}{dt}\tilde{f}_{T_{a_1,\dots,a_k}}(t) = \sum_{i=1}^{k}\sum_{r=1}^{a_i} \tilde{f}_{\mathrm{Path}_r}(t)\cdot \tilde{f}_{T_{a_1,\dots,a_i-r,\dots,a_k}}(t)\cdot(a_i-r+1),$$
$$+\sum \tilde{f}_{T_{b_1,\dots,b_k}}(t)\cdot \tilde{f}_{H_{a_1-b_1,\dots,a_k-b_k}}(t),$$

where the second sum is over $b_1,\dots,b_k$ such that $0\leq b_i\leq a_i$, for $i=1,\dots,k$. Indeed, a connected subset I of vertices of $G = T_{a_1,\dots,a_k}$ either belongs to one of the chains Path_{a_i}, or contains the central node. In the first case, the restriction is $G|_I = \mathrm{Path}_r$ and the contraction is $G/I = T_{a_1,\dots,a_i-r,\dots,a_k}$, where $r = |I|$. In the second case, the restriction $G|_I$ has the form $T_{b_1,\dots,b_k}$ and the contraction is $G/I = H_{a_1-b_1,\dots,a_k-b_k}$. Similarly, for hedgehog graphs $H_{a_1,\dots,a_k}$, one obtains the recurrence relation

$$\frac{d}{dt}\tilde{f}_{H_{a_1,\dots,a_k}}(t) = \sum_{i=1}^{k}\sum_{r=1}^{a_i} \tilde{f}_{\mathrm{Path}_r}(t)\cdot \tilde{f}_{H_{a_1,\dots,a_i-r,\dots,a_k}}(t)\cdot(a_i-r),$$
$$+\sum \tilde{f}_{H_{b_1,\dots,b_k}}(t)\cdot \tilde{f}_{H_{a_1-b_1,\dots,a_k-b_k}}(t),$$

where the second sum is over $b_1, \dots, b_k$ such that $0 \le b_i \le a_i$, for $i = 1, \dots, k$. In all cases one has the initial conditions $\tilde f_{\mathrm{Path}_n}(0) = \tilde f_{T_{a_1,\dots,a_k}}(0) = \tilde f_{H_{a_1,\dots,a_k}}(0) = 1$, except $\tilde f_{\mathrm{Path}_0}(t) = \tilde f_{H_{0,\dots,0}}(t) = 0$.

The above recurrence relations can be written in a more compact form using these generating functions:

$$\begin{aligned}
F_A(t,x) &:= \sum_{n\ge 1} \tilde f_{\mathrm{Path}_n}(t)\, x^{n+1} = x^2 + (1+2t)\,x^3 + (1+5t+5t^2)\,x^4 + \cdots,\\
F_T(t,x_1,\dots,x_k) &:= \sum_{a_1,\dots,a_k\ge 0} \tilde f_{T_{a_1,\dots,a_k}}(t)\, x_1^{a_1+1}\cdots x_k^{a_k+1},\\
F_H(t,x_1,\dots,x_k) &:= \sum_{a_1,\dots,a_k\ge 0} \tilde f_{H_{a_1,\dots,a_k}}(t)\, x_1^{a_1}\cdots x_k^{a_k}.
\end{aligned}$$

Note that F_A and F_T are related to generating functions from Section 12:

$$\begin{aligned}
F_A(t,x) &= t^{-1}x\, C(t^{-1}+1, tx)\\
F_T(t,x_1,\dots,x_k) &= t^{-k}\, T(t^{-1}+1, tx_1,\dots,tx_k).
\end{aligned}$$

The above recurrence relations can be expressed as the following partial differential equations with initial conditions at $t = 0$:

$$\frac{\partial F_A}{\partial t} = F_A \cdot \frac{\partial F_A}{\partial x}, \quad F_A|_{t=0} = \frac{x^2}{1-x}, \tag{12.5}$$

$$\frac{\partial F_T}{\partial t} = \sum_{i=1}^k F_A(t,x_i)\,\frac{\partial F_T}{\partial x_i} + F_T\cdot F_H, \quad F_T|_{t=0} = \frac{x_1\cdots x_k}{\prod_{i=1}^k (1-x_i)}, \tag{12.6}$$

$$\frac{\partial F_H}{\partial t} = \sum_{i=1}^k F_A(t,x_i)\,\frac{\partial F_H}{\partial x_i} + (F_H)^2, \quad F_H|_{t=0} = \frac{1-\prod_{i=1}^k (1-x_i)}{\prod_{i=1}^k (1-x_i)}. \tag{12.7}$$

One can actually solve these partial differential equations for arbitrary initial conditions, as follows.

Proposition 12.6. *The solutions $F(t,x)$, $G(t,x_1,\dots,x_k)$, $H(t,x_1,\dots,x_k)$, and $R(t,x_1,\dots,x_k)$ to the following system of partial differential equations with initial conditions*

$$\frac{\partial F}{\partial t} = F\cdot\frac{\partial F}{\partial x}, \quad F|_{t=0} = f_0(x), \tag{12.8}$$

$$\frac{\partial G}{\partial t} = \sum_{i=1}^k F(t,x_i)\,\frac{\partial G}{\partial x_i}, \quad G|_{t=0} = g_0(x_1,\dots,x_k), \tag{12.9}$$

$$\frac{\partial H}{\partial t} = \sum_{i=1}^k F(t,x_i)\,\frac{\partial H}{\partial x_i} + H^2, \quad H|_{t=0} = h_0(x_1,\dots,x_k), \tag{12.10}$$

$$\frac{\partial R}{\partial t} = \sum_{i=1}^k F(t,x_i)\,\frac{\partial R}{\partial x_i} + R\cdot H, \quad R|_{t=0} = r_0(x_1,\dots,x_k) \tag{12.11}$$

are given by

$$
\begin{aligned}
&f_0(x+t\cdot F)=F \text{ (implicit form)}\\
&G=g_0(\xi_1,\dots,\xi_k)\\
&H=-(t+(h_0(\xi_1,\dots,\xi_k))^{-1})^{-1}\\
&R=-r_0(\xi_1,\dots,\xi_k)\cdot(1+t\cdot h_0(\xi_1,\cdots,\xi_k))^{-1}
\end{aligned}
$$

where $\xi_i = x_i + t\cdot F(t,x_i)$, *for* $i=1,\dots,k$.

Proof. Let us first solve (12.8). For a constant C, consider the function $x(t)$ given implicitly as $F(t,x)=C$, i.e., the graph of $x(t)$ is a level curve for $F(t,x)$. The tangent vector to the graph of $x(t)$ at some point (t_0,x_0) such that $F(t_0,x_0)=C$ is $(1,\frac{dx(t_0)}{dt})$. The derivative of the function $F(t,x)$ at the point (t_0,x_0) in the direction of this vector should be 0, i.e., $1\cdot\frac{\partial F(t_0,x_0)}{\partial t}+\frac{dx(t_0)}{dt}\cdot\frac{\partial F(t_0,x_0)}{\partial x}=0$. This equation, together with the differential equation (12.8) for F, implies that $\frac{d}{dt}x(t)=-C$. Solving this trivial differential equation for $x(t)$ one deduces that $x(t)=-C\cdot t+B(C)$, where B is a function that depends only on the constant C. Since C can be an arbitrary constant, one deduces that

$$x=-F(t,x)\cdot t+B(F(t,x)),\ \text{ or, equivalently, } B^{\langle -1\rangle}(x+t\cdot F(t,x))=F(t,x).$$

Plugging the initial condition $F|_{t=0}=f_0(x)$ in the last expression, one gets

$$B^{\langle -1\rangle}(x)=f_0(x).$$

Thus the solution $F(t,x)$ is given by $f_0(x+t\cdot F)=F$, as needed.

Direct verification shows that the function $G=R(F(t,x_1),\dots,F(t,x_k))$ satisfies the differential equation (12.9), for an arbitrary $R(y_1,\dots,y_k)$. The initial condition for $t=0$ gives $R(f_0(x_1),\dots,f_0(x_k))=g_0(x_1,\dots,x_k)$. Thus $R(y_1,\dots,y_k)=g_0(B(y_1),\dots,B(y_k))$, where $B=f_0^{\langle -1\rangle}$, as above. Since $B(F(t,x))=x+t\cdot F(t,x)$, one deduces that $G=g_0(B(F(t,x_1)),\dots,B(F(t,x_k)))=g_0(\xi_1,\dots,\xi_k)$, as needed.

Making the substitution $H=-(t+G(t,x_1,\dots,x_k))^{-1}$ in differential equation (12.10) for H, one obtains equation (12.10) for G with $g_0=-(h_0)^{-1}$. By the previous calculation, one has $G=-(h_0(\xi_1,\dots,\xi_k))^{-1}$. Thus the solution for (12.10) is $H=-(t+(h_0(\xi_1,\dots,\xi_k))^{-1})^{-1}$.

Making the substitution $R=H\cdot G$ in equation (12.10) for R, where H is the solution to (12.10), one obtains equation (12.9) for G with $g_0=r_0/h_0$. By the above calculation, one has $G=r_0(\xi_1,\dots,x_k)/h_0(\xi_1,\dots,\xi_k)$. Thus,

$$R=-\frac{1}{t+(h_0(\xi_1,\dots,\xi_k))^{-1}}\cdot\frac{r_0(\xi_1,\dots,x_k)}{h_0(\xi_1,\dots,\xi_k)}=-\frac{r_0(\xi_1,\dots,x_k)}{1+t\cdot h_0(\xi_1,\cdots,\xi_k)},$$

as needed. □

Applying Proposition 12.6 to differential equation (12.5) for $F_A(t,x)$, one obtains the implicit solution:

$$\frac{(x+t\cdot F_A)^2}{1-x-t\cdot F_A}=F_A.$$

This is equivalent to the quadratic equation (12.4) for $C(t,x)$. Explicitly, one gets

$$F_A(t,x)=\frac{(1-x-2tx)-\sqrt{(1-x-2tx)^2-4t(t+1)x^2}}{2t(t+1)}. \tag{12.12}$$

Applying Proposition 12.6 to differential equations (12.6) and (12.7) for the generating functions F_T and F_H, one obtains the following result.

Theorem 12.7. *The generating functions $F_T(t, x_1, \ldots, x_k)$ and $F_H(t, x_1, \ldots, x_k)$ are given by the following expressions*

$$F_T(t, x_1, \ldots, x_k) = \frac{\xi_1 \cdots \xi_n}{(t+1)(1-\xi_1)\cdots(1-\xi_n) - t},$$
$$F_H(t, x_1, \ldots, x_k) = \frac{1-(1-\xi_1)\cdots(1-\xi_k)}{(t+1)(1-\xi_1)\cdots(1-\xi_n) - t},$$

where $\xi_i = x_i + t \cdot F_A(t, x_i)$ and F_A is given by (12.12).

Note that the above expression for F_T is equivalent to Theorem 12.1, using (2.1).

13. Graph-associahedra for path-like graphs

The goal of this section is to use Theorem 6.12 to compute the h-polynomials of the graph-associahedra of a fairly general infinite family of graphs, including all Dynkin diagrams of finite and affine Coxeter groups.

Let A and B be two connected graphs with a marked vertex in each, and let n_0 be the total number of unmarked vertices in A and B. For $n > n_0$, let $G_n = G_n(A, B)$ be the graph obtained by connecting the marked vertices in A and B by the path Path_{n-n_0} so that the total number of vertices in G_n is n. Call graphs of the form G_n *path-like graphs* because, for large n, they look like paths with some "small" graphs attached to their ends.

The following claim shows that the h-polynomials of the graph-associahedra $P_{\mathcal{B}(G_n)}$ can be expressed as linear combinations (with polynomial coefficients) of the h-polynomials $C_n(t)$ of usual associahedra; see (12.1).

Theorem 13.1. *There exist unique polynomials $g_0(t), g_1(t), \ldots, g_{n_0}(t) \in \mathbb{Z}[t]$ of degrees $\deg g_i(t) \leq i$ such that, for any $n > n_0$, one has*

$$h_{G_n}(t) = g_0(t)\, C_n(t) + g_1(t)\, C_{n-1}(t) + \cdots + g_{n_0}(t)\, C_{n-n_0}(t).$$

The polynomial $g_i(t)$ is a palindromic polynomial, that is $g_i(t) = t^i\, g_i(t^{-1})$, for $i = 0, \ldots, n_0$.

Similarly, one can express the f-polynomials of $P_{\mathcal{B}(G_n)}$ as a linear combination of the f-polynomials of usual associahedra, because $f_G(t) = h_G(t+1)$.

One can rewrite this theorem in terms the generating function $C(t, x)$ for the Narayana numbers; see (12.3).

Corollary 13.2. *There exists a unique polynomial $g(t, x) \in \mathbb{Z}[t, x]$ such that, for any $n > n_0$, one has*

$$h_{G_n}(t) = \textit{the coefficient of } x^n \textit{ in } g(t,x)\, C(t,x).$$

The polynomial $g(t, x)$ has degree $\leq n_0$ with respect to the variable x. It satisfies the equation $g(t, x) = g(t^{-1}, tx)$.

Proof. This follows from Theorem 13.1, by setting $g(t, x) = g_0(t) + g_1(t)\, x + \cdots + g_{n_0}(t)\, x^{n_0}$. ∎

Proof of Theorem 13.1. Let us first prove the existence of the linear expansion (and later prove its uniqueness and the palindromic property of the coefficients $g_i(t)$). The recurrence from Theorem 6.12 will be used to prove this clam by induction on the total number of vertices in A and B. For this argument one should drop the assumption that A and B are connected. Suppose that A or B is disconnected, say, A is a disjoint union of graphs A_1 and A_2 where A_1 contains the marked vertex. Let $\tilde{G}_n := G_n(A_1, B)$ and let r be the number of vertices in A_2. Then $h_{G_n}(t) = h_{\tilde{G}_{n-r}}(t)\, h_{A_2}(t)$, where $\deg h_{A_2}(t) \leq r-1$. By induction, $h_{\tilde{G}_{n-r}}(t)$ can be expressed as a linear combination of $C_{n-r}(t), C_{n-r-1}(t), \ldots, C_{n-n_0}(t)$, which produces the needed expression for $h_{G_n}(t)$.

Now assume that both A and B are connected graphs. Theorem 6.12(3) gives the expression for the h-polynomial as the sum $h_{G_n}(t) = \sum_L (t-1)^{|L|-1} h_{G_n \setminus L}(t)$ over nonempty subsets L of vertices of G_n, where $G_n \setminus L$ denotes the graph G_n with removed vertices in L. (Here one has shifted t by -1 to transform f-polynomials into h-polynomials.) Let us write L as a disjoint union $L = I \cup J \cup K$, where I is a subset of unmarked vertices of A, J is a subset of unmarked vertices of B, and K is a subset of vertices in the path connecting the marked vertices. The contribution of the terms with $K = \emptyset$ to the above sum is $\sum_{I,J} (t-1)^{|I|+|J|-1} h_{G_n \setminus (I \cup J)}(t)$. Note that $G_n \setminus (I \cup J) = G_{n-r}(A \setminus I, B \setminus J)$, where $r = |I| + |J|$. By induction, one can express each term $h_{G_n \setminus (I \cup J)}(t)$ as a combination of $C_{n-r}(t), \ldots, C_{n-n_0}(t)$.

The remaining terms involve a nonempty subset K of vertices in the path Path_{n-n_0}. When one removes these $k = |K|$ vertices from the path, it splits into $k+1$ smaller paths $\mathrm{Path}_{m_1}, \ldots, \mathrm{Path}_{m_{k+1}}$ with $m_i \geq 0$; cf. paragraph before (12.2). Thus the remaining contribution to $h_{G_n}(t)$ can be written as

$$\sum_{I,J} \sum_{m_1,\ldots,m_{k+1} \geq 0} (t-1)^{|I|+|J|+k-1}\, h_{G_p(A \setminus I, \circ)}(t)\, C_{m_2}(t) \cdots C_{m_k}(t)\, h_{G_q(\circ, B \setminus J)}(t),$$

where $\circ$ is the graph with a single vertex,

$$\begin{aligned} p &= m_1 + |A \setminus I| - 1, \\ q &= m_{k+1} + |B \setminus J| - 1, \text{ and} \\ k + \sum m_i &= n - n_0. \end{aligned}$$

By induction, one can express $h_{G_p(A\setminus I,\circ)}(t)$ and $h_{G_q(\circ,B\setminus J)}(t)$ as linear combinations of the $C_{m'}(t)$. So the remaining contribution to $h_{G_n}(t)$ can be written as a sum of several expressions of the form

$$s(t) \sum_{k \geq 1} \sum_{m_1', m_2, \ldots, m_k, m_{k+1}'} (t-1)^{k-1} C_{m_1'}(t)\, C_{m_2}(t) \cdots C_{m_k}(t)\, C_{m_{k+1}'}(t),$$

where the sum is over $m_1', m_2, \ldots, m_k, m_{k+1}'$ such that $m_1' \geq a$, $m_2, \ldots, m_k \geq 0$, $m_{k+1}' \geq b$, $m_1' + m_2 + \cdots + m_k + m_{k+1}' + k = n - c$. This expression depends on nonnegative integers a, b, c such that $a + b + c = n_0$ and a polynomial $s(t)$ of degree $\deg s(t) \leq c$. If one extends the summation to all $m_1', m_{k+1}' \geq 0$, one obtains the expansion (12.2) for $C_{n-c}(t)$ times $s(t)$. Applying the inclusion-exclusion principle and equation (12.2), one deduces that the previous sum equals

$$s(t)\left(C_{n-c}(t) - \sum_{m_1'=0}^{a-1} t\, C_{m_1'}(t)\, C_{n-c-m_1'-1}(t) - \cdots\right),$$

which is a combination of $C_n(t), \ldots, C_{n-n_0}(t)$ as needed.

It remains to show the uniqueness of the linear expansion and show that the $g_i(t)$ are palindromic polynomials. (Here one assumes that the graphs A and B are connected.) According to Corollary 13.2, the polynomial $H(t,x) := \sum_{n>n_0} h_{G_n}(t)\, x^n$ can be written as $H(t,x) = g(t,x)\, C(t,x) + r(t,x)$, where $g(t,x), r(t,x) \in \mathbb{Z}[t,x]$. If $H(t,x) = \tilde{g}(t,x)\, C(t,x) + \tilde{r}(t,x)$ with $\tilde{g}(t,x) \neq g(t,x)$, then this would imply that $C(t,x)$ is a rational function, which contradicts the formula (12.3) involving a square root. This proves the uniqueness claim. One has $H(t,x) = H(t^{-1},tx)/t$ and $C(t,x) = C(t^{-1},tx)/t$ because h-polynomials are palindromic. Thus

$$H(t,x) = g(t,x)\, C(t,x) + r(t,x) = g(t^{-1},tx)\, C(t,x) + \frac{1}{t} r(t^{-1},tx).$$

This implies that $g(t,x) = g(t^{-1},tx)$. Otherwise, $C(t,x)$ would be a rational function. The equation $g(t,x) = g(t^{-1},tx)$ says that the coefficients $g_i(t)$ of $g(t,x) = \sum_i g_i(t)\, x^i$ are palindromic. $\square$

Let us illustrate Theorem 13.1 by several examples. For a series of path-like graphs G_n, let $g\{G_n\}$ denote the polynomial $g(t,x)$ that appears in the generating functions $\sum_{n\geq n_0} h_{G_n}(t)\, x^n = g(t,x)\, C(t,x) + r(t,x)$. For instance, the expression $g\{D_n\} = 2-(t+1)x - tx^2$ (see the example below) is equivalent to the expression $h_{D_n}(t) = 2C_n(t) - (t+1)\, C_{n-1}(t) - t\, C_{n-2}(t)$, for $n>2$.

Examples 13.3. Define *daisy graphs* as $\mathrm{Daisy}_{n,k} := T_{n-k-1,1^k}$; see Section 12. (Here 1^k means a sequence of k ones.) They include *type D Dynkin diagrams* $D_n := \mathrm{Daisy}_{n,2}$. For fixed k, the $\mathrm{Daisy}_{n,k}$ form the series of path-like graphs for $A = K_{1,k}$ (the k-star with marked central vertex) and $B = \circ$ (the graph with a single vertex). Also define *kite graphs* as $\mathrm{Kite}_{n,k} := H_{n-k+1,1^{k-1}}$; see Section 12.4. They are path-like graphs for $A = K_k$ and $B = \circ$. The *affine Dynkin diagram of type $\widetilde{D}_{n-1}$* is the nth path-like graph in the case when both A and B are 3-paths with marked central vertices.

Here are the polynomials $g\{G_n\}$ for several series of such graphs:

$$\begin{aligned}
g\{D_n\} &= 2-(t+1)\,x - t\,x^2,\\
g\{\widetilde{D}_{n-1}\} &= 4-4(t+1)\,x + (t-1)^2\,x^2 + 2\,t(t+1)\,x^3 + t^2\,x^4,\\
g\{\mathrm{Kite}_{n,3}\} &= 2-(t+1)\,x,\\
g\{\mathrm{Daisy}_{n,3}\} &= 6-6(t+1)\,x + (1-5\,t+t^2)\,x^2 - t(t+1)\,x^3,\\
g\{\mathrm{Daisy}_{n,4}\} &= 24-36(t+1)\,x + (14-16\,t+14\,t^2)\,x^2 + \\
&\quad + (-1+3\,t+3\,t^2-t^3)\,x^3 - (t+t^2+t^3)\,x^4.
\end{aligned}$$

The formulas for D_n, $\mathrm{Kite}_{n,k}$, and $\mathrm{Daisy}_{n,k}$ were derived from Theorems 12.1 and 12.7 (using the MAPLE package). The formula for the affine Dynkin diagram $\widetilde{D}_{n-1}$ was obtained using the inductive procedure given in the proof of Theorem 13.1.

Remark 13.4. The induction from the proof of Theorem 13.1 is quite involved. It is very difficult to calculate by hand other examples for bigger graphs A and B. It would be interesting to find a simpler procedure for finding the polynomials $g\{G_n\}$. Also it would be interesting to find explicit formulas for the polynomials $g\{G_n\}$ for all daisy graphs, kite graphs, and other "natural" series of path-like graphs.

14. Bounds and monotonicity for face numbers of graph-associahedra

Section 7.2 showed that the f- and h-vectors of flag nestohedra coming from connected building sets are bounded below by those of hypercubes and bounded above by those of permutohedra. It is natural to ask for the bounds within the subclass of graph-associahedra corresponding to connected graphs, and to ask for bounds on their γ-vectors.

Permutohedra are graph-associahedra corresponding to complete graphs, and so they still provide the upper bound for the f- and h-vectors. For lower bounds on f- and h-vectors, the monotonicity discussed in Remark 6.9 implies that the f- and h-vector of any graph-associahedron $P_{\mathcal{B}(G)}$ for a connected graph G is bounded below by the graph-associahedron for any spanning tree inside G. Thus it is of interest to look at lower (and upper) bounds for f-, h- and γ-vectors of graph-associahedra for trees.

A glance at Figure 14.1 suggests that, roughly speaking, trees which are more branched and forked (that is, farther from being a path) tend to have higher entries componentwise in their γ-vectors, and hence also in their f- and h-vectors. In fact, in that figure, which shows all trees on 7 vertices grouped by their degree sequences, one sees several (perhaps misleading) features:

(i) The degree sequences are ordered linearly under the *majorization* (or *dominance*) partial ordering on partitions of $2(n-1)$ $(= 2 \cdot (7-1) = 12$ here).
(ii) The γ-vectors of these trees are linearly ordered under the componentwise partial order.
(iii) Trees whose degree sequence are lower in the majorization order have componentwise smaller γ-vectors.
(iv) The trees are distinguished up to isomorphism by their γ-vectors.

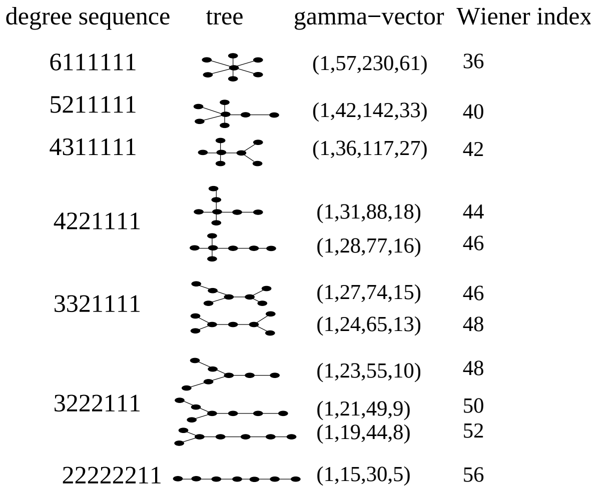

Figure 14.1. The γ-vectors $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ for graph-associahedra of all trees on 7 vertices, grouped according to their degree sequences.

Additionally, it seems that the *Wiener index* [Wie'47] for graphs has some correlation with the γ-vector. The *Wiener index* $W(G)$ of a graph G is defined as the sum of distances $d(i,j)$ over unordered pairs i,j of vertices in the graph G, where $d(i,j)$ is the number of edges in the shortest path from i to j. The Wiener index $W(T)$ of a tree is equal to the number of forbidden 312 patterns (see the remarks following Definition 9.2) provided by the tree T (plus the constant $\binom{n}{2}$). Thus, for two trees on n vertices, if one has $W(T) < W(T')$, then roughly speaking one might expect that the generalized permutohedron $P_{\mathcal{B}(T)}$ has a larger gamma vector than $P_{\mathcal{B}(T')}$.

This is exactly the case for trees on 7 vertices, as shown in Figure 14.1. It shows that as the γ-vectors decrease, the Wiener indices (weakly) increase. Note that in this case, the Wiener index together with the degree sequence completely distinguish all equivalence classes of trees.

For trees T on n vertices, the maximum and minimum values of the Wiener index are, respectively, $\sum_{i=1}^{n-1} i(n-i) = n(n^2-1)/6$ for Path_n, and $(n-1)^2$ for $K_{1,n-1}$.

None of the properties (i)-(iv) persist for all trees. For example, when looking at trees on $n = 8$ nodes, one finds that

(i) the degree sequences are only partially ordered by the majorization order on partitions of $2(n-1) = 14$:

$$\begin{aligned}22222211 < 32222111 \\ &< 33221111 < 33311111, 42221111 \\ &< 43211111 < 44111111, 52211111 \\ &< 53111111 < 62111111 < 71111111\end{aligned}$$

(ii) there are trees, such as the two shown in Figure 14.2(a) and (b), whose γ-vectors are incomparable componentwise,

(iii) there are trees, such as the two shown in Figure 14.2(d) and (c), where the degree sequence of one strictly majorizes that of the other, but its γ-vector is strictly smaller, and

(iv) there are nonisomorphic trees, such as the two shown in Figure 14.2(d) and (e), having the same γ-vector.

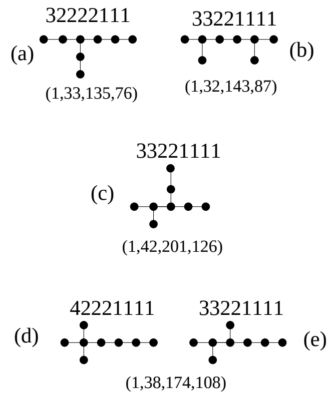

FIGURE 14.2. Some trees on 8 nodes and the γ-vectors of their graph-associahedra.

Nevertheless, we do make some monotonicity conjectures about the face numbers for graph-associahedra.

Conjecture 14.1. *There exists a partial order $\prec$ on the set of (unlabelled, isomorphism classes of) trees with n nodes, having these properties:*

- Path_n *is the unique $\prec$-minimum element,*
- $K_{1,n-1}$ *is the unique $\prec$-maximum element, and*
- $T \prec T'$ *implies $\gamma_{P_{\mathcal{B}(T)}} \leq \gamma_{P_{\mathcal{B}(T')}}$ componentwise.*

We suspect that such a partial order $\prec$ can be defined so that T, T' will, in particular, be comparable whenever T, T' are related by one of the *flossing moves* considered in [BR'04, §4.2].

Assuming Conjecture 14.1, the γ-vectors (and hence also the f-, h-vectors) of graph-associahedra for trees on n nodes would have the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ and the stellohedron $P_{\mathcal{B}(K_{1,n-1})}$ giving their lower and upper bounds. This would also imply that the f-, h-vectors of graph-associahedra for connected graphs on n nodes would have associahedra and permutohedra giving their lower and upper bounds. To make a similar assertion for γ-vectors it would be nice to have the following analogue of Stanley's monotonicity result (Theorem 4.6).

Conjecture 14.2. *When Δ, Δ' are two flag simplicial complexes and Δ' is a geometric subdivision of Δ, the γ-vector of Δ' is componentwise weakly larger than that of Δ.*

In particular, when $\mathcal{B}, \mathcal{B}'$ are building sets giving rise to flag nestohedra and $\mathcal{B} \subset \mathcal{B}'$, (such as graphical buildings $\mathcal{B}(G) \subset \mathcal{B}(G')$ for an edge-subgraph $G \subset G'$) then the γ-vector of $P_{\mathcal{B}'}$ is componentwise weakly larger than that of $P_{\mathcal{B}}$.

We close with a question suggested by the sets of permutations $\mathfrak{S}_n(\mathcal{B})$ and $\widehat{\mathfrak{S}}_n(\mathcal{B})$ for a chordal building set $\mathcal{B}$ which appeared in Corollary 9.6 and Theorem 11.6.

Question 14.3. Given a (non-chordal) building set $\mathcal{B}$, is there a way to define two sets of permutations $\mathfrak{S}'_n(\mathcal{B})$ and $\widehat{\mathfrak{S}}'_n(\mathcal{B})$ such that:

- the h-polynomial for the nestohedron $P_{\mathcal{B}}$ is given by the descent generating function for $\mathfrak{S}'_n(\mathcal{B})$, and
- the γ-polynomial is given by the peak generating function for $\widehat{\mathfrak{S}}'_n(\mathcal{B})$?

15. Appendix: Deformations of a simple polytope

The goal of this section is to clarify the equivalence between various definitions of the deformations of a simple polytope, either by

- deforming vertex positions, keeping edges in the same parallelism class, or
- deforming edge lengths, keeping them nonnegative, or
- altering level sets of facet inequalities, or but not allowing facets to move past any vertices.

There will be defined below three cones of such deformations which are all linearly isomorphic. This discussion is essentially implicit in [Post'05, Definition 6.1 and §19], but we hope the explication here clarifies this relationship.

Let P be a simple d-dimensional polytope in $\mathbb{R}^d$. Let V be its set of vertices. Let $E \subseteq \binom{V}{2}$ be its set of edge pairs. Let F be an indexing set for its facets, so that P is defined by facet inequalities $h_f(x) \leq \alpha_f$ for $f \in F$, in which each h_f is a linear functional in $(\mathbb{R}^d)^*$, and $(\alpha_f)_{f \in F} \in \mathbb{R}^F$.

Definition 15.1. (1) The *vertex deformation cone* D_P^V of P is the set of points $(x_v)_{v\in V} \in (\mathbb{R}^d)^V$ such that

$$x_u - x_v \in \mathbb{R}_{\geq 0}(u-v), \quad \text{for every edge } uv \in E. \tag{15.1}$$

(2) The *edge length deformation cone* D_P^E of P is the set of points $(y_e)_{e\in E} \in \mathbb{R}^E$ such that all $y_e \geq 0$, and, for any 2-dimensional face of P with edges $e_1 = v_1v_2$, $e_2 = v_2v_3$, $\ldots$, $e_{k-1} = v_{k-1}v_k$, $e_k = v_kv_1$, one has

$$y_{e_1}(v_1 - v_2) + y_{e_2}(v_2 - v_3) + \cdots + y_{e_k}(v_k - v_1) = 0.$$

(3) For $\beta = (\beta_f)_{f\in F} \in \mathbb{R}^F$, let $P_\beta := \{x \in \mathbb{R}^d \mid h_f(x) \leq \beta_f, \text{ for } f \in F\}$ be the polytope obtained from P by parallel translations of the facets. In particular, $P_\alpha = P$. The *open facet deformation cone*[7] $D_P^{F,open}$ for P is the set of $\beta \in \mathbb{R}^F$ for which the polytopes P_β and P have the same normal fan $\mathcal{N}(P_\beta) = \mathcal{N}(P)$. (Equivalently, P_β and P have the same combinatorial structure.) The *(closed) facet deformation cone* is the closure D_P^F of $D_P^{F,open}$ inside $\mathbb{R}^F$.

It is clear that the definitions of D_P^V and D_P^E translate into linear equations and weak linear inequalities. Thus D_P^V and D_P^E are (closed) polyhedral cones in the spaces $(\mathbb{R}^d)^V$ and $\mathbb{R}^E$, correspondingly. The following lemma shows that D_P^F is also a polyhedral cone.

Lemma 15.2. *For a simple polytope P, the facet deformation cone $D_P^{F,open}$ is a full $|F|$-dimensional open polyhedral cone inside $\mathbb{R}^F$, that is a nonempty subset in $\mathbb{R}^F$ given by strict linear inequalities. Thus D_P^F is the closed polyhedral cone in $\mathbb{R}^F$ given by replacing the strict inequalities with the corresponding weak inequalities.*

Proof. Since every polytope P_β has facet normals in directions which are a subset of those for P, the rays (=1-dimensional normal cones) in $\mathcal{N}(P_\beta)$ are a subset of those in $\mathcal{N}(P)$. Therefore, one will have $\mathcal{N}(P_\beta) = \mathcal{N}(P)$ if and only if P_β, P have the same face lattices, or equivalently, the same collection of vertex-facet incidences (v, f). This means that one can define the set $D_P^{F,open}$ inside $\mathbb{R}^F$ by a collection of strict linear inequalities on the coordinates $\beta = (\beta_f)_{f\in F}$. It is next explained how to produce one such inequality for each pair (v_0, f_0) of a vertex v_0 and facet f_0 of P such that $v_0 \notin f_0$.

If v_0 lies on the d facets $f_1, \ldots, f_d$ in P, then v_0 is the unique solution to the linear system $h_{f_j}(x) = \alpha_{f_j}$ for $j = 1, ..., d$. Its corresponding vertex x_0 in P_β is then the unique solution to $h_{f_j}(x) = \beta_{f_j}$ for $j = 1, ..., d$. Note that this implies x_0 has coordinates given by linear expressions in the coefficients β. Then the inequality corresponding to the vertex-facet pair (v_0, f_0) asserts that $h_{f_0}(x_0) < \beta_{f_0}$.

Lastly, note that this system of strict linear inequalities has at least one solution, namely the α for which $P_\alpha = P$. Hence this open polyhedral cone is nonempty. $\square$

The following theorem gives several different ways to describe deformations of a simple polytope.

Theorem 15.3. *Let P be a d-dimensional simple polytope in $\mathbb{R}^d$, with notation as above. Then the following are equivalent for a polytope P' in $\mathbb{R}^d$:*

(i) *The normal fan $\mathcal{N}(P)$ refines the normal fan $\mathcal{N}(P')$.*

[7] This is linearly isomorphic to the *type-cone* of P described by McMullen in [McM'73, §2, p. 88].

(ii) *The vertices of P' can be (possibly redundantly) labelled x_v, $v \in V$, so that $(x_v)_{v\in V}$ is a point in the vertex deformation cone D_P^V, i.e., the x_v satisfy conditions* (15.1).

(iii) *The polytope P' is the convex hull of points x_v, $v \in V$, such that $(x_v)_{v\in V}$ is in the vertex deformation cone D_P^V.*

(iv) *$P' = P_\beta$ for some β in the closed facet deformation cone D_P^F.*

(v) *P' is a Minkowski summand of a dilated polytope rP, that is there exist a polytope $Q \subset \mathbb{R}^d$ and a real number $r > 0$ such that $P' + Q = rP$.*

Proof. One proceeds by proving the following implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

(i) implies (ii). The refinement of normal fans gives rise to the redundant labelling of vertices $(x_v)_{v\in P}$ as follows: given a vertex x of P', label it by x_v for every vertex v in P having its normal cone $\mathcal{N}_v(P)$ contained in the normal cone $\mathcal{N}_x(P')$. There are then two possibilities for an edge $uv \in E$ of P: either $x_u = x_v$, in which case (15.1) is trivially satisfied, or else $x_u \neq x_v$ so that $\mathcal{N}_u(P), \mathcal{N}_v(P)$ lie in different normal cones $\mathcal{N}_{x_u}(P') \neq \mathcal{N}_{x_v}(P')$. But then since $\mathcal{N}(P)$ refines $\mathcal{N}(P')$, these latter two cones must share a codimension one subcone lying in the same hyperplane that separates $\mathcal{N}_u(P)$ and $\mathcal{N}_v(P)$. As this hyperplane has normal vector $u - v$, this forces $x_u - x_v$ to be a positive multiple of this vector, as desired.

(ii) implies (iii). Trivial.

(iii) implies (i). Let P' be the convex hull of the points x_v. Fix a vertex $u \in V$. Let $\lambda \in (\mathbb{R}^d)^*$ be a generic linear functional that belongs to the normal cone $\mathcal{N}_u(P)$ of P at the vertex u. Then the maximum of λ on P is achieved at the point u and nowhere else. Let us orient the 1-skeleton of P so that λ increases on each directed edge. This connected graph has a unique vertex of outdegree 0, namely the vertex u. Thus, for any other vertex $v \in V$, there is a directed path $(v_1, \dots, v_l)$ from $v_1 = v$ to $v_l = u$ in this directed graph. According to the conditions of the lemma, one has have $\lambda(x_{v_1}) \leq \lambda(x_{v_2}) \leq \cdots \leq \lambda(x_{v_l})$, so $\lambda(x_v) \leq \lambda(x_u)$. Thus the maximum of λ on the polytope P' is achieved at the point x_u. This implies that x_u is a vertex of P' and the normal cone $\mathcal{N}_{x_u}(P')$ of P' at this point contains the normal cone $\mathcal{N}_u(P)$ of P at u. Since the same statement is true for any vertex of P, one deduces that $\mathcal{N}(P)$ refines $\mathcal{N}(P')$.

(i) implies (iv). First, note that if $\mathcal{N}(P') = \mathcal{N}(P)$ then $P' = P_\beta$ for some β in the open facet deformation cone $D_P^{F,open}$. Indeed, the facets of P' are orthogonal to the 1-dimensional cones in $\mathcal{N}(P')$, thus they should be parallel to the corresponding facets of P.

Now assume that $\mathcal{N}(P)$ refines $\mathcal{N}(P')$. Recall the standard fact [Zieg'94, Prop. 7.12] that the normal fan $\mathcal{N}(Q_1 + Q_2)$ of a Minkowski sum $Q_1 + Q_2$ is the common refinement of the normal fans $\mathcal{N}(Q_1)$ and $\mathcal{N}(Q_2)$. Thus, for any $\epsilon > 0$, the normal fan of the Minkowski sum $P' + \epsilon P$ coincides with $\mathcal{N}(P)$. By the previous observation, one should have $P' + \epsilon P = P_{\beta(\epsilon)}$ for some $\beta(\epsilon) \in D_P^{F,open}$. Since all coordinates of $\beta(\epsilon)$ linearly depend on ϵ, one obtain $P' = P_\beta$ for $\beta = \lim_{\epsilon\to 0} \beta(\epsilon) \in D_P^F$.

(iv) implies (iii). Given $\beta \in D_P^F$, it is the limit point for some family $\{\beta(\epsilon)\} \subset D_P^{F,open}$. One may assume that $\beta(\epsilon)$ linearly depends on $\epsilon > 0$ and $\lim_{\epsilon\to 0} \beta(\epsilon) = \beta$. Hence $P' = P_\beta$ is the limit of the polytopes $P_{\beta(\epsilon)}$, which each have $\mathcal{N}(P_{\beta(\epsilon)}) = \mathcal{N}(P)$. In particular, the vertices of $P_{\beta(\epsilon)}$ can be labelled by $x_v(\epsilon)$, $v \in V$. These vertices linearly depend on ϵ and satisfy $x_u(\epsilon) - x_v(\epsilon) = \mathbb{R}_{\geq 0}(u - v)$ for any edge

$uv \in E$. Taking the limit $\epsilon \to 0$, one obtains that P' is the convex hull of points $x_v = \lim_{\epsilon\to 0} x_v(\epsilon)$ satisfying (15.1).

(iv) implies (v). Note that $P_\gamma + P_\delta = P_{\gamma+\delta}$, for $\gamma, \delta \in D_P^F$. Let $P' = P_\beta$ for $\beta \in D_P^F$. The point α (such that $P = P_\alpha$) belongs to the *open* cone $D_P^{F,open}$. Thus, for sufficiently large r, the point $\gamma = r\alpha - \beta$ also belongs to the open cone $D_P^{F,open}$. Let $Q = P_\gamma$. Then one has $P' + Q = P_\beta + P_{r\alpha-\beta} = P_{r\alpha} = rP$, as needed.

(v) implies (i). This follows from the standard fact [Zieg'94, Prop. 7.12] on normal fans of Minkowski sums mentioned above. □

Remark 15.4. We are being somewhat careful here, since Theorem 15.3 can fail when one allows a broader interpretation for a simple polytope P to deform into a polytope P' by parallel translations of its facets, e.g. if one allows facets to translate *past* vertices. For example, letting P' be a regular tetrahedron in $\mathbb{R}^3$, and P the result of "shaving off an edge" from P' with a generically tilted plane in $\mathbb{R}^3$, one finds that $\mathcal{N}(P)$ does not refine $\mathcal{N}(P')$.

Let us now describe the relationship between the three deformation cones D_P^V, D_P^E, and D_P^F. Let H be the linear subspace in $(\mathbb{R}^d)^V$ given by

$$H := \{(x_v)_{v\in V} \in (\mathbb{R}^d)^V \mid x_u - x_v \in \mathbb{R}(u-v) \text{ for any edge } uv \in E\}.$$

Clearly, the cone D_P^V belongs to the subspace H. Let us define two linear maps

$$\phi : H \to \mathbb{R}^E \quad \text{and} \quad \psi : \mathbb{R}^F \to H.$$

The map ϕ sends $(x_v)_{v\in V} \in H$ to $(y_e)_{e\in E} \in \mathbb{R}^E$, where $x_u - x_v = y_e(u-v)$, for any edge $e = uv \in E$. The map ψ sends $\beta = (\beta_f)_{f\in F}$ to $(x_v)_{v\in V}$, where, for each vertex v of P given as the intersection of the facets of P indexed $f_1, \dots, f_d$, the point $x_v \in \mathbb{R}^d$ is the unique solution of the linear system $\{h_{f_j}(x) = \beta_{f_j} \mid j = 1, \dots, d\}$. For $\beta \in D_P^{F,open}$, $\psi(\beta) = (x_v)_{v\in V}$, where the x_v are the vertices of the polytope P_β. One can easily check that $\psi(\beta) \in H$. Indeed, this is clear for $\beta \in D_P^{F,open}$ and thus this extends to all $\beta \in \mathbb{R}^F$ by linearity.

Note that the kernel of the map ϕ is the subspace $\Delta(\mathbb{R}^d) \simeq \mathbb{R}^d$ embedded diagonally into $(\mathbb{R}^d)^V$. This follows from the fact that the 1-skeleton of P is connected. The vertex deformation cone D_P^V can be reduced modulo the subspace $\Delta(\mathbb{R}^d)$ of parallel translations of polytopes. Similarly, the facet deformation cone can be reduced modulo the subspace $\Delta'(\mathbb{R}^d) = \psi^{-1}(\Delta(\mathbb{R}^d)) \simeq \mathbb{R}^d$, where $\Delta'(x) := (h_f(x))_{f\in F}$ for $x \in \mathbb{R}^d$.

Theorem 15.5. *The map ψ gives a linear isomorphism between the cones D_P^F and D_P^V. The map ϕ induces a linear isomorphism between the cones $D_P^V/\Delta(\mathbb{R}^d)$ and D_P^E. Thus one has*

$$D_P^E \simeq D_P^V/\Delta(\mathbb{R}^d) \simeq D_P^F/\Delta'(\mathbb{R}^d).$$

In particular, $\dim D_P^E = \dim D_P^V - d = \dim D_P^F - d = |F| - d$.

Proof. The claim about the map ψ follows immediately from Theorem 15.3.

Let us prove the claim about ϕ. Note that, for $(x_v)_{v\in V} \in D_P^V$, the point $(y_e)_{e\in E} = \phi((x_v)_{v\in V})$ satisfies the condition of Definition 15.1(2) because

$$\begin{aligned} &y_{e_1}(v_1 - v_2) + y_{e_2}(v_2 - v_3) + \cdots + y_{e_k}(v_k - v_1) \\ &= (x_{v_1} - x_{v_2}) + (x_{v_2} - x_{v_3}) + \cdots + (x_{v_k} - x_{v_1}) \\ &= 0. \end{aligned}$$

It remains to show that for any $(y_e)_{e\in E} \in D_P^E$ there exists a unique (modulo diagonal translations) element $(x_v)_{v\in V} \in H$ such that $x_u - x_v = y_e(u-v)$ for any edge $e = uv \in E$. Let us construct the points $x_v \in \mathbb{R}^d$, as follows. Pick a vertex $v_0 \in V$ and pick any point $x_{v_0} \in \mathbb{R}^d$. For any other $v \in V$, find a path $(v_0, v_1, \ldots, v_l)$ from v_0 to $v_l = v$ in the 1-skeleton of P and define $x_v = x_{v_0} - y_{v_0 v_1}(v_0 - v_1) - y_{v_1 v_2}(v_1 - v_2) - \cdots - y_{v_{l-1} v_l}(v_{l-1} - v_l)$. This point x_v does not depend on a choice of path from v_0 to v, because any other path in the 1-skeleton can be obtained by switches along 2-dimensional faces of P. These $(x_v)_{v\in V}$ satisfy the needed conditions.

Finally, note that $\dim D_P^F = |F|$ because D_P^F is a full-dimensional cone. □

References

[AK'06] F. Ardila, C. Klivans: The Bergman complex of a matroid and phylogenetic trees, *J. Comb. Th. B* **96** (2006), 38–49.

[AKW'06] F. Ardila, C. Klivans, L. Williams: The positive Bergman complex of an oriented matroid, *European Journal of Combinatorics* **27** (2006), no. 4, 577–591.

[AM'06] G. Agnarsson, W. Morris: On Minkowski Sums of Simplices, arXiv: `math.CO/ 0605564`.

[ARW'05] F. Ardila, V. Reiner, L. Williams: Bergman complexes, Coxeter arrangements, and graph associahedra, *Sém. Lothar. Combin.* **54a** (2006), B54Aj (electronic).

[BR'04] E. Babson, V. Reiner: Coxeter-like complexes, *Discrete Math. Theor. Comput. Sci.* **6** (2004), no. 2, 223–251 (electronic).

[Brä'04] P. Brändén: Sign-graded posets, unimodality of W-polynomials and the Charney-Davis conjecture, *Electron. J. Combin.* **11** (2004/06), Research Paper 9, 15 pp. (electronic).

[Brä'06] P. Brändén: Actions on permutations and unimodality of descent polynomials, arXiv: `math.CO/ 0610185`

[Cay'1890] A. Cayley: On the partitions of a polygon, *Proc. Lond. Math. Soc.* **22** (1890), 237–262.

[CD'06] M. Carr, S. Devadoss: Coxeter complexes and graph associahedra, *Topology and its Applications* (2006), 2155–2168.

[ChD'95] R. Charney, M. Davis: Euler characteristic of a nonpositively curved, piecewise Euclidean manifold, *Pac. J. Math.* **171** (1995), 117–137.

[DJS'98] M. Davis, T. Januszkiewicz, R. Scott: Nonpositive curvature of blow-ups, *Selecta Math. (N.S.)* **4** (1998), no. 4, 491-547.

[DJS'03] M. Davis, T. Januszkiewicz, R. Scott: Fundamental groups of minimal blow-ups, *Adv. Math.* **177** (2003), 115–179.

[DP'95] C. De Concini, C. Procesi: Wonderful models for subspace arrangements, *Selecta Math. (N.S.)* **1** (1995), 459-494.

[ER'94] P. H. Edelman, V. Reiner: h-shellings and h-complexes, *Adv. Math.* **106** (1994), no. 1, 36–64.

[FS'05] E.-M. Feichtner, B. Sturmfels: Matroid polytopes, nested sets and Bergman fans, *Portugaliae Mathematica* **62** (2005), 437–468.

[FY'04] E.-M. Feichtner, S. Yuzvinsky: Chow rings of toric varieties defined by atomic lattices, *Invent. Math.* **155** (2004), 515–536.

[FG'65] D. R. Fulkerson, O. A. Gross: Incidence matrices and interval graphs, *Pacific J. Math* **15** (1965), 835–855.

[Ful'93] W. Fulton: Introduction to toric varieties, *Annals of Mathematics Studies***131**, Princeton University Press, Princeton, NJ, 1993.

[FM'94] W. Fulton, R. MacPherson: A compactification of configuration spaces, *Ann. Math.* **139** (1994), 183–225.

[Gal'05] S. R. Gal: Real root conjecture fails for five- and higher-dimensional spheres. *Discrete Comput. Geom.* **34** (2005), no. 2, 269–284.

[GJ'83] I. P. Goulden, D. M. Jackson: *Combinatorial Enumeration,* John Wiley & Sons, 1983.

[GZ'83] C. Greene, T. Zaslavsky: On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs. *Trans. Amer. Math. Soc.* **280** (1983), no. 1, 97–126.

[Kim'06] S. W. Kim:, Shellable complexes and diagonal arrangements, Extended abstract, FPSAC 2006 (18th International Conference Formal Power Series and Algebraic Combinatorics).

[Lee'89] C. W. Lee: The associahedron and triangulations of the n-gon, *European J. Combin.* **10** (1989), no. 6, 551–560.

[Lod'04] J.-L. Loday: Realization of the Stasheff polytope, *Archiv der Mathematik (Basel)* **83** (2004), no. 3, 267-278; arXiv: `math.AT/0212126`.

[Mac'17] P. A. MacMahon: *Combinatory Analysis, Vol. I,* Cambridge, 1917; reprinted by the American Mathematical Society, 2001.

[Man'02] T. Mansour: Restricted 132-alternating permutations and Chebyshev polynomials, arXiv: math.CO/0210058.

[McM'73] P. McMullen: Representations of polytopes and polyhedral sets, *Geometriae Dedicata* **2** (1973), 83–99.

[M–W'06] J. Morton, L. Pachter, A. Shiu, B. Sturmfels, O. Wienand:, Geometry of rank tests, arXiv: math.ST/0605173

[Post'05] A. Postnikov: Permutohedra, associahedra, and beyond, arXiv: math.CO/0507163.

[RSW'03] V. Reiner, D. Stanton, V. Welker: The Charney-Davis quantity for certain graded posets, *Sém. Lothar. Combin.* **50** (2003/04), Art. B50c, 13 pp.

[SWG'83] L. Shapiro, W.J. Woan, S. Getu: Runs, slides and moments. *SIAM J. Algebraic Discrete Methods* **4** (1983), 459–466.

[Sim'94] R. Simion: Combinatorial statistics on noncrossing partitions, *J. Combin. Theory Ser. A* **66** (1994), 270–301.

[Sim'03] R. Simion: A type-B associahedron, *Adv. in Appl. Math.* **30** (2003), 2–25.

[SU'91] R. Simion, D. Ullman: On the structure of the lattice of noncrossing partitions. *Discrete Math.* **98** (1991), 193–206.

[Stan'92] R. P. Stanley: Subdivisions and local h-vectors, *J. Amer. Math. Soc.* **5** (1992), 805–851:

[Stan'96] R. P. Stanley: *Combinatorics and commutative algebra, 2nd edition*, Progress in Mathematics **41**, Birkhäuser Boston, Inc., Boston, MA, 1996.

[Stan'97] R. P. Stanley: *Enumerative combinatorics, Volume 1,* Cambridge University Press, New York, 1997.

[Stan'99] R. P. Stanley: *Enumerative Combinatorics, Volume 2,* Cambridge University Press, 1999.

[StPi'02] R. P. Stanley, J. Pitman: A polytope related to empirical distributions, plane trees, parking functions, and the associahedron, *Discrete & Computational Geometry* **27** (2002), no. 4, 603-634; arXiv: math.CO/9908029.

[Stash'63] J. D. Stasheff: Homotopy associativity of H-spaces, I, II, *Trans. Amer. Math. Soc.* **108** (1963), 275292; ibid. 293312.

[Stem'07] J. Stembridge: Coxeter cones and their h-vectors, preprint, dated March 12, 2007, 31 pages.

[Stud'05] M. Studený: *Probabilisic conditional independence structures,* Springer Series in Information Science and Statistics. Springer-Verlag, London 2005.

[Tol'05] V. Toledano-Laredo: Quasi-Coxeter algebras, Dynkin diagram cohomology and quantum Weyl groups, arXiv: math.QA/0506529.

[Wie'47] H. Wiener: Stuctural determination of paraffin boiling points, *J. Amer. Chem. Soc.* **69** (1947), 17-20.

[Zel'06] A. Zelevinsky: Nested complexes and their polyhedral realizations, *Pure and Applied Mathematics Quarterly* **2** (2006), 655–671.

[Zieg'94] G. M. Ziegler: *Lectures on polytopes,* Graduate Texts in Mathematics **152**, Springer-Verlag, 1994.

Massachusetts Institute of Technology
E-mail address: apost at math.mit.edu

University of Minnesota
E-mail address: reiner at math.umn.edu

Harvard University
E-mail address: lauren at math.harvard.edu