ALGEBRAS OF CURVATURE FORMS ON HOMOGENEOUS MANIFOLDS

ALEXANDER POSTNIKOV, BORIS SHAPIRO, AND MIKHAIL SHAPIRO

To Dmitry Borisovich Fuchs with love from former students of Moscow 'Jewish' University

Abstract. Let $\mathrm{C}(X)$ be the algebra generated by the curvature two-forms of standard holomorphic hermitian line bundles over the complex homogeneous manifold $X = G/B$. The cohomology ring of X is a quotient of $\mathrm{C}(X)$. We calculate the Hilbert polynomial of this algebra. In particular, we show that the dimension of $\mathrm{C}(X)$ is equal to the number of independent subsets of roots in the corresponding root system. We also construct a more general algebra associated with a point on a Grassmannian. We calculate its Hilbert polynomial and present the algebra in terms of generators and relations.

1. Homogeneous Manifolds

In this section we remind the reader the basic notions and notation related to homogeneous manifolds G/B and root systems, as well as fix our terminology.

Let G be a connected complex semisimple Lie group and B its Borel subgroup. The quotient space $X = G/B$ is then a compact homogeneous complex manifold. We choose a maximal compact subgroup K of G and denote by $T = K \cap B$ its maximal torus. The group K acts transitively on X. Thus X can be identified with the quotient space K/T.

By $\mathfrak{g}$ we denote the Lie algebra of G and by $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra. Also denote by $\mathfrak{g}_\mathbb{R} \subset \mathfrak{g}$ the real form of $\mathfrak{g}$ such that $i\,\mathfrak{g}_\mathbb{R}$ is the Lie algebra of K. Analogously, $\mathfrak{h}_\mathbb{R} = \mathfrak{h} \cap \mathfrak{g}_\mathbb{R}$ and $i\,\mathfrak{h}_\mathbb{R}$ is the Lie algebra of the maximal torus T. The *root system* associated with $\mathfrak{g}$ is the set Δ of nonzero vectors (roots) $\alpha \in \mathfrak{h}^*$ for which the root spaces

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

are nontrivial. Then $\mathfrak{g}$ decomposes into the direct sum of subspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

For $\alpha \in \Delta$, the spaces $\mathfrak{g}_\alpha$ and $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ are one-dimensional and there exists a unique element $h_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(h_\alpha) = 2$. The elements $h_\alpha \in \mathfrak{h}$ are called *coroots.* Actually, $\alpha \in \mathfrak{h}_\mathbb{R}^*$ and $h_\alpha \in \mathfrak{h}_\mathbb{R}$, for $\alpha \in \Delta$. Let us choose generators $e_\alpha \in \mathfrak{g}_\mathbb{R}$ of the root spaces $\mathfrak{g}_\alpha$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ for any root α. Then $[h_\alpha, e_\alpha] = 2e_\alpha$ and $[h_\alpha, e_{-\alpha}] = -2e_{-\alpha}$.

The root system Δ is subdivided into a disjoint union of sets of positive roots Δ_+ and negative roots $\Delta_- = -\Delta_+$ such that the direct sum $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$ is the

Date: January 17, 1999.

Lie algebra of Borel subgroup B. The Weyl group W is the group generated by the reflections $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$, $\alpha \in \Delta_+$, given by

$$s_\alpha : \lambda \mapsto \lambda - \lambda(h_\alpha)\,\alpha.$$

The lattice $\hat{T} = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ is called the *weight lattice.* Every weight $\lambda \in \hat{T}$ determines an irreducible unitary representations $\pi_\lambda : T \to \mathbb{C}^\times$ of the maximal torus T given by $\pi_\lambda(\exp(x)) = e^{\lambda(x)}$, for $x \in i\,\mathfrak{h}_\mathbb{R}$, and every irreducible unitary representation of T is of this form.

For a weight $\lambda \in \hat{T}$, the homomorphism $\pi_\lambda : T \to \mathbb{C}^\times$ extends uniquely to a holomorphic homomorphism $\bar{\pi}_\lambda : B \to \mathbb{C}^\times$. Thus any $\lambda \in \hat{T}$ determines a holomorphic line bundle $L_\lambda = G \times_B \mathbb{C} = K \times_T \mathbb{C}$ over $X = G/B = K/T$. The line bundle L_λ has a canonical K-invariant hermitian metric.

The classical Borel's theorem [2] describes the cohomology ring $\mathrm{H}^*(X,\mathbb{C})$ of the homogeneous manifold X in terms of generators and relations:

$$\mathrm{H}^*(X,\mathbb{C}) \cong \mathrm{Sym}(\mathfrak{h}^*)/I_W,$$

where I_W is the ideal in the symmetric algebra $\mathrm{Sym}(\mathfrak{h}^*)$ generated by the W-invariant elements without constant term. The natural projection from $\mathrm{Sym}(\mathfrak{h}^*)$ to $\mathrm{H}^*(X,\mathbb{C})$ is the homomorphism that sends a weight $\lambda \in \hat{T}$ to the first Chern class $c_1(L_\lambda)$ of the line bundle L_λ.

The purpose of this article is to extend the cohomology ring $\mathrm{H}^*(X,\mathbb{C})$ to the level of differential forms on X. It is possible to exhibit differential two-forms that represent the Chern classes $c_1(L_\lambda)$ in the de Rham cohomology of the homogeneous manifold X. Recall that for a holomorphic hermitian line bundle $L : E \to X$ there is a canonically associated connection on E. Denote by $\Theta(L)$ the *curvature form* of this connection, which is a differential two-form on X. Then the form $i\,\Theta(L)/2\pi$ represents $c_1(L)$.

In order to construct the curvature forms $\Theta(L_\lambda)$ explicitly, we define the elements $e^\alpha \in \mathfrak{g}_\mathbb{R}^*$, $\alpha \in \Delta$, by $e^\alpha(\mathfrak{h}) = 0$ and $e^\alpha(e_\beta) = \delta_{\alpha\beta}$, for any $\beta \in \Delta$. (Here $\delta_{\alpha\beta}$ is Kroneker's delta.) The space of left K-invariant differential one-forms on K can be identified with the dual to its Lie algebra, t.e., with $i\,\mathfrak{g}_\mathbb{R}^*$. Thus the elements $i\,e^\alpha$ can be regarded as one-forms on K. The differential two-form on K given by

$$\phi_\alpha = e^\alpha \wedge e^{-\alpha}, \quad \alpha \in \Delta,$$

is invariant with respect to the right translation action of the torus T. Thus ϕ_α produces a two-form on the manifold X, for which we will use the same notation ϕ_α. It is clear from the definition that $\phi_{-\alpha} = -\phi_\alpha$.

The following statement is implicit in [4].

Proposition 1. *For $\lambda \in \hat{T}$, the curvature form of the holomorphic hermitian line bundle L_λ is given by*

$$\Theta(L_\lambda) = \sum_{\alpha\in\Delta_+} \lambda(h_\alpha)\,\phi_\alpha.$$

Let Φ be the algebra generated by the two-forms ϕ_α, $\alpha \in \Delta_+$. The relations in Φ are relatively simple:

$$\phi_\alpha\phi_\beta = \phi_\beta\phi_\alpha, \qquad (\phi_\alpha)^2 = 0.$$

Thus Φ is a 2^N-dimensional algebra, where $N = |\Delta_+|$. The main object in this paper is the subalgebra of Φ generated by the curvature forms $\Theta(L_\lambda)$.

2. Main Results

Denote by $\mathrm{C}(X)$ the subalgebra in the algebra of differential forms on X that is generated by the curvature forms $\Theta(L_\lambda)$ of line bundles. Obviously, $\mathrm{C}(X)$ has the structure of a graded ring: $\mathrm{C}(X) = \mathrm{C}^0(X) \oplus \mathrm{C}^1(X) \oplus \mathrm{C}^2(X) \oplus \cdots$, where $\mathrm{C}^k(X)$ is the subspace of $2k$-forms in $\mathrm{C}(X)$. In order to formulate our main results about $\mathrm{C}(X)$ we need some extra notation from the matroid theory.

Let V be a collection of vectors $v_1, v_2, \ldots, v_N$ in a vector space E, say, over $\mathbb{C}$. A subset of vectors in V is called *independent* if they are linearly independent in E. By convention, the empty subset is independent. Let $\mathrm{ind}(V)$ be number of all independent subsets in V.

A *cycle* is a minimal by inclusion not independent subset. For a cycle $C = \{v_{i_1}, \ldots, v_{i_l}\}$, there is a unique, up to a factor, linear dependence $a_1 v_{i_1} + \cdots + a_l v_{i_l} = 0$ with non-zero a_i's. Let us fix a linear order $v_1 < v_2 < \cdots < v_N$ of all elements of V. For an independent subset S in V, a vector $v \in V \setminus S$, is called *externally active* if the set $S \cup \{v\}$ contains a cycle C and v is the minimal element of C. Let $\mathrm{act}(S)$ be the number of externally active vectors with respect to S.

Theorem 2. *The dimension of the algebra* $\mathrm{C}(X)$ *is equal to the number* $\mathrm{ind}(\Delta_+)$ *of independent subsets in the set of positive roots* Δ_+*. Moreover, the dimension of the* k*-th component* $\mathrm{C}^k(X)$ *is equal to the number of independent subsets* $S \subset \Delta_+$ *such that* $k = N - |S| - \mathrm{act}(S)$*, where* $N = |\Delta_+|$.

We remark here that, although the number $\mathrm{act}(S)$ of externally active vectors depends upon a particular order of elements in V, the total number of subsets S with fixed $|S| + \mathrm{act}(S)$ does not depend upon a choice of ordering.

We will actually prove a more general result about an arbitrary collection of vectors V. Let V and E be as above. We will assume that the elements $v_1, \ldots, v_N$ of V span the n-dimensional space E. Thus $N \geq n$. Let $F = \mathbb{C}^N$ be the linear space with a distinguished basis $\phi^1, \ldots, \phi^N$. Then V defines the projection map $p : F \to E$ that sends the ith basis element ϕ^i to v_i. The dual map $p^* : E^* \to F^*$ defines an n-dimensional plane $P = \mathrm{Im}(p^*)$ in F^*. In other words, the collection of vectors V can be identified with an element P of the Grassmannian $G(n, N)$ of n-dimensional planes in $\mathbb{C}^N$.

Let $\phi_1, \ldots, \phi_N$ be the basis in F^* dual to the chosen basis in F. Denote by Φ_N the quotient of the symmetric algebra $\mathrm{Sym}(F^*)$ modulo the relations $(\phi_i)^2 = 0$, $i = 1, \ldots, N$. Let $\mathcal{C}_V$ be the subalgebra in Φ_N generated by the elements of the n-dimensional plane $P \subset F^*$. In other words, the algebra $\mathcal{C}_V$ is the image of the induced mapping

$$\mathrm{Sym}(E^*) \longrightarrow \Phi_N = \mathrm{Sym}(F^*)/\left\langle \phi_i^2,\, i = 1, \ldots, N \right\rangle.$$

The algebra $\mathcal{C}_V$ has an obvious grading $\mathcal{C}_V = \mathcal{C}_V^0 \oplus \mathcal{C}_V^1 \oplus \mathcal{C}_V^2 \oplus \cdots$ by degree of elements.

Suppose $E = \mathfrak{h}$ and V is the collections of coroots h_α, $\alpha \in \Delta_+$. For $\lambda \in \mathfrak{h}^* = E^*$, $p^*(\lambda) = \Theta(L_\lambda) \in F^*$ is the curvature form (see Proposition 1). Then $\mathcal{C}_V = \mathrm{C}(X)$ is the algebra generated by the curvature forms $\Theta(L_\lambda)$.

In general, we have the following result.

Theorem 3. *The dimension of the algebra* $\mathcal{C}_V$ *is equal to the number* $\mathrm{ind}(V)$ *of independent subsets in* V*. Moreover, the dimension of the* k*-th component* $\mathcal{C}_V^k$ *is equal to the number of independent subsets* $S \subset V$ *such that* $k = N - |S| - \mathrm{act}(S)$.

We can also describe the algebra $\mathcal{C}_V$ as a quotient of a polynomial ring. Let us say that a hyperplane H in E is a V*-essential hyperplane* if the elements of the subset $\{v_i,\, i = 1, \ldots, N \mid v_i \in H\}$ span the hyperplane H. Obviously, an essential hyperplane is uniquely determined by the subset of indices $I_H = \{i \in \{1, \ldots, N\} \mid v_i \notin H\}$. We will call such subset I_H a V*-essential index subset.* Denote by $d(H) = d_V(H) = |I_H|$ the number of its elements. A nonzero vector $\lambda \in E^*$ determines the hyperplane $H = \{x \in E \mid \lambda(x) = 0\}$ in E. Vectors $\lambda_H \in E^*$ corresponding to essential hyperplanes H will be called V*-essential vectors*. They are defined up to a nonzero factor.

Theorem 4. *The algebra $\mathcal{C}_V$ is naturally isomorphic to the quotient of the polynomial ring $\mathrm{Sym}(E^*)/\mathcal{I}_V$, where the ideal $\mathcal{I}_V$ is generated by the powers $(\lambda_H)^{d(H)+1}$ of V-essential vectors for all V-essential hyperplanes H in E. The isomorphism is induced by the embedding $p^* : E^* \to F^*$.*

Remark 5. There are several equivalent definitions of essential subsets, as follows:
1. An index subset $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}$ is V*-essential* if and only if the following two conditions are satisfied: (i) the coordinate plane $\langle \phi_{i_1}, \ldots, \phi_{i_k} \rangle$ in F^* has one-dimensional intersection with the plane P; (ii) there is no proper subset in I that satisfies the condition (i). For an V-essential hyperplane H, the vector $p^*(\lambda_H) \in F^*$ spans the one-dimensional intersection of P and the coordinate plane associated with I_H.
2. Let $\theta_1, \ldots, \theta_n$ be any basis in P. The V-essential subsets are in one-to-one correspondence with the cycles in the vector set $\{\phi_1, \ldots, \phi_N, \theta_1, \ldots, \theta_n\}$. For a cycle $\{\phi_{i_1}, \ldots, \phi_{i_k}, \theta_{j_1}, \ldots, \theta_{j_k}\}$, the subset $\{i_1, \ldots, i_k\}$ is V-essential. Moreover, every V-essential subset is of this form.

Note that the decomposition of the Grassmannian $G(n, N)$ of all n-dimensional planes $P \subset F^*$ into strata with the same collection of essential subsets coincides with the decomposition of $G(n, N)$ into small cells of Gelfand-Serganova [3] because any two P_1, $P_2 \in G(n, N)$ with the same collection of essential subsets have the same dimensions of intersections with all coordinate subspaces. Equivalently, P_1 and P_2 are in the same strata if and only if the corresponding collections of vectors V_1 and V_2 define the same matroid, i.e., have the same collection of independent subsets.

Let us apply the Theorem 4 to $\mathrm{C}(X)$. Let $\omega_1, \omega_2, \ldots, \omega_l$ be the fundamental weights. They generate the weight lattice $\hat{T}$. Also let d_i be the number of positive roots $\alpha \in \Delta_+$ such that $\alpha(\omega_i) \neq 0$.

Corollary 6. *The algebra $\mathrm{C}(X)$ is naturally isomorphic to the quotient of the polynomial ring $\mathrm{Sym}(\mathfrak{h}^*)/\mathcal{J}$, the ideal $\mathcal{J} \in \mathrm{Sym}(\mathfrak{h}^*)$ is generated by the elements $(w \cdot \omega_i)^{d_i+1}$, where $i = 1, \ldots, l$ and w is an element of the Weyl group W. This isomorphism is induced by the projection $\mathrm{Sym}(\mathfrak{h}^*) \to \mathrm{C}(X)$ that send $\lambda \in \hat{T}$ to the curvature form $\Theta(L_\lambda)$.*

For type A flag manifolds this statement was earlier proved in [7].

Proof — The generators of the ideal $\mathcal{J}$, as described in Theorem 4, correspond to root subsystems in Δ of codimension 1. Every subsystem of codimension 1 is congruent, modulo the Weyl group, to the root subsystem in the hyperplane H_i of zeroes of a fundamental weight ω_i. Moreover, $d(H_i) = d_i$ is exactly the number of positive roots that do not belong to this hyperplane. □

3. Dual Statement and Proof of Theorems 2 and 3

Next we give a dual version of Theorem 3. Let $\mathrm{Sym}(E)$ be the symmetric algebra of the vector space E. For a subset $S \subset V$, denote $m(S) = \prod_{v \in S} v \in \mathrm{Sym}(E)$, a square-free monomial in v_i's. Let us denote by $\mathcal{S}_V$ a subspace in $\mathrm{Sym}(E)$ spanned by 2^N square-free monomials $m(S)$, where S ranges over all subsets in V. Let $\mathcal{S}_V^k$ be the k-th graded component of $\mathcal{S}_V$.

Theorem 7. *The dimension of the subspace $\mathcal{S}_V$ in $\mathrm{Sym}(E)$ is equal to the number $\mathrm{ind}(V)$ of independent subsets in V. Moreover, the dimension of $\mathcal{S}_V^k$ is equal to the number of independent subsets $S \subset V$ such that $k = N - |S| - \mathrm{act}(S)$.*

Let us choose a basis $x^1, \ldots, x^n$ in E. Let $x_1, \ldots, x_n$ be the dual basis in E^*, and let $\theta_i = p^*(x_i)$, $i = 1, \ldots, n$, be the corresponding basis in P.

For $I = (1 \le i_1 < i_2 < \cdots < i_k \le N)$ and $J = (1 \le j_1 \le j_2 \le \cdots \le j_k \le n)$, we denote by a_{IJ} the coefficient of the square-free monomial $\phi_{i_1} \cdots \phi_{i_k} \in \Phi_N$ in the product of generators $\theta_{j_1} \cdots \theta_{j_k} \in \mathcal{C}_V$. Let $A_k = (a_{IJ})$ be the $\binom{N}{k} \times \binom{n+k-1}{k}$-matrix formed by the a_{IJ}. Clearly, $\dim \mathcal{C}_V^k$ is equal to the rank of the matrix A_k.

On the other hand, a_{IJ} is also the coefficient of the monomial $x^{j_1} \cdots x^{j_k} \in \mathrm{Sym}(E)$ in the square-free monomial $v_{i_1} \cdots v_{i_k} \in \mathcal{S}(V)$. Thus $\dim \mathcal{S}_V^k$ is equal to the rank of the transposed matrix A_k^T, which is the same as the rank of A_k. We proved the following statement, which implies the equivalence of Theorems 3 and 7.

Lemma 8. *For $k = 1, \ldots, N$, we have $\dim \mathcal{C}_V^k = \dim \mathcal{S}_V^k$.*

Let us say that a subset $S \subset V$ is *robust* if there is no cycle $C \subset V$ with minimal element v such that $S \cap C = \{v\}$.

Lemma 9. *The number of robust subsets in V is equal to the number of independent subsets. Moreover, the number of k-element robust subsets is equal to the number of independent subsets S with $k = N - |S| - \mathrm{act}(S)$.*

Proof — We present an explicit bijection between robust subsets and independent subsets. A subset A is the complement to a robust subset in V if and only if for any cycle C with minimal element v, inclusion $C \setminus \{v\} \subset A$ implies $C \subset A$. We will call such subsets *antirobust.*

For an independent subset S, let M be the collection of all externally active $v \in V \setminus S$. Then $S \cup M$ is antirobust. Conversely, for an antirobust subset A, let M be the collection of all $v \in V$ such that v is a minimal element in some cycle $C \subset A$. Then $A \setminus M$ is an independent subset.

Clearly, both these mapping are inverse to each other and the statement of lemma follows. □

Theorem 10. *The set of square-free monomials $m(S)$, where S ranges over robust subsets, forms a basis of the subspace $\mathcal{S}_V$.*

First, we prove a weaker version of Theorem 10.

Lemma 11. *The square-free monomials $m(S)$, where S ranges over robust subsets, span $\mathcal{S}_V$.*

Proof — Suppose not. Let $m(R)$ be the maximal in the lexicographical order square-free monomial which cannot be expressed linearly via the monomials $m(S)$ with robust S. Then there is a cycle $C = \{v_{i_1}, v_{i_2}, \ldots, v_{i_l}\}$ with the minimal element $v = v_{i_1}$ such that $R \cap C = \{v\}$. We can replace v in the monomial $m(R)$ by a linear combination of $v_{i_2}, v_{i_3}, \ldots, v_{i_l}$. Thus $m(R)$ is a linear combination of square-free monomials which are greater than $m(R)$ in the lexicographical order. By assumption each of these monomials can be expressed via the monomials $m(S)$ with robust S. Contradiction. $\square$

We can now conclude the proof.

Proof of Theorems 7 and 10 — Lemmas 9 and 11 imply the inequality $\dim \mathcal{S}_V \leq \mathrm{ind}(V)$. In view of these two lemmas it is enough to show that $\dim \mathcal{S}_V$ is actually equal to $\mathrm{ind}(V)$.

We prove this statement by induction on $|V|$. If the linear span of vectors in V is one-dimensional, then both $\dim \mathcal{S}_V$ and $\mathrm{inv}(V)$ are equal to the number of non-zero vectors in V plus one. This establishes the base of induction.

Assume that $v = v_N$ is a nonzero vector in V. Let $V' = V \setminus \{v\} = \{v_1, \ldots, v_{N-1}\}$, and let V'' be the collection of images of vectors $v_1, \ldots, v_{N-1}$ in the quotient space $E/\langle v\rangle$. It follows from the definition of independent subset that $\mathrm{ind}(V) = \mathrm{ind}(V') + \mathrm{ind}(V'')$. Assume by induction that $\dim \mathcal{S}_{V'} = \mathrm{ind}(V')$ and $\dim \mathcal{S}_{V''} = \mathrm{ind}(V'')$.

Clearly, $\mathcal{S}_V$ is spanned by $\mathcal{S}_{V'} + \mathcal{S}_{V'}v$. Both $\mathcal{S}_{V'}$ and $\mathcal{S}_{V'}v$ have same dimensions. Hence, $\dim \mathcal{S}_V = 2\dim \mathcal{S}_{V'} - \dim(\mathcal{S}_{V'} \cap \mathcal{S}_{V'}v)$. Let $\pi : \mathcal{S}_{V'} \to \mathcal{S}_{V''}$ be the natural projection. Then $\mathcal{S}_{V'} \cap \mathcal{S}_{V'}v \subset \mathrm{Ker}(\pi)$. Thus $\dim \mathcal{S}_{V'} - \dim(\mathcal{S}_{V'} \cap \mathcal{S}_{V'}v) \geq \dim \mathcal{S}_{V''}$ and $\dim \mathcal{S}_V \geq \dim \mathcal{S}_{V'} + \dim \mathcal{S}_{V''} = \mathrm{ind}(V') + \mathrm{ind}(V'') = \mathrm{ind}(V)$. Coupled with the inequality $\dim \mathcal{S}_V \leq \mathrm{ind}(V)$, this produces the required statement.

This finishes the proof of Theorems 7 and 10 and thus of Theorems 2 and 3. $\square$

4. Proof of Theorem 4

Let Ess_V denote the set of all V-essential hyperplanes in E. Recall that $\mathcal{I}_V$ is the ideal in $\mathrm{Sym}(E^*)$ generated by the powers of essential vectors $(\lambda_H)^{d(H)+1}$, $H \in \mathrm{Ess}_V$ (see Theorem 4). The embedding $p^* : E^* \to F^*$ induces the mapping $\mathrm{Sym}(E^*) \to \Phi_N$, whose image is the algebra $\mathcal{C}_V$. Let $\widetilde{\mathcal{I}}_V$ denote the kernel of this mapping, which we will call the *vanishing ideal* of V. Theorem 4 amounts to the identity of ideals $\mathcal{I}_V = \widetilde{\mathcal{I}}_V$.

The proof relies on a couple of simple lemmas. As in the previous section we assume that v_N is a nonzero vector in V. Let $V' = V \setminus \{v_N\} = \{v_1, \ldots, v_{N-1}\}$, and let V'' be the collection of images of vectors $v_1, \ldots, v_{N-1}$ in the quotient space $E/\langle v_N\rangle$. We also denote by $\mathrm{Ess}_{V'}$ and $\mathrm{Ess}_{V''}$ the sets of V'-essential and V''-essential hyperplanes in the corresponding spaces.

The dimension of the span of vectors in V'' is $n-1$. The dimension of the span of V' can be either $n-1$ or n. For a hyperplane H in $E/\langle v_N\rangle$, let $\overline{H} = H \oplus \langle v_N\rangle$ be the hyperplane in E. Also for a collection of hyperplanes C in $E/\langle v_N\rangle$, let $\overline{C} = \{\overline{H} \mid H \in C\}$ be the collection of hyperplanes in E.

Lemma 12. (a) *If* $\dim V' = n-1$ *then* $\mathrm{Ess}_V = \{\langle V'\rangle\} \cup \overline{\mathrm{Ess}_{V''}}$.
(b) *If* $\dim V' = n$ *then* $\mathrm{Ess}_V = \mathrm{Ess}_{V'} \bigcup \overline{\mathrm{Ess}_{V''}}$. *For* $H \in (\mathrm{Ess}_{V'} \setminus \overline{\mathrm{Ess}_{V''}})$, *we have* $d_V(H) = d_{V'}(H) + 1$. *For* $H \in \mathrm{Ess}_{V''}$, *we have* $d_V(\overline{H}) = d_{V''}(H)$.

Proof — The part (a) is left as an easy exercise for the reader. In order to prove (b) we first assume that a hyperplane H contains v_N. Then H is V-essential if and only if its projection $H/\langle v_N\rangle$ is V''-essential. Suppose that a hyperplane H does not contain v_N. Then H is V-essential if and only if it is V'-essential and its projection $H/\langle v_N\rangle$ is not V''-essential. The equalities for the numbers $d(H)$ are also obvious. This proves the lemma. □

Recall that the collection of vectors V is associated with a plane $P \in G(n,N)$. Let P' and P'' be the planes associated with vector sets V' and V'', respectively. Namely, P' is the projection of P along ϕ_N onto the hyperplane $\mathcal{H} = \langle \phi_1, \ldots, \phi_{N-1}\rangle$ spanned by the first $N-1$ coordinate vectors; and P'' is the intersection of P with the same hyperplane $\mathcal{H}$.

We can choose the basis $x_1, \ldots, x_n$ in E^* and the corresponding basis $\theta_i = p^*(x_i)$, $i = 1, \ldots, n$ in P such that $\theta_1, \ldots, \theta_{n-1} \in \mathcal{H}$ and $\theta_n \in \phi_N + \mathcal{H}$. Also denote $\widetilde{\theta}_n = \theta_n - \phi_N \in \mathcal{H}$. Then $\theta_1, \ldots, \theta_{n-1}, \widetilde{\theta}_n$ span the projected space P'. The space $\mathrm{Sym}(E^*)$ can be identified with the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$

Lemma 13. *The vanishing ideal $\widetilde{\mathcal{I}}_V$ consists of all polynomials $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that both f and $\partial f/\partial x_n$ belong to the vanishing ideal $\widetilde{\mathcal{I}}_{V'}$. In particular, $\widetilde{\mathcal{I}}_V \subseteq \widetilde{\mathcal{I}}_{V'}$.*

Proof — Recall that the ideal $\widetilde{\mathcal{I}}_V$ consists of all polynomials in $x_1, \ldots, x_n$ which vanish in the algebra Φ_N upon substituting of the θ_i instead of the x_i. Taylor's expansion in the algebra Φ_N gives

$$f(\theta_1, \ldots, \theta_n) = f(\theta_1, \ldots, \widetilde{\theta}_n + \phi_N) = f(\theta_1, \ldots, \widetilde{\theta}_n) + (\partial f/\partial x_n)(\theta_1, \ldots, \theta_{n-1}, \widetilde{\theta}_n)\phi_N.$$

We have only two non-vanishing terms in the right hand side, since $\phi_N^2 = 0$.

The polynomial $f(x_1, \ldots, x_n)$ belongs to $\widetilde{\mathcal{I}}_V$ if and only if both $f(\theta_1, \ldots, \widetilde{\theta}_n)$ and $(\partial f/\partial x_n)(\theta_1, \ldots, \theta_{n-1}, \widetilde{\theta}_n)$ vanish in the algebra Φ_N. □

We now conclude the proof of the identity $\mathcal{I}_V = \widetilde{\mathcal{I}}_V$. The inclusion $\mathcal{I}_V \subseteq \widetilde{\mathcal{I}}_V$ is straightforward. Indeed, every $p^*(\lambda_H)$ is a linear combination of $d(H)$ different ϕ_i's. Thus $(\lambda_H)^{d(H)+1}$ maps to zero in the algebra Φ_N.

We prove the identity $\mathcal{I}_V = \widetilde{\mathcal{I}}_V$ by induction on $|V|$. The base of induction is the trivial case $V = \{v_1\}$. For $|V| \geq 2$, assume by induction that the statement is true for both V' and V''. Take any $f \in \widetilde{\mathcal{I}}_V$. Our goal is to show that $f \in \mathcal{I}_V$.

By Lemma 13, the polynomial $\partial f/\partial x_n$ belongs to $\widetilde{\mathcal{I}}_{V'}$. Notice that for any $H \in \mathrm{Ess}_{V'} \cap \overline{\mathrm{Ess}_{V''}}$ the coordinate expansion of the corresponding V'-essential vector λ_H does not involve x_n. Indeed, any $H \in \overline{\mathrm{Ess}_{V''}}$ contains v_N, thus $\lambda_H(v_N) = 0$. On the other hand, $x_1(v_N) = \cdots = x_{n-1}(v_N) = 0$, and $x_n(v_N) = 1$.

By inductive assumption, one has

$$\partial f/\partial x_n = \sum_{H \in \mathrm{Ess}_{V'} \setminus \overline{\mathrm{Ess}_{V''}}} p_H\, \lambda_H^{d_{V'}(H)+1} + \sum_{H \in \mathrm{Ess}_{V'} \cap \overline{\mathrm{Ess}_{V''}}} p_H\, \lambda_H^{d_{V'}(H)+1},$$

where the p_H are certain polynomials in $x_1, \ldots, x_n$. The λ_H in the second sum do not involve x_n. Thus, integrating the above expression with respect to x_n, one deduces that there exists a polynomial $\bar{f}$ of the form

$$\bar{f} = \sum_{H \in \mathrm{Ess}_{V'} \setminus \overline{\mathrm{Ess}_{V''}}} \bar{p}_H\, \lambda_H^{d_{V'}(H)+2} + \sum_{H \in \mathrm{Ess}_{V'} \cap \overline{\mathrm{Ess}_{V''}}} \bar{p}_H\, \lambda_H^{d_{V'}(H)+1}$$

satisfying $\partial \bar{f}/\partial x_n = \partial f/\partial x_n$. By Lemma 12, the polynomial $\bar{f}$ can be written as

$$\bar{f} = \sum_{H \in \mathrm{Ess}_V} \bar{p}_H\, \lambda_H^{d_V(H)+1}$$

and thus belongs to $\mathcal{I}_P$. The difference $\hat{f} = f - \bar{f}$ belongs to $\widetilde{\mathcal{I}}_P$ and is independent on x_n. Thus $\hat{f} \in \widetilde{\mathcal{I}}_{V''}$. By induction hypothesis, $\hat{f} \in \mathcal{I}_{V''} \subset \mathcal{I}_V$. Thus $f = \bar{f} + \hat{f} \in \mathcal{I}_V$. The statement follows. Q.E.D.

5. Remarks and Open Problems

The algebra $\mathrm{C}(X)$ for type A flag manifolds $X = SL(n)/B$ was studied in more details in [7] and [6] motivated by [1]. In this case, we first proved Theorem 2 in [6] using a different approach based on a presentation of $\mathrm{C}(X)$ as a quotient of a polynomial ring (cf. Theorem 4). The theorem claims that the dimension of $\mathrm{C}(X)$ is equal to the number forests on n labelled vertices whereas the dimension of $\mathrm{C}^k(X)$ is equal to the number of forests with $\binom{n}{2} - k$ inversions. The statement about forests was initially conjectured in [7] and the statement concerning inversions was then guessed by R. Stanley. In [6] we also discuss various generalizations of the ring $\mathrm{C}(X)$.

A natural open problem is to extend the results to homogeneous manifolds G/P, where P is a parabolic subgroup. Formulas for curvature forms on G/P can be found in [4].

It is also intriguing to find the links between the algebra $\mathrm{C}(X)$ and the arithmetic Schubert calculus, see H. Tamvakis [8, 9].

B. Kostant pointed out that the algebra $\mathrm{C}(X)$ is related to the $S(\mathfrak{g})$-module $\wedge\mathfrak{g}$ studied in [5]. It would be interesting to investigate this relationship.

The authors are grateful to Vladimir Igorevich Arnold, Richard Stanley, Alek Vainshtein, and Andrei Zelevinsky for stimulating discussions and helpful suggestions.

References

[1] V. I. Arnold, Remarks on eigenvalues and eigenvectors of Hermitian matrices, Berry phase, adiabatic connections and quantum Hall effect *Selecta Math.* **(1)** (1995), 1–19.

[2] A. Borel, Sur la cohomologie de espaces fibrés principaux et des espaces homogénes des groupes de Lie compacts, *Ann. of Math.* (2) **57** (1953), 115–207.

[3] I. Gelfand and V. Serganova, Combinatorial geometries and the strata of a torus on homogeneous compact manifolds, *Russ. Math. Surv.* **42, no. 2,** (1987), 107–134.

[4] P. Griffiths and W. Schmid, Locally homogeneous complex manifolds, *Acta Math.* **123** (1969), 253–302.

[5] B. Kostant, On $\wedge\mathfrak{g}$ for a semisimple Lie algebra $\mathfrak{g}$, as as equivariant module over the symmetric algebra $S(\mathfrak{g})$, preprint.

[6] A. Postnikov, B. Shapiro, and M. Shapiro, Chern forms on flag manifolds and forests, in preparation.

[7] B. Shapiro and M. Shapiro, On algebra generated by Bott-Chern 2-forms on SL_n/B, *C. R. Acad. Sci. Paris Sér. I Math.* **326** ser. 1 (1998), 75–80.

[8] H. Tamvakis, Bott-Chern forms and arithmetic intersections, *Enseign. Math.* **43** (1997), 33–54.

[9] H. Tamvakis, Arithmetic intersection theory on flag varieties, preprint alg-geom/9611006, 1996.

Dept. Math., Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.
E-mail address: apost@math.mit.edu

Dept. Math., University of Stockholm, Stockholm, S-10691, Sweden
E-mail address: shapiro@matematik.su.se

Dept. Math., Royal Institute of Technology, Stockholm, S-10044, Sweden
E-mail address: mshapiro@math.kth.se