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A COMBINATORIAL MODEL FOR CRYSTALS OF KAC-MOODY ALGEBRAS

CRISTIAN LENART AND ALEXANDER POSTNIKOV

Abstract. We present a simple combinatorial model for the characters of the irreducible representations of Kac-Moody algebras. This model can be viewed as a discrete counterpart to the Littelmann path model. We describe crystal graphs and give a Littlewood-Richardson rule for decomposing tensor products of irreducible representations.

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1. Introduction

We have recently given a combinatorial model for the characters of the irreducible representations of a complex semisimple Lie group G, and for the Demazure characters [LP1]. This model was defined in the context of the equivariant K-theory of the generalized flag variety G/B. Our character formulas were derived from a Chevalley-type formula in $K_T(G/B)$. Our model was based on enumerating certain saturated chains in the *Bruhat order* on the corresponding *Weyl group* W. This enumeration is determined by an *alcove path*, which is a sequence of adjacent alcoves for the *affine Weyl group* W_{aff} of the Langland's dual group $G^\vee$. Alcove paths correspond to decompositions of elements in the affine Weyl

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group. Our Chevalley-type formula was formulated in terms of a certain *R-matrix*, that is, in terms of a collection of operators satisfying the *Yang-Baxter equation*. This setup allowed us to easily explain the independence of our formulas from the choice of an alcove path.

There are other models for Chevalley-type formulas in $K_T(G/B)$ and for the irreducible characters of G. Most notably, there is the *Littelmann path model*. Littelmann [Li1, Li2, Li3] showed that the characters can be described by counting certain continuous paths in $\mathfrak{h}_{\mathbb{R}}^*$. These paths are constructed recursively, by starting with an initial one, and by applying certain *root operators*. By making specific choices for the initial path, one can obtain special cases which have more explicit descriptions. For instance, a straight line initial path leads to the *Lakshmibai-Seshadri paths* (LS paths). These were introduced before Littelmann's work, in the context of *standard monomial theory* [LS]. They have a nonrecursive description as weighted chains in the Bruhat order on the quotient W/W_λ of the corresponding Weyl group W modulo the stabilizer W_λ of the weight λ; therefore, we will use the term *LS chains* when referring to this description. LS paths were used by Pittie and Ram [PR] to derive a K_T-Chevalley formula. Recently, Gaussent and Littelmann [GL], motivated by the study of Mirković-Vilonen cycles, defined another combinatorial model for the irreducible characters of a complex semisimple Lie group. This model is based on *LS galleries*, which are certain sequences of faces of alcoves for the corresponding affine Weyl group. For each LS gallery, there is an associated Littelmann path, and a saturated chain in the Bruhat order on W/W_λ. In [LP1], we explained the way in which our construction, which was developed independently of LS galleries, is related (although not quite equivalent) to the latter in the case of regular weights.

In this paper, we develop the combinatorial model in [LP1] purely in the context of representation theory, and extend it to complex symmetrizable Kac-Moody algebras. Instead of alcove paths (that make sense only in finite types) we now use λ*-chains*, which are chains of roots satisfying a certain *interlacing property*. Note that Littelmann paths and, in particular, LS paths were also defined in this more general context, but LS galleries were not. In fact, we show that LS paths are a certain limiting case of a special case of our model. The latter can be viewed as a discrete counterpart to the Littelmann path model. We define root operators in our model, and study their properties. This allows us to show that our model satisfies the axioms of an *admissible system* of Stembridge [Ste]. Thus, we easily derive character formulas, a Littlewood-Richardson rule for decomposing tensor products of irreducible representations, as well as a branching rule. The approach via admissible systems was already applied to LS chains in [Ste, Section 8]. Compared to the proofs in [GL, Li2, Li3], Stembridge's approach has the advantage of making a part of the proof independent of a particular model for Weyl characters, by using a system of axioms for such models.

Our model has several advantages over the Littelmann path model and its specializations mentioned above. First of all, our formulas are equally simple for all weights, regular and nonregular. Note that the (nonrecursive) construction of LS chains and LS galleries usually involves certain choices that add to their computational complexity. Also, it is harder to work with sequences of lower dimensional faces of alcoves (in the case of LS galleries) than with sequences of roots (in our model). We refer to [LP1] for a discussion showing that the computational complexity of our model is significantly smaller than the one of Littelmann paths (constructed recursively via root operators). Our definition of root operators resembles the one for LS paths, which is simpler than the general definition of root operators for Littelmann paths. We think that our model is easier to work with in explicit computations because, being based on certain chains of roots, it has a stronger combinatorial nature than Littelmann paths and, in particular, LS chains. Indeed, even for LS chains, we do need their description as piecewise-linear paths in order to define root operators.

We believe that the properties of our model discussed in this paper represent just a small fraction of a rich combinatorial structure yet to be explored. We will investigate it in a forthcoming paper [LP2].

Let us now present our combinatorial formula for characters. Fix a complex symmetrizable Kac-Moody algebra $\mathfrak{g}$; see [Kac]. Let Φ^+ be the associated set of positive real roots. For a root $\alpha \in \Phi^+$, let

$\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ be the corresponding coroot. Let λ be a dominant integral weight, and let V_λ be the associated irreducible representation of $\mathfrak{g}$ with the highest weight λ.

Let us define a *λ-chain (of roots)* $(\beta_i)_{i\in I}$ as a map $I \to \Phi^+$, $i \mapsto \beta_i$, from a totally ordered index set I to positive roots that satisfies the following conditions:

(1) For a root $\alpha \in \Phi^+$, the number $\#\{i \in I \mid \beta_i = \alpha\}$ of occurrences of α equals $\langle\lambda,\alpha^\vee\rangle$.
(2) For a triple of roots $\alpha,\beta,\gamma \in \Phi^+$ such that $\alpha^\vee+\beta^\vee = \gamma^\vee$, the finite ordered subsequence $(\beta_j)_{j\in J}$, where $J = \{j \in I \mid \beta_j \in \{\alpha,\beta,\gamma\}\}$, is a concatenation of pairs (α,γ) and (β,γ) (in any order).

Condition (2) says that all α's, β's, and γ's in a λ-chain should satisfy the following *interlacing property*: there is exactly one element from the set $\{\alpha,\beta\}$ between any two consecutive γ's, as well as before the first γ; and there are no α's or β's after the last γ. Note that, according to condition (1), the index set I is always a countable or a finite set. However, it is not always isomorphic to a subset of $\mathbb{Z}$. For example, it may contain infinite intervals.

Lemma 1.1. *For any dominant integral weight λ, there exists a λ-chain.*

For a weight λ, there are usually many λ-chains. We will give an explicit construction of a λ-chain.

For $\alpha \in \Phi^+$, let s_α denote the associated reflection. The reflections s_α generate the Weyl group W. The covering relations $w \lessdot ws_\alpha$, for $w \in W$, $\ell(ws_\alpha) = \ell(w) + 1$, define the partial Bruhat order on W. These covering relations are labelled by roots α.

Let us fix a λ-chain $(\beta_i)_{i\in I}$. Let us say that a finite subset $J = \{j_1 < \cdots < j_l\}$ of the index set I is *admissible* if the roots $\beta_{j_1},\ldots,\beta_{j_l}$ are labels of an increasing saturated chain in the Bruhat order starting at the identity element, i.e., we have:

$$1 \lessdot s_{\beta_{j_1}} \lessdot s_{\beta_{j_1}}s_{\beta_{j_2}} \lessdot s_{\beta_{j_1}}s_{\beta_{j_2}}s_{\beta_{j_3}} \lessdot \cdots \lessdot s_{\beta_{j_1}}\ldots s_{\beta_{j_l}}.$$

For $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, let $s_{\alpha,k}$ be the affine reflection given by $s_{\alpha,k} : \mu \mapsto s_\alpha(\mu) + k\alpha$. For a λ-chain, let $(k_i)_{i\in I}$ be the associated sequence of nonnegative integers defined by $k_i := \#\{j < i \mid \beta_j = \beta_i\}$. Let us define the *weight* $\mu(J)$ of an admissible subset $J = \{j_1 < \cdots < j_k\}$ by

$$\mu(J) = s_{\beta_{j_1},k_{j_1}} \cdots s_{\beta_{j_l},k_{j_l}}(\lambda).$$

Theorem 1.2. *For a dominant integral weight λ and any λ-chain $(\beta_i)_{i\in I}$, the character $\chi(\lambda)$ of the irreducible representation V_λ is given by*

$$\chi(\lambda) = \sum_J e^{\mu(J)},$$

where the sum is over the admissible subsets J of the index set I.

Example 1.3. Let $\mathfrak{g}$ be the Lie algebra of type A_2. Let us fix a choice of simple roots α_1, α_2, and let $\lambda = \omega_1$ be the first fundamental weight. In this case, there is only one λ-chain $(\beta_1,\beta_2) = (\alpha_1,\alpha_1+\alpha_2)$ (assuming that $I = \{1,2\}$). The index set I has 3 admissible subsets: $\emptyset, \{1\}, \{1,2\}$. The subset $\{2\}$ is not admissible because the reflection $s_{\alpha_1+\alpha_2}$ does not cover the identity element. In this case, $(k_1,k_2) = (0,0)$. Theorem 1.2 gives the following expression for the character of V_{ω_1}:

$$\chi(\omega_1) = e^{\omega_1} + e^{s_{\alpha_1}(\omega_1)} + e^{s_{\alpha_1}s_{\alpha_1+\alpha_2}(\omega_1)}.$$

We will define a crystal graph structure on the collection of admissible subsets. From this, we will deduce a rule for decomposing tensor products of irreducible representations and a branching rule.

The general outline of the paper follows. In Section 2, we review basic notions related to roots systems for complex symmetrizable Kac-Moody algebras and fix our notation. In Section 3, we discuss crystals and give Stembridge's axioms of admissible systems. In Sections 4-6, we define our combinatorial model. In Section 4, we discuss λ-chains. In Section 5, we define and study *folding operators*; we use them to construct more general chains of roots from a λ-chain, which we call *admissible foldings*. In Section 6, we study combinatorial properties of admissible subsets and admissible foldings. In Section 7, we define root operators on admissible subsets/foldings. In Section 8, we prove that our combinatorial model satisfies Stembridge's axioms. This enables us to derive a character formula for complex symmetrizable Kac-Moody algebras, a Littlewood-Richardson rule for decomposing tensor products of irreducible representations, as well as a branching rule. In Section 9, we explain the connection between our model and LS chains. In Section 10, we discuss the way in which the present model specializes to the one in our previous paper [LP1] in the finite case.

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2. Preliminaries

In this section, we briefly recall the general setup for complex symmetrizable Kac-Moody algebras and their representations. We refer to [Kac, Ku] for more details.

Let V be a finite-dimensional real vector space with a nondegenerate symmetric bilinear form $\langle \cdot , \cdot \rangle$, and let $\Phi \subset V$ be a crystallographic root system of rank r with simple roots $\{\alpha_1, \dots, \alpha_r\}$. By this, we mean that Φ is the set of real roots of some complex symmetrizable Kac-Moody algebra. The finite root systems of this type are the root systems of semisimple Lie algebras. Note that, in the infinite case, it is possible for the simple roots to span a proper subspace of V; indeed, it can happen that the bilinear form is degenerate on the span of the simple roots.

Given a root α, the corresponding *coroot* is $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$. The collection of coroots $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ forms the *dual root system.* For each root α, there is a reflection $s_\alpha : V \to V$ defined by

$$s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \, \alpha.$$

More generally, for any integer k, one can consider the affine hyperplane

$$H_{\alpha,k} := \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle = k\} \,,$$

and let $s_{\alpha,k}$ denote the corresponding reflection, that is,

$$s_{\alpha,k} \,:\, \lambda \mapsto s_\alpha(\lambda) + k\alpha \,. \tag{2.1}$$

The *Weyl group* W is the subgroup of $GL(V)$ generated by the reflections s_α for $\alpha \in \Phi$. In fact, the Weyl group W is a Coxeter group, which is generated by the *simple reflections* $s_1, \dots, s_r$ corresponding to the simple roots $s_p := s_{\alpha_p}$, subject to the *Coxeter relations*:

$$(s_p)^2 = 1 \quad \text{and} \quad (s_p s_q)^{m_{pq}} = 1 \,;$$

here the relations of the second type correspond to the distinct p, q in $\{1, \dots, r\}$ for which the dihedral subgroup generated by s_p and s_q is finite, in which case m_{pq} is half the order of this subgroup. The Weyl group is finite if and only if Φ is finite.

An expression of a Weyl group element w as a product of generators $w = s_{p_1} \cdots s_{p_l}$ which has minimal length is called a *reduced decomposition* for w; its length $\ell(w) = l$ is called the *length* of w. For $u, w \in W$, we say that u *covers* w, and write $u \gtrdot w$, if $w = us_\beta$, for some $\beta \in \Phi^+$, and $\ell(u) = \ell(w) + 1$. The transitive closure "$>$" of the relation "$\gtrdot$" is called the *Bruhat order* on W.

Let us note that Φ can be characterized by the following axioms:

- (R1) $\{\alpha_1, \dots, \alpha_r\}$ is a linearly independent set.
- (R2) $\langle \alpha_p, \alpha_p \rangle > 0$ for all $p = 1, \dots, r$.
- (R3) $\langle \alpha_p, \alpha_q^\vee \rangle \in \mathbb{Z}_{\leq 0}$ for all distinct simple roots α_p and α_q.
- (R4) $\Phi = \bigcup_{p=1}^r W\alpha_p$.

Let $\Phi^+ \subset \Phi$ be the set of positive roots, that is, the set of roots in the nonnegative linear span of the simple roots. Then Φ is the disjoint union of Φ^+ and $\Phi^- := -\Phi^+$. We write $\alpha > 0$ (respectively, $\alpha < 0$) for $\alpha \in \Phi^+$ (respectively, $\alpha \in \Phi^-$), and we define $\mathrm{sgn}(\alpha)$ to be 1 (respectively, -1). We also use the notation $|\alpha| := \mathrm{sgn}(\alpha)\alpha$.

The lattice of (integral) *weights* Λ is given by

$$\Lambda := \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}. \tag{2.2}$$

This is slightly misleading terminology, since Λ is not a lattice in V, but rather in V/Z, where

$$Z := \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle = 0 \text{ for any } \alpha \in \Phi\}. \tag{2.3}$$

The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+\}.$$

If we replace the weak inequalities above with strict ones, we obtain the strongly dominant weights. It is known that every W-orbit in V has at most one dominant member. The *fundamental weights* $\omega_1, \ldots, \omega_r$ are defined by $\langle \omega_p, \alpha_q^\vee \rangle = \delta_{pq}$.

The (integral) Tits cone Λ_c is defined to be the union of all W-orbits of dominant integral weights, or, equivalently,

$$\Lambda_c := \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle < 0 \text{ for finitely many } \alpha \in \Phi^+\}.$$

We have $\Lambda = \Lambda_c$ in the finite case, but not otherwise.

We now define a ring R that contains the characters of all integrable highest weight modules for the corresponding Kac-Moody algebra. In the finite case, one may simply take R to be the group ring of Λ, but in general more care is required.

First, we choose a height function $\mathrm{ht} : V \to \mathbb{R}$, that is, a linear map assigning the value 1 to all simple roots. Second, for each $\lambda \in \Lambda$, let e^λ denote a formal exponential subject to the rules $e^\mu \cdot e^\nu = e^{\mu+\nu}$ for all $\mu, \nu \in \Lambda$. We now define the ring R to consist of all formal sums $\sum_{\lambda \in \Lambda} c_\lambda e^\lambda$ with $c_\lambda \in \mathbb{Z}$ satisfying the condition that there are only finitely many weights λ with $\mathrm{ht}(\lambda) > h$ and $c_\lambda \neq 0$, for all $h \in \mathbb{R}$. The ring R contains the formal power series ring $R_0 = \mathbb{Z}[[e^{-\alpha_1}, \ldots, e^{-\alpha_r}]]$. In particular, if $f \in R_0$ has constant term 1, then $e^\lambda f$ has a multiplicative inverse in R.

For each $\lambda \in \Lambda^+$ with a finite W-stabilizer, we define

$$\Delta(\lambda) := \sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda)},$$

where $\mathrm{sgn}(w) = (-1)^{\ell(w)}$. It is not hard to check that $\Delta(\lambda)$ is a well-defined member of R. Moreover, if λ is dominant, then $\Delta(\lambda) \neq 0$ if and only if λ is strongly dominant. In that case, $e^{-\lambda}\Delta(\lambda) \in R_0$ has constant term 1, and $\Delta(\lambda)$ is invertible in R.

Since the scalar product is nondegenerate, we may select $\rho \in \Lambda^+$ so that $\langle \rho, \alpha_p^\vee \rangle = 1$ for all $p = 1, \ldots, r$. This given, for each $\lambda \in \Lambda^+$ we define

$$\chi(\lambda) := \frac{\Delta(\lambda + \rho)}{\Delta(\rho)} = \frac{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)-\rho}}{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\rho)-\rho}} \in R.$$

It is easy to show that $w(\rho) - \rho$, and hence $\chi(\lambda)$, do not depend on the choice of ρ. By the Kac-Weyl character formula [Kac], these are the characters of the irreducible highest weight modules for the corresponding Kac-Moody algebra.

3. Crystals

This section closely follows [Ste, Section 2]. We refer to this paper for more details.

Definition 3.1. (cf. [Ste]). A *crystal* is a 4-tuple $(X, \mu, \delta, \{F_1, \ldots, F_r\})$ satisfying Axioms (A1)-(A3) below, where

- X is a set whose elements are called objects;
- μ and δ are maps $X \to \Lambda$;
- F_p are bijections between two subsets of X.

A crystal is called an *admissible system* if it satisfies Axioms (A0) and (A4). An admissible system is called a *semiperfect crystal* if it satisfies Axiom (A5).

(A0) For all real numbers h, there are only finitely many objects x such that $\mathrm{ht}(\mu(x)) > h$.

Axiom (A0) implies that the generating series $G_X := \sum_{x \in X} e^{\mu(x)}$ is a well-defined member of R.

For each $x \in X$, we call $\mu(x)$, $\delta(x)$, and $\varepsilon(x) := \mu(x) - \delta(x)$ the *weight*, *depth*, and *rise* of x.

(A1) $\delta(x) \in -\Lambda^+$, $\varepsilon(x) \in \Lambda^+$.

We define the depth and rise in the direction α_p by $\delta(x, p) := \langle \delta(x), \alpha_p^\vee \rangle$ and $\varepsilon(x, p) := \langle \varepsilon(x), \alpha_p^\vee \rangle$. In fact, we will develop the whole theory in terms of $\delta(x, p)$ and $\varepsilon(x, p)$ rather than $\delta(x)$ and $\varepsilon(x)$.

(A2) F_p is a bijection from $\{x \in X \mid \varepsilon(x, p) > 0\}$ to $\{x \in X \mid \delta(x, p) < 0\}$.

We let $E_p := F_p^{-1}$ denote the inverse map.

(A3) $\mu(F_p(x)) = \mu(x) - \alpha_p$, $\delta(F_p(x), p) = \delta(x, p) - 1$.

Hence, we also have $\varepsilon(F_p(x), p) = \varepsilon(x, p) - 1$. The maps E_p and F_p act as raising and lowering operators that provide a partition of the objects into α_p-strings that are closed under the action of E_p and F_p. For example, the *α_p-string through x* is (by definition)

$$F_p^{\varepsilon}(x),\, \ldots,\, F_p(x),\, x,\, E_p(x),\, \ldots,\, E_p^{-\delta}(x)\,,$$

where $\delta = \delta(x, p)$ and $\varepsilon = \varepsilon(x, p)$.

We define a partial order on X by

$$x \preceq_p y \quad \text{if} \quad x = F_p^k(y) \ \text{ for some } \ k \geq 0\,. \tag{3.1}$$

Any assignment of elements $t(x, p)$ of a totally ordered set to pairs (x, p) with $\delta(x, p) < 0$ is called a *timing pattern* for X. A timing pattern is called *coherent* if the following two conditions are satisfied for all pairs (x, p) such that $\delta(x, p) < 0$ and $\varepsilon(x, p) > 0$:

- $t(x, p) \geq t(F_p(x), p)$;
- for all $q \neq p$, all integers $\delta < 0$, and all $t \geq t(x, p)$, there is an object $y \succeq_q x$ such that $\delta(y, q) = \delta$ and $t(y, q) = t$ if and only if there is an object $y' \succeq_q F_p(x)$ such that $\delta(y', q) = \delta$ and $t(y', q) = t$.

The following axiom ensures the existence of a certain sign-reversing involution used to cancel the negative terms in the Kac-Weyl character formula.

(A4) There exists a coherent timing pattern for X.

Note that, compared to [Ste], here we let the timing pattern take values in any totally ordered set, and we reverse the total order previously considered. However, these minor changes, dictated by our needs, are easily taken care of by minor changes in the corresponding proofs in [Ste].

We call an object of X *maximal* if it is maximal with respect to all partial orders $\preceq_p$, for $p = 1, \ldots, r$.

(A5) X has a unique maximal object.

Theorem 3.2. [Ste] *If X is an admissible system, then*

$$\chi(\nu) \cdot G_X = \sum_{x \in X \,:\, \nu+\delta(x) \in \Lambda^+} \chi(\nu + \mu(x))\,.$$

In particular, if X is a semiperfect crystal with maximal object x_*, then $G_X = \chi(\mu(x_*))$.

Given $P \subseteq \{1, \ldots, r\}$, let Φ_P denote the root subsystem of Φ with simple roots $\{\alpha_p \mid p \in P\}$. Following [Ste], we let $W_P \subseteq W$, $\Lambda_P \supseteq \Lambda$, and R_P denote the corresponding Weyl group, weight lattice, and character ring. Provided that we use the height function inherited from Φ (in which case $R_P \supseteq R$), it is easy to see that any admissible system X can also be viewed as an admissible system relative to Φ_P using only the operators E_p and F_p for $p \in P$. Given $\lambda \in \Lambda_P^+$, we let $\chi(\lambda; P) \in R_P$ denote the Weyl character (relative to Φ_P) corresponding to λ. The following branching rule is given in [Ste].

Corollary 3.3. [Ste] *If X is an admissible system and $P \subseteq \{1, \ldots, r\}$, we have*

$$G_X = \sum_{x \,:\, \delta(x) \in \Lambda_P^+} \chi(\mu(x); P)\,.$$

Finally, note that one can define on X the structure of a directed colored graph by constructing arrows $x \to y$ colored p for each $F_p(x) = y$.

Definition 3.4. A crystal $(X, \mu, \delta, \{F_1, \ldots, F_r\})$ is called a *perfect crystal* if the associated directed colored graph is isomorphic to the *crystal graph* of an irreducible representation of a quantum group.

4. λ-Chains of Roots

Fix a dominant weight λ. Throughout this paper, we will use the term "sequence" for any map $i \mapsto a_i$ from a totally ordered set I to some other set; we will use the notation $\{a_i\}_{i \in I}$. Given an element b and an index i, we also define

$$N(b) := \#\{k \mid a_k = b\},\ N_i(b) := \#\{k < i \mid a_k = b\},\ N_{ij}(b) := \#\{i < k < j \mid a_k = b\},$$

assuming that the corresponding cardinalities are finite.

Definition 4.1. A λ-*chain (of roots)* is a sequence of positive roots $\{\beta_i\}_{i \in I}$ indexed by the elements of a totally ordered set I, which satisfies the following conditions:

(1) the number of occurrences of any positive root α, i.e., $N(\alpha)$, is $\langle \lambda, \alpha^\vee \rangle$;
(2) for each triple of positive roots (α, β, γ) with $\gamma^\vee = \alpha^\vee + \beta^\vee$, the finite sequence $\{\beta_j\}_{j \in J}$, where $J := \{j \in I \mid \beta_j \in \{\alpha, \beta, \gamma\}\}$ has the induced total order, is a concatenation of pairs (α, γ) and (β, γ) (in any order).

Note that finding a λ-chain amounts to defining a total order on the set

$$I := \{(\alpha, k) \mid \alpha \in \Phi^+,\, 0 \le k < \langle \lambda, \alpha^\vee \rangle\} \tag{4.1}$$

such that the second condition above holds, where $\beta_i = \alpha$ for any $i = (\alpha, k)$ in I. One particular example of such an order can be constructed as follows. Fix a total order on the set of simple roots $\alpha_1 < \alpha_2 < \ldots < \alpha_r$. For each $i = (\alpha, k)$ in I, let $\alpha^\vee = c_1\alpha_1^\vee + \ldots + c_r\alpha_r^\vee$, and define the vector

$$v_i := \frac{1}{\langle \lambda, \alpha^\vee \rangle}(k, c_1, \ldots, c_r) \tag{4.2}$$

in $\mathbb{Q}^{r+1}$. The map $i \mapsto v_i$ is injective. Indeed, assume that $v_i = v_{i'}$ for $i = (\alpha, k)$ and $i' = (\alpha', k')$. If $\alpha \neq \alpha'$, the root system $\Phi^\vee$ would contain two proportional positive coroots $\alpha^\vee \neq (\alpha')^\vee$, which is not possible. Also, the fact that $\alpha = \alpha'$ implies that $k = k'$.

Proposition 4.2. *Consider the total order on the set I in (4.1) defined by $i < j$ iff $v_i < v_j$ in the lexicographic order on $\mathbb{Q}^{r+1}$. The sequence $\{\beta_i\}_{i\in I}$ given by $\beta_i = \alpha$ for $i = (\alpha, k)$ is a λ-chain.*

Proof. For each p in $\{1, \ldots, r\}$, let us denote by c_p^α, c_p^β, c_p^γ the coefficients of $\alpha_p^\vee$ in $\alpha^\vee$, $\beta^\vee$, $\gamma^\vee$, respectively; clearly, $c_p^\gamma = c_p^\alpha + c_p^\beta$. Also, let $a := \langle\lambda, \alpha^\vee\rangle$, $b := \langle\lambda, \beta^\vee\rangle$, and $c := \langle\lambda, \gamma^\vee\rangle$, where $c = a + b$. The second condition in Definition 4.1 is straightforward to check if $a = 0$ or $b = 0$. Assume $a \neq 0 \neq b$. In this case, the mentioned condition is checked based on the following three claims about the finite sequence $\{\beta_j\}_{j\in J}$ (recall the definition of J above):

(1) if an entry α is followed by β, or vice versa, there is an entry γ in-between;
(2) between two entries α, or two entries β, there is an entry γ;
(3) the sequence cannot start with γ, but must end with γ.

For the first claim, let $(\alpha, k) < (\beta, m)$, which means $k/a \leq m/b$. If the inequality is strict, then the rational number $(k + m)/c = (k + m)/(a + b)$ is strictly in-between the previous two ones; therefore, we have $(\alpha, k) < (\gamma, k + m) < (\beta, m)$, which proves the claim. Otherwise, all three numbers are equal. We can now repeat our reasoning above with k and m replaced by c_p^α and c_p^β for $p = 1$. If equality still holds, we let $p = 2$ etc. At some point, we must have strict inequalities; indeed, otherwise the positive roots $\alpha^\vee$ and $\beta^\vee$ would be proportional, which is impossible.

For the second claim, we can assume that we have two consecutive entries α with corresponding indices $(\alpha, k-1)$ and (α, k). Furthermore, we can assume that we have an entry β with corresponding index (β, m) such that

$$\frac{m}{b} \leq \frac{k-1}{a} < \frac{k}{a} \leq \frac{m+1}{b} \leq 1\,.$$

It is straightforward to check that we have

$$\frac{k-1}{a} < \frac{k+m}{a+b} < \frac{k}{a}\,;$$

therefore, we have $(\alpha, k-1) < (\gamma, k+m) < (\alpha, k)$, which proves the claim.

For the last claim, note that the first three entries in the sequence $\{\beta_j\}_{j\in J}$ are α, β, γ in some order, and that the corresponding vectors in $\mathbb{Q}^{r+1}$ have their first components equal to 0. The fact that the sequence cannot start with γ follows by an argument similar to the proof of the first claim above. On the other hand, the sequence must end with γ because the largest value of the first component in the vectors involved is $(c-1)/c$, and this value only appears in v_i for $i = (\gamma, c-1)$. □

The following result about λ-chains will be needed later.

Lemma 4.3. *Let α, β, γ be three positive roots, and let $(\beta_i)_{i\in I}$ be a λ-chain. Fix i in I such that $\beta_i = \beta$.*

(1) *If $\gamma^\vee = \alpha^\vee + k\beta^\vee$ for some integer k, then we have*

$$N_i(\gamma) = N_i(\alpha) + kN_i(\beta)\,.$$

(2) *If $\gamma^\vee = -\alpha^\vee + k\beta^\vee$ for some integer k and $\alpha \neq \beta$, then we have*

$$N_i(\gamma) = -N_i(\alpha) + kN_i(\beta) + 1\,.$$

(3) *If $\alpha \neq \beta$, then we have*

$$\operatorname{sgn}(s_\beta(\alpha))\, N_i(|s_\beta(\alpha)|) = N_i(\alpha) - \langle\beta, \alpha^\vee\rangle N_i(\beta) + \frac{\operatorname{sgn}(s_\beta(\alpha)) - 1}{2}\,.$$

Proof. Condition (2) in Definition 4.1 implies that, given the roots α, β, γ with $\gamma^\vee = \alpha^\vee + \beta^\vee$, we have

$$N_j(\gamma) = \begin{cases} N_j(\alpha) + N_j(\beta) & \text{if } \beta_j = \alpha \ \text{ or } \ \beta_j = \beta \\ N_j(\alpha) + N_j(\beta) - 1 & \text{if } \beta_j = \gamma\,. \end{cases} \tag{4.3}$$

In order to prove the first statement, it is enough to consider $k > 0$. We know that $\alpha^\vee + l\beta^\vee$ are coroots for $l = 0, \ldots, k$, since this sequence is part of the $\beta^\vee$-string of coroots through $\alpha^\vee$. The proof is now reduced to an easy induction on k, which starts at $k = 0$, and is based on (4.3). The proof of the second statement is similar, with induction starting at $k = 1$. Finally, the last statement is a straightforward consequence of the previous two. □

For the rest of our construction (Sections 5–8), let us fix a dominant integral weight λ and fix an arbitrary λ-chain $\{\beta_i\}_{i\in I}$. We will use the notation r_i for the reflection s_{β_i}.

5. Folding Chains of Roots

We start by associating to our fixed λ-chain the closely related object

$$\Gamma(\emptyset) := (\{(\beta_i, \beta_i)\}_{i\in I}, \rho)\,,$$

where ρ is a fixed dominant weight satisfying $\langle \rho, \alpha_p^\vee\rangle = 1$ for all $p = 1, \ldots, r$. Here, as well as throughout this article, we let ∞ be greater than all elements in I. We use operators called *folding operators* to construct from $\Gamma(\emptyset)$ new objects of the form

$$\Gamma = (\{(\gamma_i, \gamma_i')\}_{i\in I}, \gamma_\infty)\,; \tag{5.1}$$

here (γ_i, γ_i') are pairs of roots with $\gamma_i' = \pm\gamma_i$, any given root appears only finitely many times in Γ, and γ_∞ is in the W-orbit of ρ. More precisely, given Γ as above and i in I, we let $t_i := s_{\gamma_i}$ and we define

$$\phi_i(\Gamma) := (\{(\delta_j, \delta_j')\}_{j\in I}, t_i(\gamma_\infty))\,,$$

where

$$(\delta_j, \delta_j') := \begin{cases} (\gamma_j, \gamma_j') & \text{if } j < i \\ (\gamma_j, t_i(\gamma_j')) & \text{if } j = i \\ (t_i(\gamma_j), t_i(\gamma_j')) & \text{if } j > i\,. \end{cases}$$

Let us now consider the set of all Γ that are obtained from $\Gamma(\emptyset)$ by applying folding operators; we call these objects the *foldings* of $\Gamma(\emptyset)$. Clearly, ϕ_i is an involution on the set of foldings of $\Gamma(\emptyset)$. In order to describe this set, let us note that the folding operators commute. Indeed, if Γ is as in (5.1), $i < j$, $\alpha := \gamma_i$, and $\beta := \gamma_j$, then we have

$$t_i t_j = s_\alpha s_\beta = (s_\alpha s_\beta s_\alpha) s_\alpha = s_{s_\alpha(\beta)} s_\alpha\,;$$

so $\phi_i\phi_j(\Gamma) = \phi_j\phi_i(\Gamma)$. This means that every folding Γ of $\Gamma(\emptyset)$ is determined by the set $J := \{j \mid \gamma_j' = -\gamma_j\}$. More precisely, if $J = \{j_1 < j_2 < \ldots < j_s\}$, then $\Gamma = \phi_{j_1}\ldots\phi_{j_s}(\Gamma(\emptyset))$. We call the elements of J the *folding positions* of Γ, and write $\Gamma = \Gamma(J)$.

Throughout this paper, we will use J and $\Gamma = \Gamma(J)$ interchangeably. For instance, according to the above discussion, we have

$$\phi_i(\Gamma(J)) = \Gamma(J \triangle \{i\})\,,$$

where $\triangle$ denotes the symmetric difference of sets. Hence, it makes sense to define the folding operator ϕ_i on J (compatibly with the action of ϕ_i on $\Gamma(J)$) by $\phi_i\,:\, J \mapsto J\triangle\{i\}$.

Using the notation above and the fact that $\gamma_{j_i} = r_{j_1}\ldots r_{j_{i-1}}(\beta_{j_i})$, we have

$$t_{j_1} = r_{j_1}\,,\ t_{j_2} = r_{j_1} r_{j_2} r_{j_1}\,,\ t_{j_3} = r_{j_1} r_{j_2} r_{j_3} r_{j_2} r_{j_1}\,,\ \ldots\,;$$

recall that $r_i = s_{\beta_i}$. In particular, we have

$$r_{j_1}\ldots r_{j_i} = (t_{j_1}\ldots t_{j_i})^{-1}\,, \tag{5.2}$$

for $i = 1, \ldots, s$.

Remark 5.1. Although a folding Γ of $\Gamma(\emptyset)$ is an infinite sequence if the root system is infinite, we are, in fact, always working with finite objects. Indeed, we are examining Γ by considering only one root at a time.

Given a folding Γ of $\Gamma(\emptyset)$, we associate to each pair of roots (or the corresponding index i in I) an integer l_i, which we call *level*; the sequence $L = L(\Gamma) := \{l_i\}_{i\in I}$ will be called the *level sequence* of Γ. The definition is as follows:

$$l_i := \varepsilon + \sum_{j<i,\gamma_j=\gamma_j'=\pm\gamma_i} \operatorname{sgn}(\gamma_j)\,, \tag{5.3}$$

where

$$\varepsilon := \begin{cases} 0 & \text{if } \gamma_i > 0 \\ -1 & \text{otherwise}\,. \end{cases}$$

We make the convention that the sum is 0 if it contains no terms. The definition makes sense since the sum is always finite. In particular, we have the level sequence $L_\emptyset = L(\Gamma(\emptyset)) := \{l_i^\emptyset\}_{i\in I}$ of $\Gamma(\emptyset)$. Given a root α, we will use the following notation:

$$I_\alpha = I_\alpha(\Gamma) := \{i \in I \mid \gamma_i = \pm\alpha\}\,, \quad L_\alpha = L_\alpha(\Gamma) := \{l_i \mid i \in I_\alpha\}\,.$$

Remark 5.2. It is often useful to use the following graphical representation. Let $I_\alpha = \{i_1 < i_2 < \ldots < i_n\}$, and let us define the continuous piecewise-linear function $g_\alpha \,:\, [0,n] \to \mathbb{R}$ by

$$g_\alpha(0) = -\frac{1}{2}\,, \quad g_\alpha'(x) = \begin{cases} \operatorname{sgn}(\gamma_{i_k}) & \text{if } x \in (k-1, k-\frac{1}{2}) \\ \operatorname{sgn}(\gamma_{i_k}') & \text{if } x \in (k-\frac{1}{2}, k)\,, \end{cases}$$

for $k = 1, \ldots, n$. Then $l_{i_k} = g_\alpha(k - \frac{1}{2})$. For instance, assume that the entries of Γ indexed by the elements of I_α are $(\alpha,-\alpha)$, $(-\alpha,-\alpha)$, (α,α), (α,α), $(\alpha,-\alpha)$, $(-\alpha,-\alpha)$, $(\alpha,-\alpha)$, (α,α), in this order. The graph of g_α is shown on Figure 1.

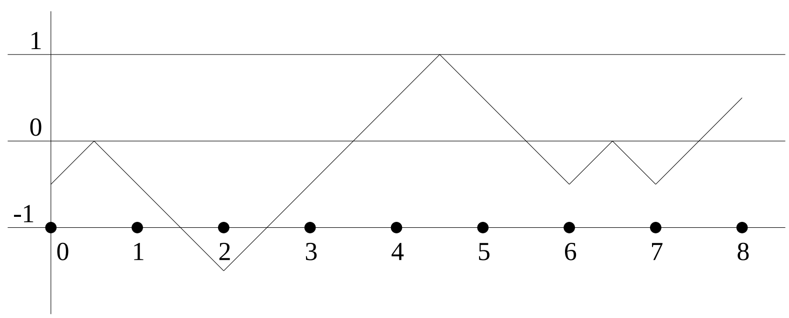

FIGURE 1.

We will now consider certain affine reflections corresponding to foldings Γ of $\Gamma(\emptyset)$. Let $\widehat{t}_i := s_{|\gamma_i|,l_i}$; recall that the latter is the reflection in the affine hyperplane $H_{|\gamma_i|,l_i}$. In particular, we have the affine reflections $\widehat{r}_i := s_{\beta_i,l_i^\emptyset}$ corresponding to $\Gamma(\emptyset)$.

Definition 5.3. Given $J = \{j_1 < j_2 < \ldots < j_s\} \subseteq I$ and $\Gamma = \Gamma(J)$, we let

$$\mu = \mu(\Gamma) = \mu(J) := \widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(\lambda)\,,$$

and call μ the *weight* of Γ (respectively J). We also use the notation $w(J) = w(\Gamma) := r_{j_1} \ldots r_{j_s}$ (recall that $r_i := s_{\beta_i}$), and

$$\widehat{I}_\alpha = \widehat{I}_\alpha(\Gamma) := I_\alpha \cup \{\infty\}\,, \quad \widehat{L}_\alpha = \widehat{L}_\alpha(\Gamma) := L_\alpha \cup \{l_\alpha^\infty\}\,,$$

where $l_\alpha^\infty := \langle \mu(\Gamma), \alpha^\vee \rangle$.

The following proposition is our main technical result, which relies heavily on the defining properties of λ-chains.

Proposition 5.4. *Let $\Gamma=\Gamma(J)$ for some $J=\{j_1<j_2<\ldots<j_s\}\subseteq I$, and let $j_p<j\leq j_{p+1}$ (the first or the second inequality is dropped if $p=0$ or $p=s$, respectively). Using the notation above, we have*

$$H_{|\gamma_j|,l_j}=\widehat{r}_{j_1}\ldots\widehat{r}_{j_p}(H_{\beta_j,l_j^\emptyset})\,.$$

Proof. It suffices to consider $p=s$. Let us define the roots $\delta_0:=\beta_j$ and

$$\delta_l:=r_{j_{s-l+1}}r_{j_{s-l+2}}\ldots r_{j_s}(\beta_j)\,,$$

for $l=1,\ldots,s$; note that $\delta_s=\gamma_j$. Throughout this proof, we use the maps N_i and N_{ij} defined at the beginning of Section 4 to count roots in $\Gamma(\emptyset)$. We need to show that

$$\widehat{r}_{j_1}\ldots\widehat{r}_{j_s}(H_{\beta_j,l_j^\emptyset})=H_{\delta_s,k}\,,$$

where, according to the definitions of folding operators and of levels, the integer $k:=\mathrm{sgn}(\delta_s)l_j$ can be expressed as

$$k=\mathrm{sgn}(\delta_s)N_{j_1}(|\delta_s|)+\mathrm{sgn}(\delta_{s-1})N_{j_1j_2}(|\delta_{s-1}|)+\ldots+N_{j_sj}(\delta_0)+\frac{1-\mathrm{sgn}(\delta_s)}{2}\,.$$

Indeed, by examining the segment of $\Gamma(\emptyset)$ between j_l and j_{l+1} for $l=1,\ldots,s$ (we let $j_{s+1}:=j$), we note that only the root $|\delta_{s-i}|$ gets changed to $\pm\delta_s$ in Γ; more precisely, the sign of the corresponding root in Γ is $\mathrm{sgn}(\delta_s)\mathrm{sgn}(\delta_{s-i})$. The definition of levels now leads us to the above formula for k.

We now use induction on s, which starts at $s=0$. Let us assume that the statement to prove holds for $s-1$. Based on the discussion above, this means that

$$\widehat{r}_{j_2}\ldots\widehat{r}_{j_s}(H_{\beta_j,l_j^\emptyset})=H_{\delta_{s-1},k'}\,,$$

where

$$k'=\mathrm{sgn}(\delta_{s-1})N_{j_2}(|\delta_{s-1}|)+\mathrm{sgn}(\delta_{s-2})N_{j_2j_3}(|\delta_{s-2}|)+\ldots+N_{j_sj}(\delta_0)+\frac{1-\mathrm{sgn}(\delta_{s-1})}{2}\,.$$

An easy linear algebra computation shows that

$$\widehat{r}_{j_1}(H_{\delta_{s-1},k'})=H_{\delta_s,k''}\,,$$

where

$$k''=k'-l_{j_1}^\emptyset\langle\beta_{j_1},\delta_{s-1}^\vee\rangle=k'-N_{j_1}(\beta_{j_1})\langle\beta_{j_1},\delta_{s-1}^\vee\rangle\,.$$

Let us now substitute into this formula the expression for k' given by the induction hypothesis, and compare the result with the expression for k given above.

Let us first assume that $\delta_{s-1}\neq\pm\beta_{j_1}$. It turns out that verifying $k=k''$ amounts to proving the following identity:

$$\mathrm{sgn}(\delta_s)N_{j_1}(|\delta_s|)-\frac{\mathrm{sgn}(\delta_s)}{2}=\mathrm{sgn}(\delta_{s-1})N_{j_1}(|\delta_{s-1}|)-\frac{\mathrm{sgn}(\delta_{s-1})}{2}-N_{j_1}(\beta_{j_1})\langle\beta_{j_1},\delta_{s-1}^\vee\rangle\,.$$

Let us now multiply both sides by $\mathrm{sgn}(\delta_{s-1})$, then substitute δ_s with $r_{j_1}(\delta_{s-1})$, and finally write α for $|\delta_{s-1}|$, β for β_{j_1}, and i for j_1. The identity to prove becomes precisely the one in Lemma 4.3 (3).

In the special case $\delta_{s-1}=\pm\beta_{j_1}$, we need to correct the identity above by adding $\mathrm{sgn}(\delta_{s-1})$ to its right-hand side. This is proved in a similar way, based on the correction of Lemma 4.3 (3) in the case $\alpha=\beta$, which amounts to adding 1 to the right-hand side of the corresponding formula. □

Corollary 5.5. *Let $\Gamma=\Gamma(J)$ for some $J=\{j_1<j_2<\ldots<j_s\}\subseteq I$. Using the notation above, we have*

$$\widehat{t}_{j_1}=\widehat{r}_{j_1},\;\widehat{t}_{j_2}=\widehat{r}_{j_1}\widehat{r}_{j_2}\widehat{r}_{j_1},\;\widehat{t}_{j_3}=\widehat{r}_{j_1}\widehat{r}_{j_2}\widehat{r}_{j_3}\widehat{r}_{j_2}\widehat{r}_{j_1},\;\ldots$$

In particular, we have

$$\widehat{r}_{j_1}\ldots\widehat{r}_{j_i}=(\widehat{t}_{j_1}\ldots\widehat{t}_{j_i})^{-1}\,,$$

for $i=1,\ldots,s$. Furthermore, if $\Gamma'=\phi_i(\Gamma)$, then $\mu(\Gamma')=\widehat{t}_i(\mu(\Gamma))$.

Proof. The first part follows from Proposition 5.4 by applying the following basic result: if $H_{\beta,m} = \widehat{r}_1 \ldots \widehat{r}_q(H_{\alpha,k})$ for some affine reflections $\widehat{r}_1, \ldots, \widehat{r}_q$, then $s_{\beta,m} = \widehat{r}_1 \ldots \widehat{r}_q s_{\alpha,k} \widehat{r}_q \ldots \widehat{r}_1$. The rest of the corollary follows easily from the first part. □

The next proposition shows that all inner products of $\mu(\Gamma)$ with positive roots can be easily read off from the level sequence $L(\Gamma) = (l_i)_{i \in I}$. Recall that, according to (5.3), the latter is computed by applying a simple counting procedure to the sequence of pairs of roots in Γ.

Proposition 5.6. *Given a positive root α, let $m := \max I_\alpha(\Gamma)$, assuming that $I_\alpha(\Gamma) \neq \emptyset$. Then we have*

$$\langle \mu(\Gamma), \alpha^\vee \rangle = \begin{cases} l_m + 1 & \text{if } \gamma'_m > 0 \text{ and } t_{j_1} \ldots t_{j_s}(\alpha) > 0 \\ l_m - 1 & \text{if } \gamma'_m < 0 \text{ and } t_{j_1} \ldots t_{j_s}(\alpha) < 0 \\ l_m & \text{otherwise}. \end{cases}$$

On the other hand, if $I_\alpha(\Gamma) = \emptyset$, then we have

$$\langle \mu(\Gamma), \alpha^\vee \rangle = \begin{cases} 0 & \text{if } t_{j_1} \ldots t_{j_s}(\alpha) > 0 \\ -1 & \text{if } t_{j_1} \ldots t_{j_s}(\alpha) < 0 . \end{cases}$$

Proof. We prove this result by induction on s, which starts at $s = 0$. If $s > 0$, we assume that the result holds for $\Gamma' := \Gamma(J \setminus \{i\})$, where $i := j_1$. Let

$$\beta := \beta_i\,, \quad \sigma := \operatorname{sgn}(s_\beta(\alpha))\,, \quad \mu := \mu(\Gamma)\,, \quad \mu' := \mu(\Gamma')\,, \quad L' = (l'_j)_{j \in I} := L(\Gamma')\,.$$

We have

$$\begin{aligned} \langle \mu, \alpha^\vee \rangle &= \langle \widehat{r}_i(\mu'), \alpha^\vee \rangle = \langle s_\beta(\mu'), \alpha^\vee \rangle + l_i^\emptyset \langle \beta, \alpha^\vee \rangle \\ &= \sigma \, \langle \mu', |s_\beta(\alpha)|^\vee \rangle + l_i^\emptyset \langle \beta, \alpha^\vee \rangle\,. \end{aligned} \tag{5.4}$$

Recall that $t_{j_1} \ldots t_{j_s} = r_{j_s} \ldots r_{j_1}$, by (5.2). We have the following cases.

Case 1.1: $I_\alpha(\Gamma) \neq \emptyset$ and $m \geq i$. We have $m = \max I_\alpha(\Gamma) = \max I_{s_\beta(\alpha)}(\Gamma')$. Based on (5.4) and induction, we have

$$\langle \mu, \alpha^\vee \rangle = \sigma \, (l'_m + \varepsilon') + l_i^\emptyset \langle \beta, \alpha^\vee \rangle\,,$$

where $\varepsilon' \in \{-1, 0, 1\}$ is the correction term in the formula for $\langle \mu', |s_\beta(\alpha)|^\vee \rangle$. On the other hand, an easy linear algebra computation shows that $\widehat{r}_i$ maps the affine hyperplane $H_{|s_\beta(\alpha)|, l'_m}$ to

$$H_{\sigma\alpha, l'_m - l_i^\emptyset \langle \beta, |s_\beta(\alpha)|^\vee \rangle} = H_{\alpha, \sigma l'_m + l_i^\emptyset \langle \beta, \alpha^\vee \rangle}\,.$$

But then, by Proposition 5.4, we have $l_m = \sigma l'_m + l_i^\emptyset \langle \beta, \alpha^\vee \rangle$. Hence, all we have to prove is that $\sigma\varepsilon' = \varepsilon$, where ε is the correction term in the formula for $\langle \mu, \alpha^\vee \rangle$. This follows from the fact that, if (γ, γ') is the pair of roots indexed by m in Γ', then we have

$$\operatorname{sgn}(s_\beta(\gamma')) = \sigma \operatorname{sgn}(\gamma')\,, \quad \operatorname{sgn}(r_{j_s} \ldots r_{j_1}(\alpha)) = \sigma \operatorname{sgn}(r_{j_s} \ldots r_{j_2}(|s_\beta(\alpha)|))\,.$$

Case 1.2: $I_\alpha(\Gamma) \neq \emptyset$ and $m < i$. It is clear that $\alpha \neq \beta$. We have two subcases, corresponding to $I_{s_\beta(\alpha)}(\Gamma')$ nonempty and empty. In the first subcase, letting $m' := \max I_{s_\beta(\alpha)}(\Gamma')$, we have $m' < i$, and, thus, $l_{m'} = N_i(|s_\beta(\alpha)|) - 1$; here the counting function N_i refers to our fixed λ-chain. Based on (5.4), induction, and Lemma 4.3 (3), we have

$$\begin{aligned} \langle \mu, \alpha^\vee \rangle &= \sigma \, (N_i(|s_\beta(\alpha)|) - 1 + \varepsilon') + \langle \beta, \alpha^\vee \rangle N_i(\beta) \\ &= N_i(\alpha) - \frac{\sigma + 1}{2} + \sigma\varepsilon'\,, \end{aligned} \tag{5.5}$$

where ε' is as above. On the other hand, note that $l_m = N_i(\alpha) - 1$, so we have to prove

$$\langle \mu, \alpha^\vee \rangle = N_i(\alpha) - 1 + \varepsilon\,. \tag{5.6}$$

Hence, it remains to show that $(1 - \sigma)/2 + \sigma\varepsilon' = \varepsilon$, where ε is as above. This follows from the fact that

$$\varepsilon = \begin{cases} 1 & \text{if } r_{j_s} \ldots r_{j_1}(\alpha) > 0 \\ 0 & \text{otherwise}, \end{cases} \qquad \varepsilon' = \begin{cases} 1 & \text{if } \sigma \, r_{j_s} \ldots r_{j_1}(\alpha) > 0 \\ 0 & \text{otherwise}. \end{cases} \tag{5.7}$$

The subcase $I_{s_\beta(\alpha)}(\Gamma') = \emptyset$ is now immediate, since (5.5) still holds, with ε' defined as in (5.7); indeed, in this case the induction is based on the second formula in the proposition to be proved.

Case 2: $I_\alpha(\Gamma) = \emptyset$. This case is easily reduced to Case 1.2 above, since the formula to be proved can still be written as in (5.6), with ε defined as in (5.7). □

Remark 5.7. Since $\gamma_\infty = r_{j_1} \dots r_{j_s}(\rho)$, we have $\operatorname{sgn}(t_{j_1} \dots t_{j_s}(\alpha)) = \operatorname{sgn}(\langle \gamma_\infty, \alpha^\vee \rangle)$. Hence, Proposition 5.6 can be restated in terms of $\langle \gamma_\infty, \alpha^\vee \rangle$. Furthermore, if $\widehat{I}_\alpha = \{i_1 < i_2 < \dots < i_n = m < i_{n+1} = \infty\}$, we can extend the definition of the function g_α in Remark 5.2 to the interval $[0, n+\frac{1}{2}]$ in order to express $l_\alpha^\infty := \langle \mu(\Gamma), \alpha^\vee \rangle$, as given by Proposition 5.6. More precisely, letting $g'_\alpha(x) = \operatorname{sgn}(\langle \gamma_\infty, \alpha^\vee \rangle)$ for $x \in (n, n+\frac{1}{2})$, we have $l_\alpha^\infty = g_\alpha(n+\frac{1}{2})$.

6. Admissible Subsets and Admissible Foldings

We will now define some special foldings $\Gamma(J)$ of $\Gamma(\emptyset)$. The notation is the same as in Section 5.

Definition 6.1. An *admissible subset* is a finite subset of I (possibly empty), that is, $J = \{j_1 < j_2 < \dots < j_s\}$, such that we have the following saturated chain in the Bruhat order on W:

$$1 \lessdot r_{j_1} \lessdot r_{j_1} r_{j_2} \lessdot \dots \lessdot r_{j_1} r_{j_2} \dots r_{j_s}\,.$$

If J is an admissible subset, we will call $\Gamma = \Gamma(J)$ an *admissible folding* (of $\Gamma(\emptyset)$). We denote by $\mathcal{A}$ the collection of all admissible subsets corresponding to our fixed λ-chain.

According to (5.2), we have the following intrinsic criterion for $\Gamma(J)$ to be an admissible folding.

Corollary 6.2. *Given arbitrary $J = \{j_1 < j_2 < \dots < j_s\} \subseteq I$, $\Gamma(J)$ is an admissible folding if and only if*

$$1 \lessdot t_{j_1} \lessdot t_{j_1} t_{j_2} \lessdot \dots \lessdot t_{j_1} t_{j_2} \dots t_{j_s}\,.$$

In this section, we will prove some results about admissible foldings. Let us fix an admissible subset $J = \{j_1 < j_2 < \dots < j_s\}$, and let Γ be the associated admissible folding; also, let $L(\Gamma) = (l_i)_{i \in I}$, $I_\alpha = I_\alpha(\Gamma)$, and $L_\alpha = L_\alpha(\Gamma)$. We start with a basic result involving the Bruhat order.

Lemma 6.3. *Let $u \lessdot w = s_\beta u$ be a covering relation in the Bruhat order, where β is some positive root, and α is a simple root. Assume that we have $u(\alpha) > 0$ and $w(\alpha) < 0$. Then $\beta = u(\alpha)$.*

Proof. For a simple root α, the following conditions are equivalent: (1) $w(\alpha) < 0$; (2) $ws_\alpha < w$; (3) there exists a reduced decomposition for w that ends with s_α. Let us pick such a reduced decomposition $w = s_{i_1} \dots s_{i_l}$, so $\alpha = \alpha_{i_l}$. All elements u that are covered by w in the Bruhat order are obtained by skipping one term in the reduced decomposition. We know that $u(\alpha) > 0$, so the element u cannot have a reduced decomposition that ends with $s_\alpha = s_{i_l}$. Thus u is obtained from w by skipping the last term s_{i_l} in the reduced decomposition. So $w = us_\alpha = s_\beta u$, where $\beta = u(\alpha)$. □

Lemma 6.4. *Assume that $r_{j_a} \dots r_{j_1}(\alpha) > 0$ and $r_{j_b} \dots r_{j_1}(\alpha) < 0$ for some simple root α and $0 \le a < b$ (if $a = 0$, then the first condition is void). Then there exists i with $a \le i < b$ such that $\gamma_{j_{i+1}} = \alpha$.*

Proof. Find i with $a \le i < b$ such that $r_{j_i} \dots r_{j_1}(\alpha) > 0$ and $r_{j_{i+1}} \dots r_{j_1}(\alpha) < 0$. By Lemma 6.3, we have $\beta_{j_{i+1}} = r_{j_i} \dots r_{j_1}(\alpha)$. This means that $\gamma_{j_{i+1}} = r_{j_1} \dots r_{j_i}(\beta_{j_{i+1}}) = \alpha$. □

Proposition 6.5. *If $\gamma_i \ne \gamma'_i$ (that is, if $\gamma_i = -\gamma'_i$), then $\gamma_i > 0$ (and thus $\gamma'_i < 0$).*

Proof. We can assume that $i = j_s$. Since $\gamma_i = w(\beta_i)$, where $w = r_{j_1} \dots r_{j_{s-1}}$, we need to prove that the right-hand side is a positive root. This follows from the fact that $w < ws_{\beta_i}$, according to a well-known property of the Bruhat order on a Coxeter group, cf. [Hu, Proposition 5.7]. □

Proposition 6.6. *Assume that α is a simple root for which $I_\alpha \neq \emptyset$. Let $m \in I_\alpha$ be either the minimum of I_α, or an element for which its predecessor k (in I_α) satisfies $(\gamma_k, \gamma'_k) = (\alpha, \alpha)$. Then we have $\gamma_m = \alpha$.*

Proof. Assume that $\gamma_m = -\alpha$. Recall that the corresponding admissible subset is $J = \{j_1 < j_2 < \ldots < j_s\}$. Note that $m \notin J$, based on Proposition 6.5 (indeed, we must have $\gamma'_m = -\alpha$). Let us define the index b by the condition $j_b < m < j_{b+1}$ (possibly, $b + 1 = s$, in which case the second inequality is dropped). We also define the index a by setting $a := 0$ if $m = \min I_\alpha$, and by the condition $j_a < k < j_{a+1}$, otherwise (if $a = 0$ in the second case, the corresponding first inequality is dropped). We clearly have $r_{j_1} \ldots r_{j_b}(\beta_m) = -\alpha$, which implies $r_{j_b} \ldots r_{j_1}(\alpha) < 0$. If $a \neq 0$, we also have $r_{j_1} \ldots r_{j_a}(\beta_k) = \alpha$, so $r_{j_a} \ldots r_{j_1}(\alpha) > 0$. But then Lemma 6.4 applies and leads to a contradiction. □

Proposition 6.7. *Assume that, for some simple root α, we have either $I_\alpha = \emptyset$, or $(\gamma_m, \gamma'_m) = (\alpha, \alpha)$ for $m = \max I_\alpha$. Then we have $\langle \gamma_\infty, \alpha^\vee \rangle > 0$.*

Proof. Assume that the conclusion fails, which means that $r_{j_s} \ldots r_{j_1}(\alpha) < 0$ (cf. Remark 5.7). Define the index a by setting $a := 0$ if $I_\alpha = \emptyset$, and by the condition $j_a < m < j_{a+1}$, otherwise (if $a = 0$ or $a = s$ in the second case, one of the two inequalities is dropped). If $a \neq 0$, we have $r_{j_1} \ldots r_{j_a}(\beta_m) = \alpha$, so $r_{j_a} \ldots r_{j_1}(\alpha) > 0$. It means that the hypotheses of Lemma 6.4 are satisfied for $b := s$. This lemma now leads to a contradiction. □

Proposition 6.8. *Given $\Gamma = \phi_{j_1} \ldots \phi_{j_s}(\Gamma(\emptyset))$ and $\mu = \mu(\Gamma)$, the following hold.*

1. *If $s = 0$, then $\max L_\alpha < \langle \mu, \alpha^\vee \rangle$ for all simple roots α with $I_\alpha \neq \emptyset$.*
2. *If $s > 0$, then there is a simple root α in Γ such that $I_\alpha \neq \emptyset$ and $\max L_\alpha > \langle \mu, \alpha^\vee \rangle$.*

Proof. The first part follows directly from the definitions. If $s > 0$, we can find a simple root α such that $r_{j_s} \ldots r_{j_1}(\alpha) < 0$. Hence, Proposition 6.7 applies, and, letting $m := \max I_\alpha$, we have $\gamma'_m = -\alpha$. But then $\langle \mu, \alpha^\vee \rangle = l_m - 1$, by Proposition 5.6, so the second statement is verified. □

Let us now fix a simple root α, and recall the description of the sets $\widehat{I}_\alpha(\Gamma)$ and $\widehat{L}_\alpha(\Gamma)$ based on the continuous piecewise-linear function g_α, which was introduced in Remarks 5.2 and 5.7. We will rephrase some of the above results in a simple way in terms of g_α, and we will deduce some consequences (to be used in the subsequent sections), which are formulated in the same language. Assume that $I_\alpha = \{i_1 < i_2 < \ldots < i_n\}$, so that g_α is defined on $[0, n + \frac{1}{2}]$, and let M be the maximum of g_α. Note first that the function g_α is determined by the sequence $(\sigma_1, \ldots, \sigma_{n+1})$, where $\sigma_j := (\mathrm{sgn}(\gamma_{i_j}), \mathrm{sgn}(\gamma'_{i_j}))$ for $1 \leq j \leq n$, and $\sigma_{n+1} := \mathrm{sgn}(\langle \gamma_\infty, \alpha^\vee \rangle)$. We have the following restrictions.

(C1) $\sigma_j \in \{(1, 1), (-1, -1), (1, -1)\}$ for $1 \leq j \leq n$ (by Proposition 6.5).
(C2) $j = 0$ or $\sigma_j = (1, 1)$ implies $\sigma_{j+1} \in \{(1, 1), (1, -1), 1\}$ (by Propositions 6.6 and 6.7).

Corollary 6.9. *We have $M \geq 0$. If $g_\alpha(x) = M$, then $x = m + \frac{1}{2}$ for $0 \leq m \leq n$, and $\sigma_{m+1} \in \{(1, -1), 1\}$.*

Proof. The fact that $M \geq 0$ follows from Condition (C2) for $j = 0$. Thus, $g_\alpha(0) = -\frac{1}{2} \neq M$. If $g_\alpha(m) = M$ for $m \in \{1, \ldots, n\}$, then $\sigma_m = (1, 1)$ by Condition (C1). But then Condition (C2) leads to a contradiction. The last statement is obvious. □

Corollary 6.10. *Assume that $M > 0$, and let m be such that $m + \frac{1}{2} = \min g_\alpha^{-1}(M)$. We have $m > 0$, $\sigma_m = (1, 1)$, and $g_\alpha(m - \frac{1}{2}) = M - 1$. Moreover, we have $g_\alpha(x) \leq M - 1$ for $0 \leq x \leq m - \frac{1}{2}$.*

Proof. If $m = 0$, then $g_\alpha(m + \frac{1}{2}) = 0$, so $M = 0$. It is also easy to see that both $\sigma_m = (-1, -1)$ and $\sigma_m = (1, -1)$ contradict the definition of m. Now assume that the last statement fails. Then there exists $1 \leq k \leq m - 1$ such that $g_\alpha(k - 1) = M - \frac{1}{2}$ and $\sigma_k = (-1, -1)$. Condition (C2) implies $k \geq 2$ and $\sigma_{k-1} \in \{(-1, -1), (1, -1)\}$, which is a contradiction to the definition of m. □

The next corollary can be proved in a similar way to Corollary 6.10.

Corollary 6.11. *Assume that* $M > g_\alpha(n+\frac{1}{2})$*, and let* k *be such that* $k-\frac{1}{2} = \max g_\alpha^{-1}(M)$*. We have* $k \le n$*,* $\sigma_{k+1} = (-1,-1)$*, and* $g_\alpha(k+\frac{1}{2}) = M-1$*. Moreover, we have* $g_\alpha(x) \le M-1$ *for* $k+\frac{1}{2} \le x \le n+\frac{1}{2}$*.*

7. Root Operators

We will now define *root operators* on the collection $\mathcal{A}$ of admissible subsets corresponding to our fixed λ-chain. Let J be such an admissible subset, let Γ be the associated admissible folding, and $L(\Gamma) = (l_i)_{i\in I}$ its level sequence, denoted as in Section 5.

We will first define a partial operator F_p on admissible subsets J for each p in $\{1,\dots,r\}$, that is, for each simple root α_p. Let p in $\{1,\dots,r\}$ be fixed throughout this section. Let $M = M(\Gamma) = M(\Gamma,p) = M(J,p)$ be the maximum of the finite set of integers $\widehat{L}_{\alpha_p}(\Gamma)$. We know that $M \ge 0$ from Corollary 6.9. Assume that $M > 0$. Let $m = m_F(\Gamma) = m_F(\Gamma,p)$ be the minimum index i in $I_{\alpha_p}(\Gamma)$ for which we have $l_i = M$. If no such index exists, then $M = \langle \mu(\Gamma), \alpha_p^\vee\rangle$; in this case, we let $m = m_F(\Gamma) = m_F(\Gamma,p) := \infty$. Now let $k = k_F(\Gamma) = k_F(\Gamma,p)$ be the predecessor of m in $\widehat{I}_{\alpha_p}(\Gamma)$. By Corollary 6.10, this always exists and we have $l_k = M-1 \ge 0$.

Let us now define

$$F_p(J) := \phi_k \phi_m(J)\,, \tag{7.1}$$

where ϕ_∞ is the identity map. Note that the folding of $\Gamma(\emptyset)$ associated to $F_p(J)$, which will be denoted by $F_p(\Gamma) = (\{(\delta_i,\delta_i')\}_{i\in I}, \delta_\infty)$, is defined by a similar formula. More precisely, we have

$$(\delta_i,\delta_i') = \begin{cases} (\gamma_i,\gamma_i') & \text{if } i<k \text{ or } i>m \\ (\gamma_i, s_p(\gamma_i')) & \text{if } i=k \\ (s_p(\gamma_i), s_p(\gamma_i')) & \text{if } k<i<m \\ (s_p(\gamma_i),\gamma_i') & \text{if } i=m\,, \end{cases}$$

and

$$\delta_\infty = \begin{cases} \gamma_\infty & \text{if } m\neq\infty \\ s_p(\gamma_\infty) & \text{if } m=\infty\,. \end{cases}$$

In other words, based on Proposition 7.1 below, we can say that applying the root operator F_p amounts to performing a "folding" in position k, and, if $m\neq\infty$, an "unfolding" in position m.

Proposition 7.1. *Given the above setup, the following hold.*

1. *If* $m\neq\infty$*, then* $\gamma_m' = -\gamma_m = -\alpha_p$*.*
2. *We have* $\gamma_k = \gamma_k' = \alpha_p$*.*
3. *We have* $\mu(F_p(J)) = \mu(J) - \alpha_p$*.*

Proof. Let $\mu = \mu(\Gamma)$. The first two statements follow immediately from Corollaries 6.9 and 6.10. For the third statement, note that, by Corollary 5.5, the weight of $F_p(J)$ is $\widehat{t}_k\widehat{t}_m(\mu)$ if $m\neq\infty$, and $\widehat{t}_k(\mu)$ otherwise. Using the formula $\widehat{t}_k(\nu) = s_p(\nu) + l_k\alpha_p$, and the similar one for $\widehat{t}_m$, we compute (in both cases)

$$\mu(F_p(J)) = \mu + (l_k - M)\alpha_p = \mu - \alpha_p\,.$$

□

We now intend to define a partial inverse E_p to F_p. Assume that $M > \langle\mu(\Gamma),\alpha_p^\vee\rangle$. Let $k = k_E(\Gamma) = k_E(\Gamma,p)$ be the maximum index i in $I_{\alpha_p}(\Gamma)$ for which we have $l_i = M$. Note that such indices always exist, by Corollary 6.11. Now let $m = m_E(\Gamma) = m_E(\Gamma,p)$ be the successor of k in $\widehat{I}_{\alpha_p}(\Gamma)$. Corollary 6.11 implies that, if $m=\infty$, then we have $\langle\mu(\Gamma),\alpha_p^\vee\rangle = M-1$, while, otherwise, we have $l_m = M-1$. Finally, we define $E_p(J)$ by the same formula as $F_p(J)$, namely (7.1). Hence, the folding of $\Gamma(\emptyset)$ associated to $E_p(J)$ is also defined in the same way as above. The following analog of Proposition 7.1 is proved in a similar way, by invoking Corollaries 6.9 and 6.11.

Proposition 7.2. *Given the above setup, the following hold.*

(1) *We have* $\gamma'_k = -\gamma_k = -\alpha_p$.
(2) *If* $m \neq \infty$, *then* $\gamma_m = \gamma'_m = -\alpha_p$.
(3) *We have* $\mu(E_p(J)) = \mu(J) + \alpha_p$.

Proposition 7.3. *If* $F_p(J)$ *is defined, then it is also an admissible subset. Similarly for* $E_p(J)$. *Moreover, if* $F_p(J)$ *and* $E_p(J)$ *are both defined, then* $w(F_p(J)) = w(J)$.

Proof. We consider F_p first. The cases corresponding to $m \neq \infty$ and $m = \infty$ can be proved in similar ways, so we only consider the first case. Let $J = \{j_1 < j_2 < \ldots < j_s\}$, as usual, and, based on Proposition 7.1 (1)-(2), let $a < b$ be such that

$$j_a < k < j_{a+1} < \ldots < j_b = m < j_{b+1}\,;$$

if $a = 0$ or $b+1 > s$, then the corresponding indices j_a, respectively j_{b+1}, are missing. If $a+1 = b$, there is nothing to prove, so we assume $a + 1 < b$.

We use the criterion in Corollary 6.2. Based on it, we have

$$1 \lessdot t_{j_1} \lessdot t_{j_1}t_{j_2} \lessdot \ldots \lessdot t_{j_1}t_{j_2}\ldots t_{j_s}\,. \tag{7.2}$$

Let $w := t_{j_1}\ldots t_{j_a}$; if $a = 0$, then set $w := 1$. We need to prove that

$$w \lessdot ws_p \lessdot ws_p t'_{j_{a+1}} \lessdot \ldots \lessdot ws_p t'_{j_{a+1}}\ldots t'_{j_{b-1}} = t_{j_1}\ldots t_{j_b}\,, \tag{7.3}$$

where $t'_{j_i} := s_p t_{j_i} s_p$ for $i = a+1, \ldots, b-1$. Indeed, the pair of roots with index i in $F_p(\Gamma)$ is $(s_p(\gamma_i), s_p(\gamma'_i))$, for $k < i < m$. Note that (7.3) also implies that $w(F_p(J)) = w(J)$.

Recall that $r_{j_1}\ldots r_{j_i} = t_{j_i}\ldots t_{j_1}$, by (5.2). On the other hand, based on Proposition 7.1 (2), we have $r_{j_1}\ldots r_{j_a}(\beta_k) = \alpha_p$, which implies $w(\alpha_p) > 0$. Hence, we have $w \lessdot ws_p$, which gives us the first covering relation in (7.3). But we also have $w \lessdot wt_{j_{a+1}}$, by (7.2), and $t_{j_{a+1}} \neq s_p$, by the choice of k. Based on a well-known property of the Bruhat order on a Coxeter group [De, Theorem 1.1 (IV) (iii)], we deduce that $wt_{j_{a+1}} \lessdot wt_{j_{a+1}}s_p$ and $ws_p \lessdot wt_{a_{j+1}}s_p = ws_p t'_{a_{j+1}}$. The latter statement gives us the second covering relation in (7.3). We can proceed in this way until we get the whole chain (7.3). The proof of the result for $E_p(J)$ is completely similar, based on Proposition 7.2. □

Proposition 7.4. (1) *Assume that* $F_p(J)$ *is defined. Then we have*

$$M(F_p(\Gamma)) = M(\Gamma) - 1 > \langle \mu(F_p(\Gamma)), \alpha_p^\vee \rangle, \text{ and}$$
$$k_E(F_p(\Gamma)) = k_F(\Gamma), \;\; m_E(F_p(\Gamma)) = m_F(\Gamma)\,.$$

Hence, $E_p(F_p(J))$ *is defined and equal to* J.

(2) *Assume that* $E_p(J)$ *is defined. Then we have*

$$M(E_p(\Gamma)) = M(\Gamma) + 1 > 0, \text{ and}$$
$$k_F(E_p(\Gamma)) = k_E(\Gamma), \;\; m_F(E_p(\Gamma)) = m_E(\Gamma)\,.$$

Hence $F_p(E_p(J))$ *is defined and equal to* J.

(3) *Let* a *be maximal such that* $F_p^a(J)$ *is defined, and let* b *be maximal such that* $E_p^b(J)$ *is defined. Then* $a - b = \langle \mu, \alpha_p^\vee \rangle$.

The proof is straightforward based on the results already proved in this section, and is left to the interested reader. We note that that it is convenient to use the graphical representation described in Remarks 5.2 and 5.7.

8. Admissible Subsets Form a Semiperfect Crystal

In this section, we derive our main result. We start with the following definitions and lemma. Given an admissible folding $\Gamma = \Gamma(J)$, denoted as in Section 5, we denote by $\Gamma|_{\geq i}$ the sequence indexed by $j \geq i$ given by $i \mapsto \gamma_i'$ and $j \mapsto (\gamma_j, \gamma_j')$ for $j > i$. Let us define

$$\varepsilon(J,p) = \varepsilon(\Gamma,p) := M(J,p)\,, \quad \delta(J,p) = \delta(\Gamma,p) := \langle \mu(J), \alpha_p^\vee \rangle - M(J,p)\,.$$

Lemma 8.1. *Let Δ and Γ be two admissible foldings, with $L(\Delta) = (l_j)_{j\in I}$ and $L(\Gamma) = (l_j')_{j\in I}$. Assume that $w(\Delta) = w(\Gamma)$, that $E_p(\Delta)$ is defined, and that $\Delta_{\geq i} = \Gamma|_{\geq i}$ for $i = k_E(\Delta, p)$. Then $\delta(\Gamma,p) \leq \delta(\Delta,p)$. Moreover, equality holds if and only if $l_i' = \max L_{\alpha_p}(\Gamma)$.*

Proof. Let $j_1,\ldots,j_s$ and $j_1',\ldots,j_t'$ be the folding positions of Δ and Γ, respectively, where

$$\begin{aligned} &j_1 < \ldots < j_a = i < j_{a+1} < \ldots < j_s\,,\\ &j_1' < \ldots < j_b' \leq i < j_{b+1}' = j_{a+1} < \ldots < j_t' = j_s\,. \end{aligned}$$

Note that the pair of roots in Δ indexed by i is $(\alpha_p, -\alpha_p)$, by Proposition 7.2 (1). Hence, we have $r_{j_1}\ldots r_{j_a}(\beta_i) = r_{j_1'}\ldots r_{j_b'}(\beta_i) = -\alpha_p$, so $r_{j_s}\ldots r_{j_1}(\alpha_p) = r_{j_t'}\ldots r_{j_1'}(\alpha_p)$. By Proposition 5.6, we have

$$\begin{aligned} \delta(\Delta,p) &= \langle \mu(\Delta), \alpha_p^\vee\rangle - M(\Delta,p) = \langle \mu(\Delta), \alpha_p^\vee\rangle - l_i\\ &= \langle \mu(\Gamma), \alpha_p^\vee\rangle - l_i' \geq \langle \mu(\Gamma), \alpha_p^\vee\rangle - M(\Gamma,p) = \delta(\Gamma,p)\,. \end{aligned}$$

□

Recall that $\mathcal{A}$ is the collection of all admissible subsets corresponding to our fixed λ-chain.

Theorem 8.2. *The collection $\mathcal{A}$ of admissible subsets together with the root operators form a semiperfect crystal. Thus we have the following character formula:*

$$\chi(\lambda) = \sum_{J\in\mathcal{A}} e^{\mu(J)}\,.$$

Proof. As usual, we denote by J a generic element of $\mathcal{A}$, by μ its weight, and by $\Gamma = \Gamma(J)$ the corresponding admissible folding. Also, α_p will be a generic simple root.

Axiom (A1) follows from Corollary 6.9. Axiom (A2) is the content of Proposition 7.3 and Proposition 7.4 (1)-(2). Axiom (A3) is the content of Proposition 7.1 (3) and Proposition 7.4 (1).

According to Proposition 6.8, the only maximal object in our system is $\Gamma(\emptyset)$. Hence, Axiom (A5) is satisfied. Based on this, we will now check Axiom (A0). If Γ is such that $\mathrm{ht}(\mu) \geq h$, then apply the root operators E_p as long as we can. The uniqueness of the maximal object implies that we have to end up with $\Gamma(\emptyset)$. On the other hand, each application of E_p increases the height of the weight by 1, and either preserves the number of folding positions or decreases them. So the number of folding positions of Γ is bounded above by $\mathrm{ht}(\lambda) - h$. Thus, we are led to counting the admissible subsets with bounded size. There are a finite number of saturated chains in Bruhat order which start at the identity and have bounded length. Consider such a chain $1 \lessdot r_{j_1} \lessdot \ldots \lessdot r_{j_1}\ldots r_{j_s}$, represented as a sequence of roots $(\beta_{j_1},\ldots,\beta_{j_s})$. There are a finite number of ways to retrieve this sequence as a subsequence of our fixed λ-chain, since each root appears a finitely many times in the λ-chain. Thus Axiom (A0) is verified.

The definition of a coherent timing pattern and the related verification of Axiom (A4) are analogous to those for LS chains in [Ste, Theorem 8.3]; nevertheless, there are some features specifically related to our setup, such as the reversal of the total order on the set in which the timing pattern takes values, and the use of Proposition 5.6 in Lemma 8.1 (used below). Assume that $\delta(J,p) < 0$, so that $E_p(J)$ is defined. We define $t(J,p) = t(\Gamma,p) := k_E(\Gamma,p)$. Assuming that $F_p(J)$ is defined, and applying Proposition 7.4 (1), we have

$$t(\Gamma,p) = k_E(\Gamma,p) \geq m_F(\Gamma,p) > k_F(\Gamma,p) = k_E(F_p(\Gamma),p) = t(F_p(\Gamma),p)\,.$$

By iteration, it follows that $\Gamma'|_{\geq t(\Gamma,p)} = \Gamma|_{\geq t(\Gamma,p)}$ for $\Gamma' \preceq_p \Gamma$. Therefore, given $\Delta \succeq_q \Gamma$ for $q \neq p$ with $\delta(\Delta, q) = \delta < 0$ and $t = t(\Delta, q) \geq t(\Gamma, p)$, we have

$$\Delta|_{\geq t} = \Gamma|_{\geq t} = F_p(\Gamma)|_{\geq t} .$$

We have $w(\Delta) = w(\Gamma) = w(F_p(\Gamma))$, by Proposition 7.3. Hence, by Lemma 8.1, we have $\delta(F_p(\Gamma), q) \leq \delta$, so there exists $\Delta' \succeq_q F_p(\Gamma)$ such that $\delta(\Delta', q) = \delta$. We claim that $t(\Delta', q) = t(\Delta, q)$. Let $t' := t(\Delta', q)$ and $t_* := \max(t, t')$. We have

$$\Delta'|_{\geq t_*} = F_p(\Gamma)|_{\geq t_*} = \Gamma|_{\geq t_*} = \Delta|_{\geq t_*} .$$

Also note that $w(\Delta) = w(F_p(\Gamma)) = w(\Delta')$, by Proposition 7.3. Assume that $t < t'$. Then the fact that $\delta(\Delta, q) = \delta(\Delta', q)$ implies, based on Lemma 8.1, that the maximum of $L_{\alpha_q}(\Delta)$ is attained at t' as well; but this contradicts the definition of t. The case $t' < t$ is similar. On the other hand, similar reasoning proves conversely that given $\Delta' \succeq_q F_p(\Gamma)$ such that $\delta(\Delta', q) = \delta < 0$ and $t(\Delta', q) \geq t(\Gamma, p)$, there is $\Delta \succeq_q \Gamma$ such that $\delta(\Delta, q) = \delta$ and $t(\Delta, q) = t(\Delta', q)$.

The formula for $\chi(\lambda)$ now follows from Theorem 3.2. □

Corollary 8.3. *(Littlewood-Richardson rule). We have*

$$\chi(\lambda) \cdot \chi(\nu) = \sum \chi(\nu + \mu(J)) ,$$

where the summation is over all J in $\mathcal{A}$ satisfying $\langle \nu + \mu(J), \alpha_p^\vee \rangle \geq M(J, p)$ for all $p = 1, \ldots, r$.

Proof. This follows immediately from Theorem 3.2. There, the condition for J to contribute to the summation was $\langle \nu, \alpha_p^\vee \rangle + \delta(J, p) \geq 0$, which is easily seen to be equivalent to the condition stated above. □

Corollary 8.4. *(Branching rule). Given $P \subseteq \{1, \ldots, r\}$, we have the following rule for decomposing $\chi(\lambda)$ as a sum of Weyl characters relative to Φ_P:*

$$\chi(\lambda) = \sum \chi(\mu(J); P) ,$$

where the summation is over all J in $\mathcal{A}$ satisfying $\langle \mu(J), \alpha_p^\vee \rangle = M(J, p)$ for all $p \in P$.

Proof. Immediate, based on Corollary 3.3. □

9. Lakshmibai-Seshadri Chains

In this section, we explain the connection between our model and LS chains. We start with the relevant definitions.

The Bruhat order on the orbit $W\lambda$ of a dominant or antidominant weight is defined by

$$s_\alpha(\mu) < \mu \quad \text{if} \quad \langle \mu, \alpha^\vee \rangle > 0 \quad (\mu \in W\lambda,\, \alpha \in \Phi^+) .$$

The Bruhat orders on $W\lambda$ and $-W\lambda$ are dual isomorphic; in fact, $\mu < \nu$ if and only if $-\nu < -\mu$. As usual, we write $\nu \lessdot \mu$ to indicate that μ covers ν; this happens only if $\nu = s_\alpha(\mu)$ for some $\alpha \in \Phi^+$, but not conversely. Given $\pm\lambda \in \Lambda^+$ and a fixed real number b, one defines the *b-Bruhat order* $<_b$ as the transitive closure of the relations

$$s_\alpha(\mu) <_b \mu \quad \text{if} \quad s_\alpha(\mu) \lessdot \mu \;\text{ and }\; b\langle \mu, \alpha^\vee \rangle \in \mathbb{Z} \quad (\mu \in W\lambda,\, \alpha \in \Phi^+) .$$

Thus, μ covers ν in b-Bruhat order if and only if μ covers ν in the usual Bruhat order and $b(\mu - \nu)$ is an integer multiple of a root.

Definition 9.1. Given $\pm\lambda \in \Lambda^+$, we say that a pair consisting of a chain $\mu_0 < \mu_1 < \ldots < \mu_l$ in the W-orbit of λ and an increasing sequence of rational numbers $0 < b_1 < \ldots < b_l < 1$ is a *Lakshmibai-Seshadri chain* (LS chain) if

$$\mu_0 <_{b_1} \mu_1 <_{b_2} \ldots <_{b_l} \mu_l .$$

Following [Ste], we identify an LS chain (denoted as above) with the map $\gamma\,:\,(0,1]\to W\lambda$ given by $\gamma(t):=\mu_k$ for $b_k<t\le b_{k+1}$, where $k=0,\dots,l$ and $b_0:=0$, $b_{l+1}:=1$. Note that the piecewise-constant left-continuous maps that correspond to LS chains can be characterized by the property

$$\gamma(t)\le_t\gamma(t^+)\quad\text{for}\quad 0<t<1\,,$$

where $\gamma(t^+)$ denotes the right-hand limit of γ at t. To each LS chain γ, we associate the continuous piecewise-linear path $\pi\,:\,[0,1]\to\mathfrak{h}_{\mathbb{R}}^*$ given by

$$\pi(t):=\int_0^t\gamma(s)\,ds\,.$$

In other words, we define

$$\pi(t):=(t-b_k)\mu_k+\sum_{i=0}^{k-1}(b_{i+1}-b_i)\mu_i\,,$$

for $b_k\le t\le b_{k+1}$.

The root operators F_p and E_p on LS chains (for a simple root α_p) were defined by Littelmann [Li1, Li2] as follows. Let m_p be the minimum of the function $h_p\,:\,[0,1]\to\mathbb{R}$ given by $t\mapsto\langle\pi(t),\alpha_p^\vee\rangle$. If $m_p>-1$, then E_p is undefined. Otherwise, we let $t_1\in[0,1]$ be minimal such that $h_p(t_1)=m_p$, and we let $t_0\in[0,t_1]$ be maximal such that $h_p(t)\ge m_p+1$ for $t\in[0,t_0]$. We define

$$E_p(\gamma)(t):=\begin{cases} s_p(\gamma(t)) & \text{if } t_0<t\le t_1\\ \gamma(t) & \text{otherwise}\,.\end{cases}\tag{9.1}$$

The definition of F_p is similar. More precisely, if $h_p(1)-m_p<1$, then F_p is undefined. Otherwise, we let $t_0\in[0,1]$ be maximal such that $h_p(t_0)=m_p$, and we let $t_1\in[t_0,1]$ be minimal such that $h_p(t)\ge m_p+1$ for $t\in[t_1,1]$. Given the latter values for t_0 and t_1, we define $F_p(\gamma)$ by the same formula as $E_p(\gamma)$.

Remark 9.2. The above definition of E_p and F_p applies to any continuous piecewise-linear path π with $\pi(0)=0$, if we replace the map γ above with the left-hand derivative of π (which is a piecewise-constant left-continuous map defined on $(0,1]$).

With each LS chain γ (denoted as above), one can associate the *dual* LS chain γ^*, which is defined as

$$-\mu_l<_{a_l}\dots<_{a_2}-\mu_1<_{a_1}-\mu_0\,,$$

where $a_i:=1-b_i$. It is easy to see that $E_p(\gamma^*)=F_p(\gamma)^*$ and $F_p(\gamma^*)=E_p(\gamma)^*$.

The paths corresponding to LS chains are special cases of *Littelmann paths*. The latter were defined by Littelmann [Li2], still in the setup of complex symmetrizable Kac-Moody algebras. More precisely, Littelmann defined root operators E_p and F_p on continuous paths $\pi\,:\,[0,1]\to\mathfrak{h}_{\mathbb{R}}^*$ with $\pi(0)=0$. In fact, the operator E_p (respectively F_p) is defined as in Remark 9.2 if the function $h_p\,:\,[0,1]\to\mathbb{R}$ given by $t\mapsto\langle\pi(t),\alpha_p^\vee\rangle$ is weakly decreasing (respectively weakly increasing) between t_0 and t_1. In general, the definition is more involved; for convenience, we stated it in Section 10. However, only the simpler version of the definition is needed for an LS chain, since it is known that the corresponding function h_p satisfies the condition stated above. Littelmann considered the collection $\mathcal{P}_\lambda$ of all paths obtained by applying the operators F_p to a fixed continuous path from 0 to λ which lies inside the dominant Weyl chamber. He showed that these paths form a crystal, that the associated colored directed graph does not depend on the initial path, and that one can express

$$\chi(\lambda)=\sum_{\pi\in\mathcal{P}_\lambda}e^{\pi(1)}\,;\tag{9.2}$$

moreover, there is a corresponding Littlewood-Richardson rule [Li1, Li2, Li3]. Stembridge [Ste] reproved the special case of the above results corresponding to LS chains by showing that they form an admissible system. Kashiwara [Kas], Lakshmibai [La], and Joseph [Jos] proved independently that Littelmann paths (obtained from a fixed path via root operators) have the structure of a perfect crystal.

Let us now return to LS chains, and fix λ in Λ^+. Recall the set I in (4.1), and the λ-chain $(\beta_i)_{i\in I}$ given by Proposition 4.2, which depends on a total order on the set of simple roots $\alpha_1 < \cdots < \alpha_r$. We will now describe a bijection between the corresponding admissible subsets (cf. Definition 6.1) and the LS chains corresponding to the antidominant weight $-\lambda$.

Given an index $i = (\alpha, k)$, we let $\beta_i := \alpha$ and $t_i := k/\langle\lambda, \alpha^\vee\rangle$. We have an order-preserving map from I to $[0, 1)$ given by $i \mapsto t_i$. Recall the notation $r_i := s_{\beta_i}$ and $\widehat{r}_i := s_{\beta_i, l_i^\emptyset}$; we also let $\widehat{r}'_i := s_{\beta_i, -l_i^\emptyset}$. Consider an admissible subset $J = \{j_1 < j_2 < \ldots < j_s\}$ and let

$$\{0 = a_0 < a_1 < \ldots < a_l\} := \{t_{j_1} \leq t_{j_2} \leq \ldots \leq t_{j_s}\} \cup \{0\}\,.$$

Let $0 = n_0 \leq n_1 < \ldots < n_{l+1} = s$ be such that $t_{j_h} = a_k$ if and only if $n_k < h \leq n_{k+1}$, for $k = 0, \ldots, l$. Define Weyl group elements u_h for $h = 0, \ldots, s$ and w_k for $k = 0, \ldots, l$ by $u_0 := 1$, $u_h := r_{j_1} \ldots r_{j_h}$, and $w_k := u_{n_{k+1}}$. Let also $\mu_k := w_k(\lambda)$. For any $k = 1, \ldots, l$, we have the following saturated chain in Bruhat order of minimum (left) coset representatives modulo W_λ:

$$w_{k-1} = u_{n_k} \lessdot u_{n_k+1} \lessdot \ldots \lessdot u_{n_{k+1}} = w_k\,;$$

indeed, none of the reflections $r_{j_1}, \ldots, r_{j_s}$ lies in W_λ, since $\langle\lambda, \beta_i^\vee\rangle \neq 0$ for all $i \in I$. The above chain gives rise to a saturated increasing chain from $-\mu_{k-1}$ to $-\mu_k$ in the Bruhat order on $-W\lambda$. We will now show that this chain is, in fact, a chain in a_k-Bruhat order. Let $\widetilde{\beta}_h := u_{h-1}(\beta_{j_h})$, so that $u_h := s_{\widetilde{\beta}_h} u_{h-1}$, for $h = 1, \ldots, s$. We need to check that $a_k\langle u_{h-1}(\lambda), \widetilde{\beta}_h^\vee\rangle \in \mathbb{Z}$, for $n_k < h \leq n_{k+1}$. But

$$\langle u_{h-1}(\lambda), \widetilde{\beta}_h^\vee\rangle = \langle\lambda, \beta_{j_h}^\vee\rangle\,,$$

while, by definition, $a_k = t_{j_h}$ is a fraction with denominator $\langle\lambda, \beta_{j_h}^\vee\rangle$. Hence

$$-\mu_0 <_{a_1} -\mu_1 <_{a_2} \ldots <_{a_l} -\mu_l\,,$$

is an LS chain in the W-orbit of $-\lambda$. We denote this by $\gamma(J)$, and the associated continuous piecewise-linear path by $\pi(J)$.

Note that $\gamma(\emptyset)$ is the LS chain consisting only of $-\lambda$, while $\pi(\emptyset) : [0, 1] \to \mathfrak{h}_\mathbb{R}^*$ is the path $t \mapsto -t\,\lambda$. The path $\pi(\emptyset)$ intersects the affine hyperplane $H_{\beta_i, -l_i^\emptyset}$ at $t = t_i$ for $i \in I$; moreover, these and $t = 1$ are the only intersections of $\pi(\emptyset)$ with affine hyperplanes $H_{\alpha,k}$, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$. It is not hard to see that the path $\pi(J)$ can be described using folding operators as follows:

$$\pi(J) := \phi_{j_1} \ldots \phi_{j_s}(\pi(\emptyset))\,; \tag{9.3}$$

here, the folding operators ϕ_i are defined as follows on the relevant paths π:

$$\phi_i(\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \leq t \leq t_i \\ \widehat{r}'_i(\pi(t)) & \text{if } t_i < t \leq 1\,. \end{cases} \tag{9.4}$$

Remark 9.3. Based on (9.3), we can easily show that $\pi(J)(1) = -\mu(J)$ (cf. Definition 5.3). Indeed, we have $\widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(\lambda) = -\widehat{r}'_{j_1} \ldots \widehat{r}'_{j_s}(-\lambda)$.

Theorem 9.4. *We have*

$$E_p(\pi(J)) = \pi(F_p(J))$$

for all admissible subsets J (here E_p is the root operator for paths, while F_p in the one defined in Section 7).

Proof. Consider the point $P_\varepsilon := \varepsilon\,\omega_1 + \varepsilon^2\omega_2 + \cdots + \varepsilon^r\omega_r$, where ε is a small positive real number. Let $\pi_\varepsilon : [0, 1] \to \mathfrak{h}_\mathbb{R}^*$ be the path $t \mapsto -t\,\lambda + P_\varepsilon$. Given $i = (\alpha, k)$ in I and a sufficiently small ε, the path π_ε crosses the affine hyperplane $H_{\alpha,-k}$ at $t_{\varepsilon,i} := (k + \sum_{p=1}^r (\omega_p, \alpha^\vee)\,\varepsilon^p)/\langle\lambda, \alpha^\vee\rangle$. Fix a simple root α_p and an admissible subset $J = \{j_1 < j_2 < \ldots < j_s\}$. Let $\Gamma = \Gamma(J)$ be the corresponding admissible folding, and $L(\Gamma) = (l_i)_{i\in I}$ the corresponding level sequence. We can find $\varepsilon_0 < 1$ such that, for all $i < j$ in $J \cup I_{\alpha_p}(\Gamma)$ and $\varepsilon < \varepsilon_0$, the points $t_{\varepsilon,i}$, $t_{\varepsilon,j}$ exist, and we have $t_{\varepsilon,i} < t_{\varepsilon,j}$. Let us now extend this path by the segments from 0 to P_ε and from $-\lambda+P_\varepsilon$ to $-\lambda$. More precisely, we consider the path $\widehat{\pi}_\varepsilon : [0,1] \to \mathfrak{h}_{\mathbb{R}}^*$ given by

$$\widehat{\pi}_\varepsilon(t) := \begin{cases} tP_\varepsilon & \text{if } 0 \le t < \varepsilon \\ \pi_\varepsilon\left(\frac{t-\varepsilon}{1-2\varepsilon}\right) & \text{if } \varepsilon \le t \le 1-\varepsilon \\ -\lambda + \frac{1-t}{\varepsilon} P_\varepsilon & \text{if } 1-\varepsilon < t \le 1\,. \end{cases}$$

Since $\varepsilon_0 < 1$, the point $\widehat{\pi}_\varepsilon(t)$ does not lie on an affine hyperplane $H_{\alpha_p,k}$, for $k \in \mathbb{Z}$, whenever $t \in (0,\varepsilon] \cup [1-\varepsilon, 1)$ and $\varepsilon < \varepsilon_0$. From now on, we assume that $\varepsilon < \varepsilon_0$. Let $\widehat{t}_{\varepsilon,i}$ be the value of t in $[\varepsilon,\, 1-\varepsilon]$ for which $\widehat{\pi}_\varepsilon$ crosses the affine hyperplane $H_{\alpha,-k}$ for $i = (\alpha, k)$ in I (assuming that such a value exists). Clearly, it is still true that, for all $i < j$ in $J \cup I_{\alpha_p}(\Gamma)$, the points $\widehat{t}_{\varepsilon,i}$, $\widehat{t}_{\varepsilon,j}$ exist, and we have $\widehat{t}_{\varepsilon,i} < \widehat{t}_{\varepsilon,j}$.

Now consider the path

$$\widehat{\pi}_\varepsilon(J) := \phi_{j_1} \ldots \phi_{j_s}(\widehat{\pi}_\varepsilon)\,,$$

where the folding operators ϕ_i are defined as in (9.4), except that t_i is replaced by $\widehat{t}_{\varepsilon,i}$. It is easy to see that the only intersections of $\widehat{\pi}_\varepsilon(J)$ with affine hyperplanes $H_{\alpha_p,k}$, for $k \in \mathbb{Z}$, occur at $t = 0$, $t = 1$, and $t = \widehat{t}_{\varepsilon,i}$ for $i \in I_{\alpha_p}(\Gamma)$. Moreover, note that the signs of the pairs of roots $(\pm\alpha_p, \pm\alpha_p)$ in Γ, as well as the sign of $\langle \gamma_\infty, \alpha_p^\vee \rangle$ indicate the sides of the mentioned hyperplane on which the path $\widehat{\pi}_\varepsilon$ lies before and/or after the intersection. Corollary 6.9 shows that the minimum of the function $t \mapsto \langle \widehat{\pi}_\varepsilon(J)(t), \alpha_p^\vee \rangle$ is $-M(\Gamma, p)$; furthermore, this minimum is attained at $t = \widehat{t}_{\varepsilon,i}$ for $i \in I_{\alpha_p}(\Gamma)$ with $l_i = M(\Gamma, p)$, as well as at $t = 1$ if $l_{\alpha_p}^\infty = M(\Gamma, p)$. Let $k := k_F(\Gamma, p)$, $m := m_F(\Gamma, p)$, and assume $m \neq \infty$. We define $a < b$ by $j_a < k < j_{a+1}$ and $j_b = m$ (possibly $a = 0$, in which case the corresponding inequality is dropped). Let us define $E_p(\widehat{\pi}_\varepsilon(J))$ as in Remark 9.2. By Corollary 6.10, the corresponding points t_0 and t_1 are as follows:

$$t_0 = \widehat{t}_{\varepsilon,k}\,, \quad t_1 = \widehat{t}_{\varepsilon,m}\,.$$

Consider $i \in \{k < j_{a+1} < \ldots < j_{b-1}\}$, and let i' be its successor in $J \cup \{k\}$. Given a subset $A = \{a_1 < a_2 < \ldots < a_l\}$ of I, we will use the notation r_A for $r_{a_1} \ldots r_{a_l}$; we also set $r_\emptyset := 1$. Let $J' := \{j \in J \mid j < j_{a+1}\}$ and $J'' := \{j \in J \mid j_{a+1} \le j \le i\}$. By (9.1), the direction of $E_p(\widehat{\pi}_\varepsilon(J))$ for $t \in [\widehat{t}_{\varepsilon,i},\, \widehat{t}_{\varepsilon,i'}]$ is $s_p r_{J'} r_{J''}(-\lambda)$. Since $F_p(J) = (J \cup \{k\}) \setminus \{m\}$, the direction of $\widehat{\pi}_\varepsilon(F_p(J))$ for the same values of t is $r_{J'} r_k r_{J''}(-\lambda)$. But $r_{J'}(\beta_k) = \alpha_p$, so $s_p = r_{J'} r_k r_{J'}^{-1}$, and, therefore, the two directions above coincide. The directions of the two paths also coincide for $t \ge \widehat{t}_{\varepsilon,m}$, since $w(F_p(J)) = w(J)$ (cf. Proposition 7.3). The case $m = \infty$ is similar. All this shows that

$$E_p(\widehat{\pi}_\varepsilon(J)) = \widehat{\pi}_\varepsilon(F_p(J))\,. \tag{9.5}$$

Now let us take the limit as $\varepsilon \to 0$. Note that $\widehat{\pi}_\varepsilon(J)$ converges uniformly to $\pi(J)$, since $\widehat{t}_{\varepsilon,j_i}$ converges to t_{j_i}, for $i = 1, \ldots, s$; so the minimum of $\pi(J)$ is also $-M(\Gamma, p)$, and the points t_0, t_1 in the construction of $E_p(\pi(J))$ are precisely t_k, t_m (which are the limits of $\widehat{t}_{\varepsilon,k}$, $\widehat{t}_{\varepsilon,m}$). This implies that E_p commutes with limits, so the proposition follows by taking the limit in (9.5). □

Corollary 9.5. *The map $J \mapsto \gamma(J)$ is a bijection between the admissible subsets considered above and the LS chains corresponding to the antidominant weight $-\lambda$.*

Proof. Surjectivity follows directly from Theorem 9.4, based on the fact that all LS chains corresponding to $-\lambda$ can be obtained from the one consisting only of $-\lambda$ by applying the root operators F_p. Injectivity then follows from the character formula (9.2), since $\pi(J)(1) = -\mu(J)$, as noted in Remark 9.3. □

Remarks 9.6. (1) The proof of Theorem 9.4 contains the justification of the fact that the minima of the paths associated to LS chains are integers. This justification is based only on the combinatorics in Section 6. Note that the same fact was proved by Littelmann in [Li1] using different methods.

(2) The proof of Theorem 9.4 shows that LS chains can be viewed as a limiting case of a special case of our construction. The special choices of λ-chains that lead to LS chains represent a very small fraction of all possible choices.

Based on the independent results of Kashiwara [Kas], Lakshmibai [La], and Joseph [Jos], which were discussed above, we deduce the following corollary.

Corollary 9.7. *Given a complex symmetrizable Kac-Moody algebra $\mathfrak{g}$, consider the colored directed graph defined by the action of root operators (cf. Section 7) on the admissible subsets corresponding to the special choice of a λ-chain above. This graph is isomorphic to the crystal graph of the irreducible representation with highest weight λ of the associated quantum group $U_q(\mathfrak{g})$.*

We make the following conjecture, which is the analog of a result due to Littelmann, that was discussed above.

Conjecture 9.8. The colored directed graph defined by the action of root operators on the admissible subsets corresponding to any λ-chain does not depend on the choice of this chain.

This conjecture would imply that any choice of a λ-chain leads to a perfect crystal.

10. The Finite Case

In this section, we discuss the way in which the model in this paper specializes to the one in [LP1] in the case of finite irreducible root systems.

Let Φ be the root system of a simple Lie algebra. Let W_{aff} be the *affine Weyl group* for $\Phi^\vee$, that is, the group generated by the affine reflections $s_{\alpha,k}$ (defined in (2.1)). The corresponding affine hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_{\mathbb{R}}$ into open regions, called *alcoves.* The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

We say that two alcoves are adjacent if they are distinct and have a common wall. For a pair of adjacent alcoves, let us write $A \xrightarrow{\alpha} B$ if the common wall of A and B is of the form $H_{\alpha,k}$ and the root $\alpha \in \Phi$ points in the direction from A to B.

Definition 10.1. An *alcove path* is a sequence of alcoves $(A_0, A_1, \ldots, A_l)$ such that A_{j-1} and A_j are adjacent, for $j = 1, \ldots, l$. We say that an alcove path is *reduced* if it has minimal length l among all alcove paths from A_0 to A_l.

Let $A_\lambda = A_\circ + \lambda$ be the alcove obtained via the affine translation of the fundamental alcove $A_\circ$ by a weight λ. The reduced alcove paths from $A_\circ$ to A_λ are in bijection with the reduced decompositions of the element v_λ in W_{aff} defined by $v_\lambda(A_\circ) = A_\lambda$; see [LP1]. Let us fix a dominant weight λ.

Proposition 10.2. *The sequence of roots $\{\beta_i\}_{i\in I}$ with $I = \{1, \ldots, l\}$ is a λ-chain (cf. Definition 4.1) if and only if there exists a reduced alcove path $A_0 = A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l = A_{-\lambda}$.*

This proposition is an analog of the fact that the normal ordering of roots can be described in terms of dihedral subgroups. Each alcove A is given by the inequalities

$$A = \{v \in V \mid n_\alpha < \langle v, \alpha^\vee \rangle < n_\alpha + 1, \text{ for all roots } \alpha \in \Phi^+\},$$

where $n_\alpha = n_\alpha(A)$ are some integers. We need the following characterization due to Shi of the collection of integers $(n_\alpha)_{\alpha\in\Phi^+}$ associated with alcoves.

Proposition 10.3. [Shi] *An arbitrary collection of integers $(m_\alpha)_{\alpha\in\Phi^+}$ corresponds to some alcove A, i.e., $m_\alpha = n_\alpha(A)$ for all $\alpha \in \Phi^+$, if and only if, for any triple of roots $\alpha, \beta, \gamma \in \Phi^+$ such that $\gamma^\vee = \alpha^\vee + \beta^\vee$, we have $m_\gamma - m_\alpha - m_\beta \in \{0, 1\}$.*

Proof of Proposition 10.2. For a sequence of positive roots $(\beta_1, \ldots, \beta_l)$, define $m^i_\alpha := -\#\{j \le i \mid \beta_j = \alpha\}$, for $\alpha \in \Phi^+$ and $i = 0, \ldots, l$. The axioms of a λ-chain (Definition 4.1) can be rewritten in terms of the integers m^i_α as follows: (1) $0 = m^0_\alpha \ge m^1_\alpha \ge \cdots \ge m^l_\alpha = -\langle \lambda, \alpha^\vee \rangle$, for $\alpha \in \Phi^+$; and (2) for any triple

$\alpha, \beta, \gamma \in \Phi^+$ such that $\alpha^\vee + \beta^\vee = \gamma^\vee$ and $i = 0, \ldots, l$, we have $m^i_\gamma - m^i_\alpha - m^i_\beta \in \{0, 1\}$ (interlacing condition). Shi's result implies that these conditions are equivalent to the fact that $A_0 = A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l = A_{-\lambda}$ is a reduced alcove path, where A_i is the alcove associated with the collection of integers $(m^i_\alpha)_{\alpha \in \Phi^+}$, i.e., $n_\alpha(A_i) = m^i_\alpha$. □

Note that, in [LP1], (reduced) λ-chains were defined as chains of roots determined by a reduced alcove path. As we have seen, the mentioned definition is equivalent to the one in this paper.

Definition 10.4. A *gallery* is a sequence $\gamma = (F_0, A_0, F_1, A_1, F_2, \ldots, F_l, A_l, F_{l+1})$ such that $A_0, \ldots, A_l$ are alcoves; F_j is a codimension one common face of the alcoves A_{j-1} and A_j, for $j = 1, \ldots, l$; F_0 is a vertex of the first alcove A_0; and F_{l+1} is a vertex of the last alcove A_l. Furthermore, we require that $F_0 = \{0\}$, $A_0 = A_\circ$, and $F_{l+1} = \{\mu\}$ for some weight $\mu \in \Lambda$, which is called the *weight* of the gallery. The folding operator ϕ_j is the operator which acts on a gallery by leaving its initial segment from A_0 to A_{j-1} intact and by reflecting the remaining tail in the affine hyperplane containing the face F_j. In other words, we define

$$\phi_j(\gamma) := (F_0, A_0, F_1, A_1, \ldots, A_{j-1}, F'_j = F_j, A'_j, F'_{j+1}, A'_{j+1}, \ldots, A'_l, F'_{l+1}),$$

where $F_j \subset H_{\alpha,k}$, $A'_i := s_{\alpha,k}(A_i)$, and $F'_i := s_{\alpha,k}(F_i)$, for $i = j, \ldots, l+1$.

The galleries defined above are special cases of the generalized galleries in [GL].

Let us fix a reduced alcove path $A_0 = A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l = A_{-\lambda}$, which determines the λ-chain $\{\beta_i\}_{i \in I}$ with $I := \{1, \ldots, l\}$. The alcove path also determines an obvious gallery

$$\gamma(\emptyset) = (F_0, A_0, F_1, \ldots, F_l, A_l, F_{l+1})$$

of weight $-\lambda$. We use the same notation as in Sections 4-7. For instance, $r_i := s_{\beta_i}$ and $\widehat{r}_i := s_{\beta_i, l_i^\emptyset}$. We also let $\widehat{r}'_i$ be the affine reflection in the hyperplane containing F_i.

Definition 10.5. Given an admissible subset $J = \{j_1 < \cdots < j_s\} \subseteq I$ (cf. Definition 6.1), we define the gallery $\gamma(J)$ as $\phi_{j_1} \cdots \phi_{j_s}(\gamma(\emptyset))$, and call it an *admissible folding* of $\gamma(\emptyset)$.

Remark 10.6. The weight of the gallery $\gamma(J)$ is $-\mu(J)$ (cf. Definition 5.3). Indeed, we have $\widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(\lambda) = -\widehat{r}'_{j_1} \ldots \widehat{r}'_{j_s}(-\lambda)$. Hence, the model in this paper specializes to the one in [LP1], whose construction was based on the geometry of the generalized flag variety.

Since we assumed that Φ is irreducible, there is a unique *highest coroot* $\theta^\vee \in \Phi^\vee$, i.e., a unique coroot that has maximal height. The *dual Coxeter number* of $\Phi^\vee$ is $h^\vee := \langle \rho, \theta^\vee \rangle + 1$ (in the finite case, the dominant weight ρ considered at the end of Section 2 is unique, and is given by $\frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$). Let Z be the set of the elements of the lattice $\Lambda / h^\vee$ that do not belong to any affine hyperplane $H_{\alpha,k}$. Each alcove A contains precisely one element ζ_A of the set Z (cf. [Kos, LP1]); this will be called the *central point* of A. In particular, $\zeta_{A_\circ} = \rho / h^\vee$.

Proposition 10.7. [LP1] *For a pair of adjacent alcoves $A \xrightarrow{\alpha} B$, we have $\zeta_B - \zeta_A = \alpha / h^\vee$.*

Let us now associate to the gallery $\gamma(\emptyset)$ a continuous piecewise-linear path. Consider the points $\eta_0 := 0$, $\eta_{2i+1} := \zeta_{A_i}$ for $i = 0, \ldots, l$, $\eta_{2i} := \frac{1}{2}(\eta_{2i-1} + \eta_{2i+1})$ for $i = 1, \ldots, l$, and $\eta_{2l+2} := -\lambda$. Note that η_{2i} lies on F_i for $i = 0, \ldots, l+1$. Let $\pi(\emptyset)$ be the piecewise-linear path obtained by joining $\eta_0, \eta_1, \ldots, \eta_{2l+2}$. Given an admissible subset J, let $\eta'_0 = 0$, $\eta'_1 = \rho / h^\vee$, $\eta'_2, \ldots, \eta'_{2l+2} = -\mu(J)$ be the points on the faces of the gallery $\gamma(J)$ that are obtained (in the obvious way) from $\eta_0, \eta_1, \eta_2, \ldots, \eta_{2l+2}$ in the process of constructing $\gamma(J)$ from $\gamma(\emptyset)$ via folding operators. Clearly, η'_{2i+1} are the central points of the corresponding alcoves in $\gamma(J)$, for $i = 0, \ldots, l$. By joining $\eta'_0, \eta'_1, \ldots, \eta'_{2l+2}$, we obtain a piecewise-linear path that we call $\pi(J)$. Note that $\pi(J)$ can be described using folding operators, as in (9.3), once these operators are appropriately defined.

Remark 10.8. The maps $J \mapsto \gamma(J)$ and $J \mapsto \pi(J)$ are one-to-one. Indeed, given the gallery $\gamma(J) = (F_0, A_0, F_1, \ldots, A_l, F_{l+1})$, we have $J = \{j \mid A_{j-1} = A_j\}$. Also, the gallery $\gamma(J)$ can be easily recovered from $\pi(J)$.

The next result follows from Proposition 10.7 and the definition of folding operators on chains of roots and galleries.

Proposition 10.9. *Let $\Gamma(J) = (\{(\gamma_i, \gamma_i')\}_{i\in I}, \gamma_\infty)$. Then, for all $i \in I$, we have*

$$\eta'_{2i-1} - \eta'_{2i} = \frac{\gamma_i}{2h^\vee}\,, \qquad \eta'_{2i} - \eta'_{2i+1} = \frac{\gamma_i'}{2h^\vee}\,, \qquad \eta'_{2l+1} - \eta'_{2l+2} = \frac{\gamma_\infty}{h^\vee}\,.$$

It turns out that, in general, the collection of paths $\pi(J)$, for J ranging over admissible subsets, does *not* coincide with the collection of Littelmann paths obtained from $\pi(\emptyset)$ by applying the root operators E_p. Indeed, it is not true in general that $E_p(\pi(J)) = \pi(F_p(J))$, as was the case with the paths corresponding to LS chains (cf. Theorem 9.4). The reason is that, given $\pi = \pi(J)$, the function $h_p : [0, 1] \to \mathbb{R}$ given by $t \mapsto \langle \pi(t), \alpha_p^\vee \rangle$ is usually not weakly decreasing between the corresponding points t_0 and t_1 (see Section 9 for the definition of these points). This happens, for instance, when applying E_2 to the path $\pi(\emptyset)$ in Example 10.11 below. Such situations can also arise if we define $\pi(\emptyset)$ by joining the centers of the faces F_i, or the centers of both the alcoves A_i and the faces F_i (in the order they appear in the gallery $\gamma(\emptyset)$). In all these situations, we need the general definition of E_p for Littelmann paths, which we now recall from [Li2].

As in Section 9, the definition is easier to state if we replace the continuous piecewise-linear paths π (satisfying $\pi(0) = 0$) with their left-hand derivatives γ (which are piecewise-constant left-continuous maps defined on $(0, 1]$). Recall that we denoted the minimum of the function h_p by m_p. If $m_p > -1$, then $E_p(\gamma)$ is undefined, as before, so assume that $m_p \le -1$. Choose $t_0 = x_0 < x_1 < \ldots < x_l = t_1$ such either

(1) $h_p(x_{i-1}) = h_p(x_i)$ and $h_p(t) \ge h_p(x_{i-1})$ for $t \in [x_{i-1}, x_i]$;
(2) or h_p is strictly decreasing on $[x_{i-1}, x_i]$ and $h_p(t) \ge h_p(x_{i-1})$ for $t \le x_{i-1}$.

We now define

$$E_p(\gamma)(t) := \begin{cases} s_p(\gamma(t)) & \text{if } x_{i-1} < t \le x_i \text{ and } h_p \text{ behaves on } [x_{i-1}, x_i] \text{ as in (2)} \\ \gamma(t) & \text{otherwise}\,. \end{cases} \tag{10.1}$$

Remark 10.10. Our model can be based on *any* λ-chain (that is, not necessarily on the ones given by Proposition 4.2), and we still have a definition of root operators for admissible subsets/admissible foldings that corresponds to the simpler version of their definition on paths, given in (9.1).

Example 10.11. Suppose that the root system Φ is of type G_2. The positive roots are $\gamma_1 = \alpha_1,\ \gamma_2 = 3\alpha_1 + \alpha_2,\ \gamma_3 = 2\alpha_1 + \alpha_2,\ \gamma_4 = 3\alpha_1 + 2\alpha_2,\ \gamma_5 = \alpha_1 + \alpha_2,\ \gamma_6 = \alpha_2$. The corresponding coroots are $\gamma_1^\vee = \alpha_1^\vee,\ \gamma_2^\vee = \alpha_1^\vee + \alpha_2^\vee,\ \gamma_3^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee,\ \gamma_4^\vee = \alpha_1^\vee + 2\alpha_2^\vee,\ \gamma_5^\vee = \alpha_1^\vee + 3\alpha_2^\vee,\ \gamma_6^\vee = \alpha_2^\vee$.

Suppose that $\lambda = \omega_2$. Proposition 4.2 gives the following ω_2-chain:

$$(\beta_1, \ldots, \beta_{10}) = (\gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_2, \gamma_5, \gamma_3, \gamma_4, \gamma_5, \gamma_3)\,.$$

Thus, we have $\widehat{r}_1 = s_{\gamma_6,0}$, $\widehat{r}_2 = s_{\gamma_5,0}$, $\widehat{r}_3 = s_{\gamma_4,0}$, $\widehat{r}_4 = s_{\gamma_3,0}$, $\widehat{r}_5 = s_{\gamma_2,0}$, $\widehat{r}_6 = s_{\gamma_5,1}$, $\widehat{r}_7 = s_{\gamma_3,1}$, $\widehat{r}_8 = s_{\gamma_4,1}$, $\widehat{r}_9 = s_{\gamma_5,2}$, $\widehat{r}_{10} = s_{\gamma_3,2}$. There are six saturated chains in the Bruhat order (starting at the identity) on the corresponding Weyl group that can be retrieved as subchains of the ω_2-chain. We indicate each such chain and the corresponding admissible subsets in $\{1, \ldots, 10\}$.

(1) 1: $\{\}$;
(2) $1 < s_{\gamma_6}$: $\{1\}$;
(3) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5}$: $\{1, 2\}$, $\{1, 6\}$, $\{1, 9\}$;
(4) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}$: $\{1, 2, 3\}$, $\{1, 2, 8\}$, $\{1, 6, 8\}$;

(5) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}s_{\gamma_3}$: $\{1,2,3,4\}$, $\{1,2,3,7\}$, $\{1,2,3,10\}$, $\{1,2,8,10\}$, $\{1,6,8,10\}$;

(6) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}s_{\gamma_3} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}s_{\gamma_3}s_{\gamma_2}$: $\{1,2,3,4,5\}$.

The weight of each admissible subset is now easy to compute (by applying the corresponding affine reflections above to ω_2, cf. Definition 5.3). This leads to the expression for the character $\chi(\omega_2)$ as the sum over admissible subsets:

$$\chi(\omega_2) = e^{\omega_2} + e^{\widehat{r}_1(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_2(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_6(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_9(\omega_2)} + \cdots + e^{\widehat{r}_1\,\widehat{r}_6\,\widehat{r}_8\,\widehat{r}_{10}(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_2\,\widehat{r}_3\,\widehat{r}_4\,\widehat{r}_5(\omega_2)}.$$

Figure 2 displays the galleries $\gamma(J)$ corresponding to the admissible subsets J indicated above, the associated paths $\pi(J)$, as well as the action of the root operators F_p on J. For each path, we shade the fundamental alcove, mark the origin by a white dot "$\circ$", and mark the endpoint of a black dot "$\bullet$". Since some linear steps in $\pi(J)$ might coincide, we display slight deformations of these paths, so that no information is lost in their graphical representations. As discussed above, the weights of the irreducible representation V_{ω_2} are obtained by changing the signs of the endpoints of the paths $\pi(J)$ (marked by black dots). The roots in the corresponding admissible foldings $\Gamma(J)$ can also be read off; see Proposition 10.9. At each step, a path $\pi(J)$ either crosses a wall of the affine Coxeter arrangement or bounces off a wall. The associated admissible subset J is the set of indices of bouncing steps in the path.

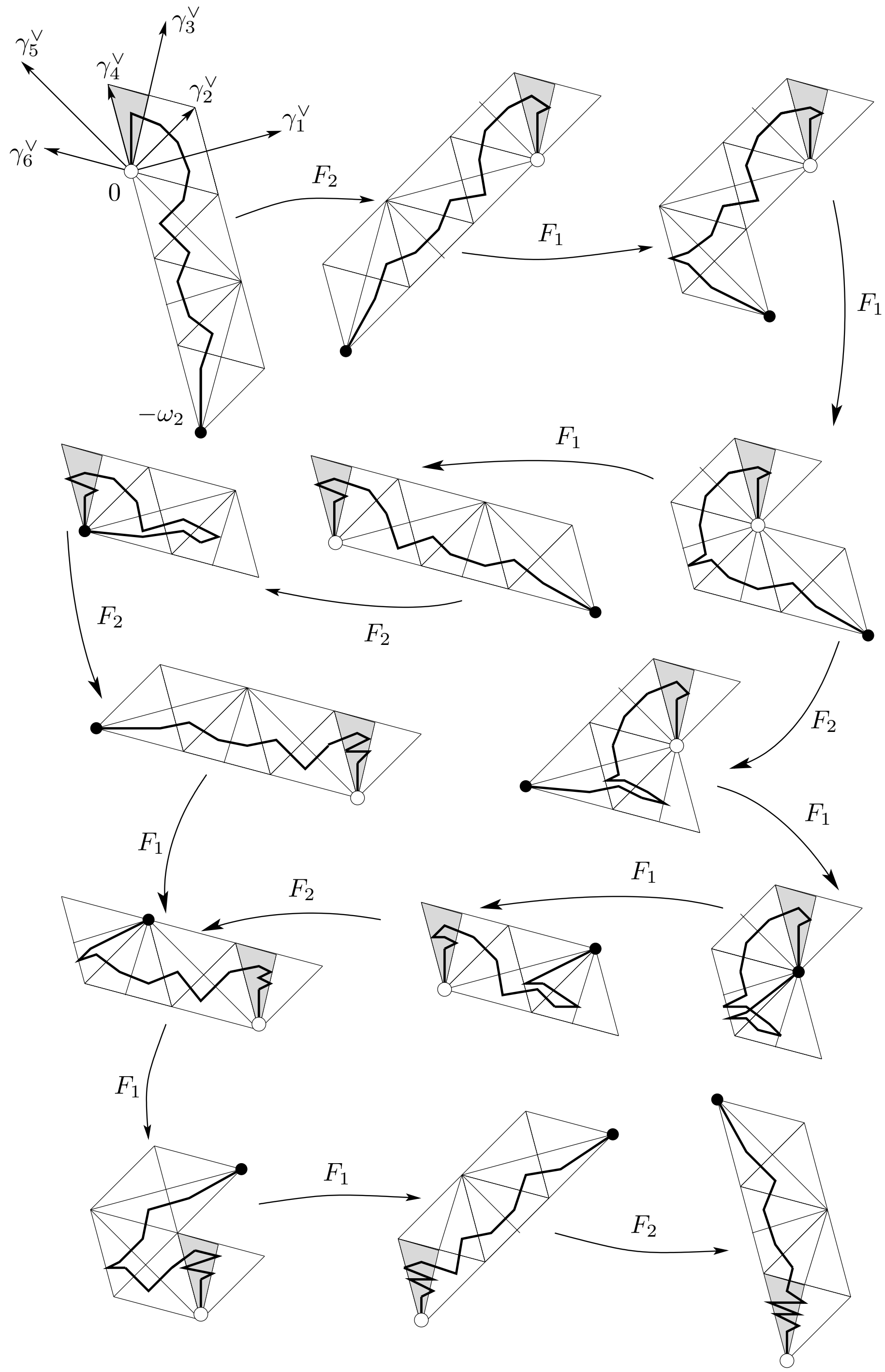

Figure 2. The crystal for the fundamental weight ω_2 for type G_2.

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Department of Mathematics and Statistics, State University of New York, Albany, NY 12222

E-mail address: `lenart@csc.albany.edu`

Department of Mathematics, M.I.T., Cambridge, MA 02139

E-mail address: `apost@math.mit.edu`