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TREE AND FOREST VOLUMES OF GRAPHS

by

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ABSTRACT

The *tree volume of a weighted graph* G is the "sum" of the tree volumes of all spanning trees of G, and the *tree volume of a weighted tree* T is the product of the edge weights of T times the "product" of the letters of the Prüfer code of T where the vertices of G are viewed as independent indeterminants that can be multiplied and commute. The *forest volume of* G is the tree volume of the graph G^c obtained from G by adding a new vertex c and connecting every vertex of G with c by an arc of weight 1. We show that the forest volume is a natural generalization of the Laplacian polynomial of graphs and that it also can be expressed as the characteristic polynomial of a certain matrix similar to the Laplacian matrix. It turns out that the forest volumes of graphs possesses many important properties of the Laplacian polynomials, for example, the reciprocity theorem holds also for the forest volumes. We describe two constructions of graph compositions, and show that the forest volume of a composition can be easily found if the "structure" of the composition and the forest volumes of the graph–bricks are known. As an illustration of the results on the forest volume of graph–compositions we give a combinatorial interpretation and proof of Hurwitz's identity.

Keywords: graph, tree, forest spanning tree, Laplacian matrix and polynomial, tree and forest volume.

Contents

1 Introduction

Enumerating and/or finding the number of spanning trees of a graph is one of the classical problems in enumerative combinatorics and graph theory. The first result to be mentioned in this respect is the classical matrix–tree theorem that in its general form expresses, as the determinant of a certain graph matrix, not only the number of spanning trees of a graph but also a certain polynomial that can be viewed as an enumerator (a generating function) of all spanning trees of a graph [4, 22] (see also [2, 6]). This theorem provides an efficient (polynomial time) algorithm for finding the number of spanning trees of a graph.

Another aspect of the spanning tree enumeration is the problem of finding formulas for the number of spanning trees of graphs of special type (as a function of certain parameters of the graph). There are various natural classes of graphs whose descriptions are (asymptotically) much smaller than $O(n^2)$ which is the size of descriptions of arbitrary graphs. It is natural to expect that for some of these classes the number of spanning trees of their members also have a short description (formula) as a function of certain graph parameters. The first example of such classes is the class of complete graphs.

An elegant formula n^{n-2} for the number of spanning trees of a complete graph on n vertices was found in the last century [3, 5, 28] (see also [2, 6]).

There are various approaches for finding such "good" classes of graphs and the corresponding formulas for graphs of these classes.

One approach is to use directly the matrix–tree theorem. If a class of graphs under consideration is "good enough", one may use his skill and the properties of the corresponding matrices of such special graphs and succeed in finding formulas for the determinant of these matrices because of their special structure. For example, the Sylvester–Borchardt–Cayley formula (usually called the Cayley formula) for the complete graphs can easily be found that way.

One more approach was developed by Moon [23] in 1967. This approach views every graph as obtained from the corresponding complete graph by deleting some edges, and accordingly uses the inclusion–exclusion principle. The success of this approach depends essentially on one's skill in finding closed formulas for the corresponding series sums, and formulas for some special graphs have been found that way.

Another approach was developed by A. Kelmans [9] in 1964. In this approach the characteristic polynomial of the Laplacian matrix (so called *Laplacian polynomial*) of a graph plays an essential role. It turns out that the Laplacian polynomial of graphs has very interesting and useful combinatorial properties that can be used to approach the above problem [9]. Many papers have since been devoted to the Laplacian polynomials (see, for example, [6]). One important observation [9] is that the number of spanning trees times the number of vertices of the graph is the product of all eigenvalues of the Laplacian matrix excluding one zero eigenvalue. Another essential property is the so called reciprocity theorem concerning the relation between the Laplacian polynomial (and the Laplacian spectrum) of a graph and its complement [9, 11]. This theorem gives, in particular, an expression for the number of spanning trees of a graph in terms of the Laplacian polynomial of the complement graph. A remarkable property of this expression is that it turns out to be nothing but (it can be read term by term as) the basic inclusion–exclusion relation in the previous approach [12]. Therefore this approach contains the previous approach but says much more about the basic inclusion–exclusion relation.

As we mentioned above, the class of complete graphs was the first "good" class in this respect. The next natural classes were the classes of complete bipartite graphs, and more generally, complete multipartite graphs. The question arises as to what would be a natural direction to develop these results.

The main idea suggested and explored in [9] was to find a certain composition $\mathcal{C}$ of graph–components with the property that if we know the Laplacian polynomials (the Laplacian spectrum) of the components and the "structure" of the composition $\mathcal{C}$ then we can find the same information for the result of the composition.

The reciprocity theorem mentioned above turned out to be a natural basis to develop such a composition. It suggested certain operations on graphs that induced the corresponding operations on the Laplacian polynomials. Therefore the Laplacian polynomial (the Laplacian spectrum) of any graph obtained from the graph–components by a series of such operations can be easily found if we know the same information on the graph–components and the series of operations (the composition "structure"). This development resulted in an algorithm providing formulas of the Laplacian polynomial (and spectrum) for so called *decomposable graphs* [9, 11] (see also Sections 9 and 10 below). It turns out that many formulas for the number of spanning trees found so far by either approach can be obtained by this algorithm. In particular, the corresponding formulas for the complete mulipartite graphs can be easily found by applying the above mentioned algorithm. A natural class of graphs that are uniquely defined by their degree functions (so called threshold graphs) turns out to be a small subclass of the class of decomposable graph. Therefore the above approach provides the formulas for the number of spanning trees of threshold graphs in terms of their vertex degrees [7].

When studying the combinatorial structure of the set of spanning trees of a graph it is natural to classify spanning trees by their degree functions. For that reason A. Kelmans introduced in [17, 18] (see also [15]) a notion of the *spanning tree volume* (or simply $\mathcal{T}$*–volume*) of a graph which is a generating function of the graph spanning trees that reflects the above mentioned classification.

In this paper we consider the so called *forest volume* (or simply $\mathcal{F}$*–volume*) of a graph which is a modification of the spanning tree volume (each of these two notions uniquely defines the other, see Section 5). We show that the forest volume is a natural generalization of the Laplacian polynomial of graphs and that it also can be expressed as the characteristic polynomial of a certain matrix similar to the Laplacian matrix (see Section 5). It turns out that many important properties of the Laplacian polynomial can be generalized to the forest volume of a graph.

The main notions and notation are given in Section 2.

In Section 3 we recall the matrix–tree theorem.

In Section 4 we outline the main properties of the Laplacian polynomials of graphs.

The notions of the tree and forest volumes of graphs are defined and discussed in Section 5.

Recursive properties of the forest volumes are given in Sections 6 and are used in Sections 8 and 11.

In Section 7 we show that the forest volume is the characteristic polynomial of a matrix similar to the Laplacian matrix of a graph. We also discuss the relation between the forest volumes and the Laplacian polynomials.

In Section 8 we prove the reciprocity theorem for the forest volumes similar to that for the Laplacian polynomials.

Two constructions of graph compositions (T*–aggregates* and G*–compositions*) are described in Section 9.

In Section 10 we adopt the algorithm of finding the Laplacian polynomials of decomposable graphs in [9, 11] to find the forest volumes of T–aggregates.

In Section 11 we consider the G–compositions of graphs and describe the relation between the forest volumes of a G–composition and its graph–components. This relation shows that the forest volume of a G–composition is uniquely defined by the forest volumes of its graph–components and the "structure" G of the composition.

In Section 12 we find formulas of the forest volumes for some special weighted digraphs.

In Section 13 we use the results of Sections 11 and 12 to give a combinatorial interpretation and proof of Hurwitz's identity.

In another paper we use the relation between the forest volumes of a G–composition and its graph–components in Section 11 to obtain more general results on identities.

2 Main notions and notation

The notions that are used but not defined here can be found in [2].

An *undirected graph* or simply a *graph* G is a pair (V, E) where V is a finite non–empty set of elements (called *vertices*), $E \subseteq \binom{V}{2}$ where $\binom{V}{2}$ is the set of unordered pairs of different elements of V (the elements of E are called *edges* of G). Let $V(G) = V$ and $E(G) = E$.

Let $E(u, G) = \{[u, v] : [u, v] \in E(G)$. The number $d(u, G) = |E(u, G)|$ is called the *degree of the vertex v in G*.

A *directed graph* or simply a *digraph* G is a pair (V, E) where V is a finite non–empty set of elements (called *vertices* of G), $E \subseteq (V^2)$ where (V^2) is the set of ordered pairs of different elements of V. Let $V(G) = V$ and $E(G) = E$.

Note that in the above definitions both $\binom{V}{2}$ and (V^2) do not contain pairs of type (v, v) which means that the above defined graphs do not have *loops*.

For a digraph G, let $E_{in}(u, G) = \{(v, u) : (v, u) \in E(G)$ and $E_{out}(u, G) = \{(u, v) : (u, v) \in E(G)\}$.

The numbers $d_{in}(u, G) = |E_{in}(u, G)|$, $d_{out}(u, G) = |E_{out}(u, G)|$, and $d(u, G) = d_{in}(u, G) + d_{out}(u, G)$ are called, respectively, the *indegree*, *outdegree*, and *degree* of the vertex v in G.

A *source* (a *sink*) of a digraph G is a vertex v having no incoming (respectively, outgoing) edges in G. Let $S(G)$ and $R(G)$ denote the sets of sources and sinks of G, respectively.

A digraph is *acyclic* if it has no directed cycles.

A graph S is a *subgraph* of G, written $S \subseteq G$ if $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$. A subgraph S of G is *spanning* if $V(S) = V(G)$.

Two spanning subgraphs S_1 and S_2 of G are *different* if $E(S_1) \neq E(S_2)$.

For digraph the corresponding notions are defined similarly.

A *forest* is a graph with no cycles. A *tree* is a connected graph with no cycles (i.e. a connected forest).

A *spanning tree* of G is a spanning subgraph of G which is a tree.

Let $\mathcal{F}(G)$ and $\mathcal{T}(G)$ denote the sets of different spanning forests and spanning trees of G, respectively. Put $t(G) = |\mathcal{T}(G)|$, i.e. $t(G)$ is the number of different spanning trees of G.

A *ditree* (or an *arborescence*) A is a digraph with the properties:
$(a1)$ A has no directed cycles, and
$(a2)$ for every vertex v in $V(A)$ except for one vertex, say r, there exists a unique arc $e_v = (v, t_v)$ starting at v.

The vertex r is called the *root of* A, and A is also called a *ditree rooted at* r. A *leaf* of a ditree A is a vertex having no incoming edge in A (or equivalently, a vertex of degree one in A). Clearly $R(A) = \{r\}$ and $S(A)$ is the set of leaves of A.

It is clear that a ditree is a special acyclic digraph. It is easy to see that a ditree can be obtained from a tree T by specifying one of its vertices, say r, and assigning orientations to its edges such that for every edge e of T if we walk in T along a path starting at r and containing e then we traverse e in the direction opposite to its orientation.

A *diforest* is a digraph such that every its component is a ditree.

For a diforest A and a vertex $u \in V(A \setminus R(A))$ let $f(u) = v$ if $(u, v) \in E(A)$. Since A is a diforest, clearly $f : V(A \setminus R(A)) \to V(A)$ is a function. We call f the *pointer* of a diforest A.

A *spanning ditree* (*spanning diforest*) of a digraph G is a spanning subdigraph of G which is a ditree (respectively, a diforest).

Let $\mathcal{T}_r(G)$ denote the set of different spanning ditrees of G rooted at $r \in V(G)$. Similarly let $\mathcal{F}(G)$ denote the set of different spanning diforests of G. Put $t_r(G) = |\mathcal{T}_r(G)|$, i.e. $t_r(G)$ is the number of different spanning ditrees of G rooted at r.

A weighted graph G is a pair (V, g) where $V = V(G)$ is a finite set of elements called *vertices*, and $g : \binom{V}{2} \to R_e$ is a function that prescribes to every undirected pair $[u, v]$ with $u, v \in V$ and $u \neq v$, a weight $g[u, v]$ which is an element of a commutative ring R_e. Let $\grave{G}$ denote the graph such that $V(\grave{G}) = V(G)$ and $[u, v] \in E(G)$ if $g[u, v] \neq 0 \in R_e$ (i.e. $E(\grave{G})$ is the support of function g). If $e = [u, v] \in E(G)$ then put $g(e) = g[u, v]$. The graph $\grave{G}$ is called the *skeleton* of the weighted graph G. Let

$$t(G) = \sum\{g(T) : T \in \mathcal{T}(\grave{G})\} \text{ where } g(T) = \prod\{g(e) : e \in E(T)\}.$$

A *weighted digraph* $G = (V, g)$ and its graph–skeleton $\grave{G}$ are defined similarly. The only difference is that the set $\binom{V}{2}$ of all 2–subsets of V is replaced by the set (V^2) of all ordered pairs of distinct elements of V and $\grave{G}$ is the corresponding digraph.

Note that in the definition of a weighted digraph the set (V^2) of all ordered pairs of distinct elements of V can be replaced by the set V^2 of all ordered pairs with the additional condition that $g(v, v) = 0$ for each $v \in V$.

For a weighted digraph $G = (V, g)$ and $e = (b, a)$, $a, b \in V$, let $G \setminus e = (V, g|_{g(e)=0})$. Let G/e denote the graph $H = (V_h, h)$ such that $V_h = V \setminus b$ and $h(a, v) = g(a, v) + g(b, v)$, $h(v, a) = g(v, a) + g(v, b)$, and $h(u, v) = g(u, v)$ if $u, v \in V_h \setminus a$. The operations $\setminus$ and $/$ are called *deletion* and *contraction* an edge e in G.

Let

$$t_v(G) = \sum\{g(T) : T \in \mathcal{T}_v(\grave{G})\}$$

where $g(T) = \prod\{g(e) : e \in E(T)\}$.

A weighted digraph is *acyclic* if its skeleton is an acyclic digraph.

In many situations weighted graph G can be viewed as the weighted digraph $\vec{G} = (G, \vec{g})$ such that $V(\vec{G}) = V(G)$ and $\vec{g}(u,v) = \vec{g}(v,u) = g[u,v]$.

If in particular, R_e is the set of integers, and $g(e) \geq 0$ for every $e \in E(G)$ then G can be viewed as the graph obtained from $\grave{G}$ by replacing every edge e of $\grave{G}$ by $g(e)$ parallel (labeled) edges, and so $t(G)$ is the number of spanning trees of the graph G. A weighted digraph has a similar interpretation in this case.

A *p–complete weighted digraph* K^p is a pair (V,k) where $V = V(K^p)$ is a finite set of elements (*vertices of* K^p), and $k(u,v) = p \in R_e$ for every ordered pair of distinct vertices (u,v) of V.

Let $A = (V_a, a)$ and $B = (V_b, b)$ be two weighted digraphs such that $V_b \subseteq V_a$. Let $B - A$ denote the weighted digraph $G = (V,g)$ such that $V = V_a$, and $g(u,v) = a(u,v) - b(u,v)$ if $u,v \in V_b$, and $g(u,v) = a(u,v)$ otherwise. If, in particular, $A = K^p$ and $V_a = V_b$ then $K^p - B$ is called the *weighted digraph p–complement to* B, written $\bar{G}^p = K^p - B$.

Let $A \setminus B$ denote the weighted digraph $G = (V,g)$ such that $V = V_a \setminus V_b$ and g is the restriction of a on V^2.

A complete graph K^p is a very particular case of so called totally decomposable graphs [9, 11] (see Section 9).

Let $G_1 = (V_1, g_1)$ and $G_2 = (V_1, g_1)$ be two disjoint weighted digraphs, and let $p \in R_e$. Let $G_1(p)G_2$ denote the weighted digraph $G = (V,g)$ such that $V = V_1 \cup V_2$, and $g(u,v) = g_i(u,v)$ if $u,v \in V_i$, $i = 1,2$, and $g(u,v) = p$ if $u \in V_1$ and $v \in V_2$ or $v \in V_1$ and $u \in V_2$. We denote $G_1(0)G_2$ by $G_1 + G_2$, and $G_1(1)G_2$ by $G_1 \times G_2$.

For example, we can describe K^p_n using the above p–operation as follows:

$K^p_n = v_1(p)v_2 \ldots (p)v_n = (P)\{v_i : i \in I^n_1$ and, in particular,

$K^1_n = v_1 \times v_2 \ldots \times v_n = \prod\{v_i : i \in I^n_1\}$, and

$K^0_n = v_1 + v_2 \ldots + v_n = \sum\{v_i : i \in I^n_1\}$

where $V(K^p_n) = \{v_1, \ldots v_n\}$. We can abbreviate:

$K^p_n = v^{(p)n}$, $K^1_n = v^n$, and $K^0_n = v^{(0)n} = nv$.

Here and after $I^s_k = \{k, \ldots, s\}$ where k and s are integers and $k \leq s$.

Let $G^* = (V^*, w^*)$ be the weighted digraph obtained from $G = (V, w)$ as follows: $V^* = V \cup *$, $w^*(v, *) = 1 \in R_e$, $w^*(*, v) = 0 \in R_e$ for $v \in V$, and $w^*(u, v) = w(u, v)$ for $u, v \in V$. We call G^* the *cone* of G.

Let $f : A \to R$ be a function, and let $S \subseteq A$. Then let $f|^S = f|_{A \setminus S}$ and $f \downarrow_a^U$ be the function obtained from f by putting $f(u) = a \in R$ for every $u \in U$.

3 Laplacian matrix and the matrix–tree theorem

The first result to mention concerning enumeration of the spanning trees in a graph is the classical matrix–tree theorem.

Let $G = (V, g)$ be an (undirected) weighted graph with $V = \{v_1, \ldots, v_n\}$ and with the function $g : \binom{V}{2} \to R_e$. Let $L(H) = \{l_{ij}\}$ where $l_{ij} = -g[v_i, v_j]$ for $i \neq j$, and $l_{ii} = -\sum\{l_{ij} : j \neq i, j \in I_1^n$. The matrix $L(G)$ is called the *Laplacian matrix of the weighted graph* G. Let $L_v(G)$ denote the matrix obtained from $L(G)$ by deleting the row and the column of $L(G)$ corresponding to a vertex v of G.

Let $t(G) = \sum\{g(T) : T \in \mathcal{T}(\grave{G})\}$ where $G(T) = \prod\{g(e) : e \in E(\grave{T})\}$.

The matrix–tree theorem states the following:

3.1 [4, 22] (see also [2, 6]) *Let G be a weighted graph. Then $t(G) = \det(L_v(G))$ for every vertex v of G.*

If, in particular, R_e is the set of integers, and $g(e) \geq 0$ for every $e \in E(G)$ then G can be viewed as the multigraph obtained from $\grave{G}$ by replacing every edge e of $\grave{G}$ by $g(e)$ parallel edges, and so $t(G)$ is the number of spanning trees of the multigraph G.

A more general form of the matrix–tree theorem concerns directed weighted graphs.

Let $G = (V, g)$ be a weighted digraph. Put $t_v(G) = \sum\{g(T) : T \in \mathcal{T}_v(\grave{G})\}$ where $G(T) = \prod\{g(e) : e \in E(T)\}$ and $\mathcal{T}_v(G)$ is the set of all different spanning ditrees of a simple digraph G rooted at v. Let $L(G) = \{l_{ij}\}$ where $l_{ij} = -g(v_i, v_j)$ for $i \neq j$, and $l_{ii} = -\sum\{l_{ij} : j \neq i, j \in I_1^n\}$. The matrix $L(G)$ is called the *Laplacian matrix of* G. Let as above $L_v(G)$ denote the matrix obtained from $L(G)$ by deleting the row and the column of $L(G)$ corresponding to a vertex v of G.

3.2 [4] (see also [2, 6]) *Let G be a weighted digraph and $v \in V(G)$. Then*

$$t_v(G) = \det(L_x(G)).$$

This small formula opens a world of opportunities. We will use some of these opportunities in this paper.

4 Laplacian polynomial of a graph

The Laplacian polynomial of a graph was introduced in [9] (see also [6]). This polynomial has very interesting combinatorial properties (e.g. [9, 10, 12, 13], see also [6]). Because of these properties the Laplacian polynomial of a graph plays an important role in problems concerning the enumeration of spanning trees of graphs as well as the comparison of graphs by their number of spanning trees. We will see that many properties of the Laplacian polynomial can be generalized to the so called *forest volumes* of graphs (see Section 5). In this section we will list some important properties of the Laplacian polynomial of a graph.

Let $G = (V, g)$ be an undirected weighted graph with the function $g : E(G) \to R_e$ where R_e is a commutative ring. Let $L(G)$ be the Laplacian matrix of G (see Section 3). The *Laplacian polynomial* $L(\lambda, G)$ of a graph G is the characteristic polynomial of the Laplacian matrix $L(G)$, i.e. $L(\lambda, G) = \det(\lambda I_n - L(G))$ where $n = |V(G)|$.

By using **3.1**, one can prove that

4.1 [10, 12] *Let $G = (V, g)$ be a weighted graph with n vertices.*

$$L(\lambda, G) = \sum\{(-1)^i a_i(G)\lambda^{n-i} : i \in I_0^{n-1}\}$$

where

$$a_i(G) = \sum\{\gamma(F)g(F) : F \in \mathcal{F}(\grave{G}), |E(F)| = i\},$$
$$g(F) = \prod\{g(e) : e \in E(F)\},$$

and $\gamma(F)$ is the product of the numbers of vertices of the components of a forest F.

In particular, $a_0(G) = 1$, $a_1(G) = 2W(G)$ where $W(G) = \sum\{w(e) : e \in E(\grave{G})\}$, and $a_{n-1}(G) = nt(G)$.

Let $\Phi(\lambda, G) = \lambda^{m-n+1} L(\lambda, G)$ where $n = |V(G)|$ and $m = |E(G)|$. Note that $\Phi(\lambda, G)$ does not depend on the number of isolated vertices of G, i.e.
$\Phi(\lambda, G) = \Phi(\lambda, G + v)$.

The Laplacian polynomial has the following recursive property:

4.2 [12] *Let $G \setminus e = (V, w|_{w(e)=0})$. Then*

$$\Phi'_\lambda(\lambda, G) = \sum\{\Phi(\lambda, G \setminus e) : e \in E(\grave{G})\}$$

or equivalently,

$$\Phi(\lambda, G) = \Phi(a, G) + \int_a^z \sum\{\Phi(\lambda, G \setminus e) : e \in E(\grave{G})\} d\lambda.$$

This theorem was used in [12, 13] to find graphs having the maximum number of spanning trees among graphs of the same number of vertices and the same number of edges.

Suppose that $R_e = C$ is the set of complex numbers. Let $S(G) = \{\lambda_0(G), \ldots, \lambda_{n-1}(G)\}$ be the list of eigenvalues of G.

From **3.1** and **4.1** we have:

4.3 [9] *Let $G = (V, g)$ be a weighted graph with n vertices where $g : \binom{V}{2} \to C$. Then*

$$t(G) = n^{-1} \prod \{\lambda_i(G) : i \in I_1^{n-1}\}.$$

The Laplacian polynomial has the following important property:

4.4 [9, 11] *Let $G = (V, g)$ be a weighted graph with n vertices where $g : \binom{V}{2} \to C$. Let $\lambda_0(G) = 0$. Then there is a bijection $\alpha : I_1^{n-1} \to I_1^{n-1}$ such that*

$$\lambda_i(G) + \lambda_{\alpha(i)}(\bar{G}^p) = np$$

for every $i = I_1^{n-1}$.

Different proofs of this theorem were given in [9, 11]. One of them did not use the symmetry of $L(G)$ and allowed us to prove similar theorem for directed graphs. We will adopt later this proof to obtain a generalization of this theorem for the forest volumes of weighted digraphs. Another proof is very short and uses the symmetry of $L(G)$. Here we give this short proof.

Proof of 4.4. Since for every row of $L(G)$ the sum of its entries is equal to 0, it follows that the vector $\vec{1}$ is an eigenvector of $L(G)$ corresponding to the eigenvalue 0.

Since G is an undirected graph, the matrix $L(G)$ is symmetric. For every eigenvector x_0 of a symmetric $n \times n$–matrix A there exists a list $(x_0, x_1, \ldots, x_{n-1})$ of n mutually orthogonal eigenvectors (i.e. $x_i \cdot x_j = 0$ for $i \neq j$ and $i, j \in I_0^{n-1}$). Let $A = L(G)$ and $x_0 = \vec{1}$. We can assume that the eigenvector x_0 corresponds to $\lambda_0 = \lambda_0(G) = 0$ and that x_i corresponds to $\lambda_i = \lambda_i(G)$. Then $L(G)x_i = \lambda_i x_i$.

Since $\bar{G}^p$ is the graph p–complement to G, we have:

$L(G) + L(\bar{G}^p) = L(K^p) = npI_n + pJ_n$

where J_n is the $n \times n$–matrix with every entry equal 1. Therefore

$L(G)x_i + L(\bar{G}^p)x_i = npI_n x_i + pJ_n x_i$.

Since $x_0 \cdot x_i = 0$, we have $J_n x_i = 0$ for every $i \neq 0$ and $i \in I_1^{n-1}$. Therefore we obtain from the last equation: $L(\bar{G}^p)x_i = (np - \lambda_i)x_i$. Thus x_i is an eigenvector of $L(\bar{G}^p)$ with the eigenvalue $np - \lambda_i$. Let $\lambda_{\alpha(i)}$ denote the the eigenvalue of $L(\bar{G}^p)$ corresponding to its eigenvector x_i. Then clearly $\alpha : I_1^{n-1} \to I_1^{n-1}$ is a bijection and $\lambda_i(G) + \lambda_{\alpha(i)}(\bar{G}^p) = np$ for every $i = I_1^{n-1}$. □

Theorem **4.4** is equivalent to the following:

4.5 [9, 11] *Let G be a weighted graph with n vertices. Then*

$$L(\lambda, \bar{G}^p) = (-1)^{n-1} L(np - \lambda, G).$$

We will call this theorem the *Reciprocity Theorem* for the Laplacian polynomials.

From **4.3** and **4.5** we have:

4.6 [9] *Let $G = (V, g)$ be a weighted graph, and let $V(G) \subseteq V(K^p)$. Then*

$$t(K^p - G) = (sp)^{s-n-2} L(sp, G)$$

where $n = |V(G)|$ and $s = |V(K^p)|$.

This relation was used in [12, 13, 16, 19, 20] to obtain various results on the comparison of graphs by their number of spanning trees (see also [14]).

By using **4.1** and **4.6**, we can obtain the following interesting combinatorial interpretation of formula **4.6**:

4.7 [12] *Equation* **4.6** *is an inclusion–exclusion formula for $t(K^p - G)$.*

Suppose now that $G = (V, g)$ and g is a real valued non-negative function. Since $L(G)$ is symmetric, all eigenvalues of $L(G)$ are real numbers. It is easy to show [10] (see also [6]) that in this case $L(G)$ is a positive semi–definite matrix, and so all its eigenvalues are non-negative real numbers, say $0 \le \lambda_0(G) \le \lambda_1(G) \le \ldots \le \lambda_{n-1}(G)$ where $\lambda_0(G) = 0$. Therefore we have from **4.4**:

4.8 [9, 11] *Let $G = (V, g)$ be a weighted graph with n vertices where $g : \binom{V}{2} \to R_+$. Then $\lambda_i(G) + \lambda_{n-i}(\bar{G}^p) = np$ for every $i = I_1^{n-1}$.*

From **4.8** it follows that

4.9 [10] *Let $G = (V, g)$ be a weighted graph with n vertices where $g : \binom{V}{2} \to [0, p]$. Then $0 \le \lambda_i(G) \le np$ for every $i = I_1^{n-1}$.*

By using **4.5**, one can easily obtain:

4.10 [9, 11] *Let G_1 and G_2 be weighted graphs with n_1 and n_2 vertices, respectively. Then*

$$\lambda L(\lambda, G_1(p)G_2) = (\lambda - n_1 p - n_2 p) L(\lambda - n_2 p, G_1) L(\lambda - n_1 p, G_2).$$

The last relation was used in [9, 11] to give an algorithm for finding the Laplacian spectra and the Laplacian polynomials of so called *decomposable* weighted graphs. We will describe the analogue of this algorithm in Section 10 for finding the tree and forest volumes of decomposable weighted graphs.

Similar results were shown to be true for weighted digraphs [11].

5 Tree and forest volumes of graphs

The notion of the *spanning tree volume* (or simply *tree volume*) of a graph was introduced in [17, 18]. For a simple graph G the tree volume of G is actually the "sum" of the "Prüfer code volumes" of all spanning trees of G where the labels of the vertices of G are viewed as independent indeterminants that can be multiplied and commute, and the "volume of a Prüfer code" is the product of its entries.

We recall that the *digraph–skeleton* $\grave{G}$ of G is a simple digraph such that $V(\grave{G}) = V(G)$ and $(u, v) \in E(\grave{G})$ if and only if $w(u, v) \neq 0 \in R_e$. If $e = (u, v) \in E(\grave{G})$ we put $w(e) = w(u, v)$.

Given a simple digraph Q and a vertex $r \in V(Q)$, let as above $\mathcal{T}_r(Q)$ denote the set of spanning ditrees of Q rooted at r. Let $\mathcal{T}(Q) = \cup\{\mathcal{T}_r(Q) : r \in V(Q)\}$. As usual, let $t_r(Q) = |\mathcal{T}_r(Q)|$ and $t(Q) = |\mathcal{T}(Q)|$.

A *weighted ditree* is a weighted digraph whose skeleton is a ditree.

For a weighted digraph $G = (V, g)$, let $\mathcal{T}_r(G)$ denote the set of weighted ditrees $T = (V_t, t)$ such that r is the root of T and $t(u, v) = g(u, v)$ for $(u, v) \in V_t$.

Let $x : V(G) \to R_v$ be a function where R_v is a commutative ring. We assume as above that a commutative and distributive operation ab is defined for $a \in R_v$ and $b \in R_e$.

Let $T_r = (V, w)$ be a weighted ditree rooted at r. Put

$X(T_r) = \prod\{x(v)^{d(v,\grave{T}_r)-1} : v \in V(\grave{T}_r)\}$ and $W(T_r) = \prod\{w(e) : e \in E(\grave{T}_r)\}$.

Clearly $X(T_r) = x_r^{d_{in}(r,\grave{T}_r)-1} \prod\{x(v)^{d_{in}(v,\grave{T}_r)} : v \in V(\grave{T}_r) \setminus r\}$.

Let $\mathcal{T}_r(T_r, x) = X(T_r)W(T_r)$. We call $\mathcal{T}_r(T_r, x)$ the *volume* of a weighted tree T_r.

The *spanning tree volume* (or $\mathcal{T}$*–volume*) of an arbitrary weighted digraph $G = (V, w)$ *with respect to a given vertex* $r \in V(G)$ is

$$\mathcal{T}_r(G, x) = \sum\{\mathcal{T}_r(T, x) : T \in \mathcal{T}_r(G)\}.$$

Clearly $\mathcal{T}_r(G, x)$ is a polynomial in variables $x(v)$, $v \in V(G)$.

The *spanning tree volume* (or simply $\mathcal{T}$*–volume*) of a weighted digraph $G = (V, w)$ is

$$\mathcal{T}(G, x) = \sum\{x(r)\mathcal{T}_r(G, x) : r \in V(G)\}. \tag{5.1}$$

The spanning tree volume of a weighted digraph G can also be viewed as a generating function of weighted spanning ditrees of G classified by their roots, degree functions, and numbers of edges (if the edge weights are non-negative integers).

Let $\bar{1}$ be a function $x : V(G) \to R_v$ such that $x(v) = 1 \in R_v$ for every $v \in V(G)$. Clearly $t_r(F) = \mathcal{T}_r(G, \bar{1})$ and $t(F) = \mathcal{T}(G, \bar{1})$.

Let $G^* = (V^*, w^*)$ be the weighted digraph obtained from $G = (V, w)$ as follows: $V^* = V \cup *$ (where $* \not\in V$), $w^*(v, *) = 1 \in R_e$, $w^*(*, v) = 0 \in R_e$ for $v \in V$, and $w^*(u, v) = w(u, v)$ for $u, v \in V$. For a function $x : V(G) \to R_v$ let $x^* : V(G^*) \to R_v$ be a function such that $x^*(v) = x(v)$ if $v \in V(G)$ and $x^*(*) = z \in R_v$, and so x is a restriction of x^* on $V(G)$. We call G^* the *cone* of G.

The *forest volume* or simply *$\mathcal{F}$–volume* of $G = (G, w)$ is

$$\mathcal{F}(z, G, x) = \mathcal{T}_*(G^*, x^*). \tag{5.2}$$

The forest volume of a weighted digraph G (with positive integer weights) can also be viewed as a generating function of weighted spanning diforests of G classified by their numbers of edges and degree functions.

We recall that a *diforest* is a digraph in which every component is a ditree. A *weighted diforest* is a weighted digraph whose skeleton is a diforest.

For a weighted digraph $G = (V, g)$, let $\mathcal{F}(G)$ denote the set of weighted diforests $F = (V_f, f)$ such $V_f = V$ and $f(u, v) = g(u, v)$ for every $(u, v) \in E(\grave{F})$.

Let F be a weighted diforest. Let $w(F) = \prod\{w(e) : e \in E(\grave{F})\}$. Put

$$\mathcal{F}(F, x) = w(F) \prod\{x(v)^{d_{in}(v, \grave{F})} : v \in V(F)\}.$$

From the definition of $\mathcal{F}(z, G, x)$ it follows that

5.1 *Let $G = (V, w)$ be a weighted digraph, $n = |V(G)|$ and let $x : V(G) \to R_e$ be a function. Then*

$$\mathcal{F}(z, G, x) = \sum\{z^{n-1-i} f_i(G, x) : i \in I_0^{n-1}\} \tag{5.3}$$

where

$$f_i(G, x) = \sum\{\mathcal{F}(F, x) : F \in \mathcal{F}(G), |E(F)| = i\} \tag{5.4}$$

and, in particular, $f_0(G, x) = 1$, $f_1(G, x) = \sum\{w(u, v)x(v) : (u, v) \in E(\grave{G})$, and

$$f_{n-1}(G, x) = \mathcal{F}(0, G, x) = \mathcal{T}(G, x). \tag{5.5}$$

Clearly $\mathcal{F}(z, G, x)$ is a polynomial in z and $x(v)$, $v \in V(G)$. As a polynomial $\mathcal{F}(z, G, x)$ has the following useful property.

5.2 *$\mathcal{F}(z, G, x)$ is a homogeneous polynomial of degree $|V(G)| - 1$ in variables $x(v)$: $v \in V(G)$ and z.*

Proof By the definition, the forest volume of G is $\mathcal{F}(z, G, x) = \mathcal{T}_*(G^*, s)$. Let T be a spanning tree of $\grave{G}^*$. Then $V(T) = V(G^*) = V(G) \cup \{*\}$, and clearly

$\sum\{d(v, T) : v \in V(T)\} = 2|V(G^*)|$. Therefore

$\deg(\mathcal{T}_*(T, x)) = \sum\{d(v, T) - 1 : v \in V(T)\} = |V(G^*)| - 2 = |V(G)| - 1.$ □

6 Recursive properties of graph volumes

In this section we establish some recursive relations for the forest volumes of digraphs. We will use some of them in Sections 8 and 11.

For a vertex u of G, let $d_u(G,x) = \sum\{x(v)w(u,v) : v \in V(G) \setminus u\}$. It is natural to call $d_u(G,x)$ the *outdegree of* u in (G,x). We recall that for $U \subseteq V$, $x|^U = x|_{V \setminus U}$ and $x \downarrow_0^U$ is obtained from x by putting $x(u) = 0$ for every $u \in U$.

6.1 *Let $r \in V(G)$ and $U \subseteq V(G)$. If $r \notin U$ then*

$\mathcal{T}_r(G, x \downarrow_0^U) = \prod\{d_u(G \setminus (U \setminus u), x) : u \in U\}\mathcal{T}_r(G \setminus U, x|^U)$.

If $r \in U$ then

$\mathcal{T}_r(G, x \downarrow_0^U) = \sum\{x_v w(v,r)\mathcal{T}_v(G \setminus U, x|^U) : v \in V(G \setminus U)\}$

$\prod\{d_u(G \setminus (U \setminus u), x) : u \in U \setminus r\}\mathcal{T}_r(G \setminus U, x|^U)$.

Proof By the definitions of $\mathcal{T}_r(G,x)$,

$\mathcal{T}_r(G, x \downarrow_0^U) = \sum\{\mathcal{T}_r(T,x) : T \in \mathcal{T}_r(\grave{G}), d(u,T) = 1, u \in U\}$.

If u is not the root of T, then the equality $d(u,T) = 1$ implies $d_{in}(u,T) = 0$. If $u = r$ is the root of T, then the equality $d(r,T) = 1$ implies $d_{in}(r,T) = 1$. Therefore if $r \notin U$, then

$\mathcal{T}_r(G, x \downarrow_0^U) = \prod\{d_u(G \setminus (U \setminus u), x) : u \in U\} \sum\{\mathcal{T}_r(T, x|^U) : T \in \mathcal{T}_r(\grave{G} \setminus U)\} =$

$\prod\{d_u(G \setminus (U \setminus u), x) : u \in U\}\mathcal{T}_r(G \setminus U, x|^U)$.

Now suppose that $r \in U$. Then

$\mathcal{T}_r(G, x \downarrow_0^U) = \sum\{x_v w(v,r)\mathcal{T}_v(G \setminus U, x|^U) : v \in V(G \setminus U)\}$

$\prod\{d_u(G \setminus (U \setminus u), x) : u \in U \setminus r\}\mathcal{T}_r(G \setminus U, x|^U)$. □

Note that the formula for $f_{n-1}(H,x)$ in **5.1** is a particular case of the last formula in **6.1** when $r = *$, $G = H^*$ (and so $G \setminus r = H$), and $w(v,r) = w(v,*) = 1$ for every $v \in V(H)$.

From **6.1** we have in particular:

6.2 *Let $U \subset V(G)$. Then*

$$\mathcal{F}(z, G, x \downarrow_0^U) = \prod\{(z + d_u(G \setminus U, x) : u \in U\}\mathcal{F}(z, G \setminus U, x|^U).$$

Proof (uses **6.1**). Let $x^*(u,v) = x(u,v)$ if $u, v \in V(G)$, $x^*(v,*) = 1 \in R_e$, and $x^*(*,v) = 0 \in R_e$ for every $v \in V(G)$. By the definitions of G^* and a forest volume of a digraph,

$\mathcal{F}(z, G, x \downarrow_0^U) = \mathcal{T}_*(G^*, x^* \downarrow_0^U)$. Clearly $d_u(G^*, x) = z + d_u(G,x)$. Therefore by **6.1**,

$\mathcal{F}(z, G, x \downarrow_0^U) = \prod\{(z + d_u(G \setminus U, x) : u \in U\}\mathcal{F}(z, G \setminus U, x|^U)$. □

Next statement is a generalization of the recursive relation **4.2** for the Laplacian polynomial. Let $|V(G)| = n$ and $|E(G)| = m$. It is sometimes more convenient to consider the polynomial

$$\Phi(z, G, x) = z^{m-n+1}\mathcal{F}(z, G, x)$$

instead of $\mathcal{F}(z, G, x)$ because $\Phi(z, G, x)$ does not depend on the number of isolated vertices of G, i.e. $\Phi(z, G, x) = \Phi(z, G + g, x)$.

6.3 *Let $G \setminus e = (V, w|_{w(e)=0})$. Then*

$$\Phi'_z(z, G, x) = \sum\{\Phi(z, G \setminus e, x) : e \in E(G)\}$$

or equivalently,

$$\Phi(z, G, x) = \Phi(a, G, x) + \int_a^z \sum\{\Phi(z, G \setminus e, x) : e \in E(G)\}dz.$$

Proof (uses **5.1**). Let $|V(G)| = n$ and $|E(G)| = m$. From (5.4) in **5.1**, we have:

$$(m - n + 1 + i) \cdot f_i(G, x) = \sum\{f_i(G \setminus e, x) : e \in E(G)\}. \tag{6.1}$$

By (5.3) in **5.1**,

$$\Phi(z, G, x) = z^{m-n+1}\mathcal{F}(z, G_n^m, x) = \sum\{z^{m-n+1+i}f_i(G, x) : i \in I_1^{n-1}\}. \tag{6.2}$$

By (6.1) and (6.2),

$\Phi'_z(z, G, x) = \sum\{(m - n + 1 + i)z^{m-n+i}f_i(G, x) : i \in I_1^{n-1}\} =$

$\sum\{z^{m-n+i}\sum\{f_i(G \setminus e, x) : e \in E(G)\} : i \in I_1^{n-1}\} =$

$\sum\sum\{\{z^{m-1-n+1+i}f_i(G \setminus e, x) : i \in I_1^{n-1}\} : e \in E(G)\} =$

$= \sum\{\Phi(z, G \setminus e, x) : e \in E(G)\}$. □

We recall the deletion and contraction operations for weighted digraph.

For a weighted digraph $G = (V, g)$ and $e = (b, a)$, $a, b \in V$, let $G \setminus e = (V, g|_{g(e)=0}$. Let G/e denote the graph $H = (V_h, h)$ such that $V_h = V \setminus b$ and $h(u, v) = g(u, v)$ if $u, v \in V_h \setminus a$, $h(a, v) = g(a, v) + g(b, v)$, and $h(v, a) = g(v, a) + g(v, b)$. Let $G_b = G \setminus \{(b, v) : v \in V_h \setminus b$.

It is easy to see that

6.4 *Let $G = V, g)$ be a digraph, $r \in V$, and $e = (b, a) \in \grave{G}$. Then*

$$t_r(G) = t_rG \setminus e) + g(e)t_r(G_b/e).$$

7 Graph volumes and Laplacian polynomials

It turns out that there are natural relations between the tree and forest volumes of a digraph and its Laplacian matrix and Laplacian polynomial.

The first important observation is that a theorem similar to the matrix–tree theorem **3.2** turns out to be true for the tree volume of a weighted digraph.

Let $I_k^s = \{k, \dots s\}$ for $k \le s$. Let $V(G) = \{v_1, \dots, v_n\}$. Let $L^x(G) = \{l_{ij}\}$ where $l_{ij} = -x(v_i)x(v_j)w(v_i, v_j)$ if $i \ne j$, and $l_{ii} = -\sum\{l_{ij} : j \ne i, j \in I_1^n\}$.

Let $M^x(G) = \{m_{ij}\}$ where $i, j \in I_1^n$, $m_{ij} = -x(v_j)w(v_i, v_j)$ if $i \ne j$ and $m_{ii} = -\sum\{m_{ij} : j \ne i, j \in I_1^n\}$. In other words, $L^x(G)$ is obtained from $M^x(G)$ by multiplying every entry of the i-th row of $M^x(G)$ by $x(v_i)$. If all $x(v_i)$'s are 1 then $L^x(G) = M^x(G) = L(G)$. Thus $M^x(G)$ is a generalization of the Laplacian matrix of G. We call $M^x(G)$ the *x–Laplacian matrix* of G. Clearly $det(L^x(G)) = det(M^x(G)) = 0$. Let $L_v^x(G)$ and $M_v^x(G)$ denote the matrices obtained, respectively, from $L^x(G)$ and $M^x(G)$ by deleting the row and the column of $L^x(G)$ corresponding to a vertex v of G.

Let $\pi^x(G) = \pi(x) = \prod\{x(v) : v \in V(G)\}$.

From **3.2** we have:

7.1 $\pi^x(G)x(r)\mathcal{T}_r(G, x) = det(L_r^x(G)) = \pi^x(G)det(M_r^x(G))$ *where* $r \in V(G)$.

From **7.1** we have

7.2 $z\pi^x(G)\mathcal{F}(z, G, x) = det(L_*^x(G^*)) = \pi^x(G)det(M_*^x(G^*))$.

It is easy to see that

7.3 $det(L_*^x(G^*)) = \pi^x(G)det(zI_n + M^x(G))$ *where* $n = |V(G)|$.

As in Section 4, $L^x(\lambda, G) = det(\lambda I_n - L^x(G))$ and $\check{L}^x(\lambda, G) = \lambda^{-1}L^x(\lambda, G)$. Similarly let $M^x(\lambda, G) = det(\lambda I_n - M^x(G))$ and $\check{M}^x(\lambda, G) = \lambda^{-1}M^x(\lambda, G)$. We call $M^x(\lambda, G)$ an *x–Laplacian polynomial* of G.

Since $det(M^x(G) = 0$, clearly $\check{M}^x(\lambda, G)$ is a polynomial in λ. We recall that $x : V(G) \to R_v$ and $\lambda \in R_v$ where R_v is a commutative ring.

It turns out that there is a natural relation between the forest volume and x–Laplacian polynomial of a graph.

From **7.2** and **7.3** we have:

7.4 $\mathcal{F}(z, G, x) = (-1)^{n-1}\check{M}(-z, G, x)$ *where* $n = |V(G)|$.

If $x(v) = 1$ for every $v \in V(G)$ then clearly $M(\lambda, G, x) = L(\lambda, G)$.
Therefore from **7.4** we have in particular:

7.5 $\mathcal{F}(z, G, \bar{1}) = (-1)^{n-1}\check{L}(-z, G)$ *where* $n = |V(G)|$.

The last statement shows that the forest volume is a natural generalization of the Laplacian polynomial of a graph. We will see that many properties of the Laplacian polynomials can be generalized to the forest volumes.

Suppose that R_v is the set of real or complex numbers. Then we can consider the list $S(G,x) = (\lambda_0(G,x), \ldots, \lambda_{n-1}(G,x))$ of eigenvalues of $M^x(G)$ where $\lambda_i(G,x) \in C$. We call $S(G,x)$ the x–Laplacian spectrum of G. Since $det(M^x(G)) = 0$, one of $\lambda_i(G,x)$ is 0. We will assume that $\lambda_0(G,x) = 0$. Clearly $M^x(\lambda, G) = \prod\{\lambda - \lambda_i(G,x) : i \in I_1^{n-1}\}$.

Thus from **5.1** and **7.2** we have:

7.6 *Suppose that R_v is the set of real or complex numbers. Then*

$$\mathcal{T}(G,x) = \prod\{\lambda_i(G,x) : i \in I_1^{n-1}\}$$

where $n = |V(G)|$.

8 Reciprocity theorem for graph volumes

In this section we obtain a generalization **11.2** of the Reciprocity Theorem **4.5** on the Laplacian polynomials to the x–Laplacian polynomials (and therefore to the forest volumes) of graphs. We use this generalization to establish a relation between the forest volume of a graph and the tree volume of its complement (see **8.5**). We also use this theorem in Section 10 to give an algorithm for finding the forest volumes of so called decomposable graphs.

We recall that $\bar{G}^p = K_p - G$ where $V(K^p) = V(G)$.

8.1 *Let $n = |V(G)|$. Then*

$$\check{M}(\lambda, \bar{G}^p, x) = (-1)^{n-1}\check{M}(px(G) - \lambda, G, x).$$

Proof This proof is similar to the proof of theorem **4.5** in [9, 11] on the Laplacian matrices of graphs.

Let $M^x(G) = \{m_{ij}\}$ and $M^x(\bar{G}^p) = \{\bar{m}_{ij}\}$. Let $A = \{a_{ij}\}$ be the $n \times n$–matrix such that $a_{i1} = 1$, $a_{ii} = \lambda - m_{ii}$, and $a_{ij} = -m_{ij}$ for $j \neq 1$, $i \neq j$ and $i, j \in I_1^n$. The matrix $\bar{A} = \{\bar{a}_{ij}\}$ is defined similarly by $M^x(\bar{G}^p)$. Clearly $\check{M}(\lambda, G, X) = det(A)$ and $\check{M}(\lambda, \bar{G}^p, X) = det(\bar{A})$. Put $x(v_i) = x_i$. Since $\bar{G}^p$ is p–complement of G we have: $m_{ii} + \bar{m}_{ii} = px(G) - px_i$ and $m_{ij} + \bar{m}_{ij} = -px_j$ for $i \neq j$. Therefore $a_{ii} + \bar{a}_{ii} = 2\lambda - px(G) + px_j$ and $a_{ij} + \bar{a}_{ij} = px_j$ for $j \neq 1$ and $i \neq j$.

Let $\bar{B} = \{\bar{b}_{ij}\}$ where $\bar{b}_{i1} = 1$, and $\bar{b}_{ij} = \bar{a}_{ij} - px_j$ for $i \neq 1$ and $i \neq j$. In other words, B is obtained from A by adding the first column times px_j to the j-th column. Clearly $det(\bar{A}) = det(\bar{B})$. From the above equations we have: $\bar{b}_{ii} = 2\lambda - px(G) - a_{ii} = \lambda - px(G) + m_{ii}$ and $\bar{b}_{ij} = -a_{ij} = m_{ij}$ for $j \neq 1$ and $i, j \in I_1^n$. Therefore $det(\bar{B}) = (-1)^{n-1}\check{M}(px(G) - \lambda, G, x)$. Since $det(\bar{A}) = det(\bar{B})$ and $\check{M}(\lambda, \bar{G}^p, x) = det(\bar{A})$, we have $\check{M}(\lambda, \bar{G}^p, x) = (-1)^{n-1}\check{M}(px(G) - \lambda, G, x)$. □

Suppose that both R_v and R_e are the sets of real or complex numbers. Then we can consider the list
$S(G,x) = (\lambda_0(G,x), \ldots, \lambda_{n-1}(G,x))$ of eigenvalues of $M^x(G)$ where $\lambda_i(G,x) \in C$.
Since $M^x(\lambda, G) = \prod\{(\lambda - \lambda_i(G,x)) : i \in I_0^{n-1}\}$, we have from **8.1**:

8.2 *Let both R_v and R_e be the sets of real or complex numbers. Then there is a bijection $\alpha : I_1^{n-1} \to I_1^{n-1}$ such that*

$$\lambda_i(G,x) + \lambda_{\alpha(i)}(G,x) = px(G)$$

for every $i \in I_1^{n-1}$.

From **8.1** we obtain the corresponding Reciprocity Theorem for the forest volumes:

8.3 *Let $n = |V(G)|$. Then*

$$\mathcal{F}(z, \bar{G}^p, x) = (-1)^{n-1}\mathcal{F}(-z - px(G), G, x).$$

Proof (uses **7.4** and **8.1**). By **7.4**,

$\mathcal{F}(z,G,x) = (-1)^{n-1}\check{M}(-z,G,x)$. Therefore by **8.1**, $\mathcal{F}(z,\bar{G}^p,x) =$

$(-1)^{n-1}\check{M}(-z,\bar{G},x) = \check{M}(z + px(G), G, x) = (-1)^{n-1}\mathcal{F}(-z - px(G), G, x)$. □

This proof uses the relation between $\mathcal{F}(z,G,x)$ and the characteristic polynomial of the matrix $M^x(G)$ (see **7.4**). By using **5.2** and **6.2**, we can give an alternative proof of the reciprocity theorem **8.3** for $\mathcal{F}(z,G,x)$. The idea of this proof is similar to that A. Réney used in [27].

We need the following simple statement on polynomials.

8.4 *Let p be a polynomial in $|A|$ variables in $x(a) : a \in A$. Suppose that*

($h1$) the degree of p is less than $|A|$: $\deg(p) < |A|$, and

($h2$) $p|_{x(a)=0} \equiv 0$ for every $a \in A$.

Then $p \equiv 0$.

Proof Suppose on the contrary that $p \not\equiv 0$. Then p has a monomial, say M, with a non-zero coefficient. Since by ($h1$), $\deg(p) < |A|$, there exists $b \in A$ such that $x(b)$ is not an entry of M. Therefore $p|_{x(b)=0} \not\equiv 0$. This contradicts ($h2$). □

Now we are ready to give an alternative

Proof of Theorem 8.3 (uses **5.2**, **6.2**, and **8.4**).
Let as above $x|_{V\setminus u} = x|^u$ and $x|_{x(u)=0} = x\downarrow_0^u$. Let

$\Delta_p(z,G,x) = \mathcal{F}(z,\bar{G}^p,x) - (-1)^{n-1}\mathcal{F}(-z - px(G), G, x)$.

We will prove **8.3** by induction on $|V(G)|$ that $\Delta_p(z,G,x) \equiv 0$. Suppose that $|V(G)| = 1$. There is only one graph K_1 with one vertex, and $\mathcal{F}(z,K_1,x) \equiv 1$. Therefore

$\Delta_p(z, K_1, x) \equiv 0$.

Now suppose that $|V(G)| \geq 2$. Let $u \in V(G)$. By **6.2**,

$\Delta_p(z, G, x \downarrow_0^u) = \mathcal{F}(z, \bar{G}^p, x \downarrow_0^u) - (-1)^{n-1}\mathcal{F}(-z - px(G), G, x \downarrow_0^u) =$

$(z + \sum\{x(v)(p - w(v,u)) : v \in V(G) \setminus u\})\mathcal{F}(z, \bar{G}^p \setminus u, x|^u)-$

$(-1)^{n-1}(-z - px(G) + \sum\{x(v)w(v,u) : v \in V(G)\})\mathcal{F}(-z - px(G), G \setminus u, x|^u) =$

$(z + \sum\{x(v)(p - w(v,u)) : v \in V(G) \setminus u\})(\mathcal{F}(z, \bar{G}^p \setminus u, x|^u)-$

$(-1)^{n-2}\mathcal{F}(-z - px(G), G \setminus u, x|^u) =$

$(z + \sum\{x(v)(p - w(v,u)) : v \in V(G) \setminus u\})\Delta_p(z, G \setminus u, x|^u)$.

By the induction hypothesis, $\Delta_p(z, G \setminus u, x|^u) \equiv 0$. Therefore $\Delta_p(z, G, x \downarrow_0^u) \equiv 0$ for every $u \in V(G)$. By **5.2**, the degree of $\Delta_p(z, G, x)$, as a polynomial in $|V(G)|$ variables $x(v)$, $v \in V(G)$, is less than $|V(G)|$. Therefore by **8.4**, $\Delta_p(z, G, x) \equiv 0$. □

We recall some notations. Let $F = (V, f)$ be a weighted diforest. Put $f(F) = \prod\{f(e) : e \in E(\grave{F})\}$, and

$$\mathcal{F}(F, x) = f(F) \prod\{x(v)^{d_{in}(v,\grave{F})} : v \in V(F)\}.$$

If $V(G) \subseteq V(K^p)$ then clearly $K^p - G = \overline{G + dg}^p$ where $d = |V(K_p| - |V(G)|$.

8.5 *Let G be a weighted digraph, K^p be a p–complete weighted digraph, $|V(G)| = n$, $|V(K^p)| = s$, $V(G) \subseteq V(K^p)$ (and so $n \leq s$). Let $x : V(K^p) \to R_v$. We will write $\mathcal{F}(z, G, x)$ instead of $\mathcal{F}(z, G, x|_G)$. Then*

$\mathcal{T}((K^p - G), x) = (-1)^{n-1}(px(K^p))^{s-n}\mathcal{F}(-px(K^p), G, x) =$

$$(-1)^{n-1} \sum\{(-px(K^p))^{s-i-1} f_i(G, x) : i \in I_0^{n-1}\}$$

where

$$f_i(G, x) = \sum\{\mathcal{F}(F, x) : F \in \mathcal{F}(G), |E(\grave{F})| = i\}.$$

Proof (uses **5.1**, and **8.3**, **10.4** below). We write + instead of the graph operation (0).

$\mathcal{T}(K^p - G, x) = \mathcal{F}(0, K^p - G, x) = \mathcal{F}(0, \overline{G + (s-n)g}^p, x) =$

$(-1)^{s-1}\mathcal{F}(-px(K^p), G + (s-n)g, x) = (-1)^{s-1}(-px(K^p))^{s-n}\mathcal{F}(-px(K^p), G, x)$

$(-1)^{n-1}(px(K^p))^{s-n}\mathcal{F}(-px(K^p), G, x)$. □

Since $t(G) = \mathcal{T}(G, \bar{1})$, we have from **5.1** and **8.5** the following analogue of **4.6** for weighted digraphs:

8.6 [11] *Suppose that the hypothesis of* **8.5** *is satisfied. Then*

$$t(K^p - G) = (-1)^s(ps)^{s-n}\mathcal{F}(-ps, G, \bar{1}) = (ps)^{s-n-1}L(ps, G).$$

9 Graph constructions

In this section we describe some constructions that give new weighted digraphs from a given set of weighted digraph–bricks. We will show in Sections 10 and 11 that these constructions have the following important property: if we know the forest volumes for the bricks and the "structure" of the construction then we can easily find the forest volume of the result of this construction.

We recall that a graph $\bar{G}^p$ is *p–complement* to G if $V(G) = V(\bar{G}^p)$ and $\bar{G}^p = K^p - G$. Given two disjoint weighted digraphs $G_1 = (V_1, w_1)$ and $G_2 = (V_1, w_1)$, let $G_1(p)G_2$ denote the weighted digraph $G = (V, w)$ such that $V = V_1 \cup V_2$, and $w(u, v) = w_i(u, v)$ if $u, v \in V_i$, $i = 1, 2$, and $w(u, v) = p$ if $u \in V_1$ and $v \in V_2$ or $v \in V_1$ and $u \in V_2$.

Clearly

9.1 *Let $G_1 = (V_1, w_1)$ and $G_2 = (V_1, w_1)$ be two disjoint weighted digraphs. Then*

$$\overline{G_1(p)G_2}^p = \bar{G_1}^p + \bar{G_2}^p.$$

A digraph G is called *decomposable* if G can be obtained from some digraphs by a serious of operations $(p_1), \ldots (p_k)$. A digraph G is called *totally decomposable* if G can be obtained from one-vertex graphs by a serious of operations $(p_1), \ldots (p_k)$. Here each $p_i \in R_e$.

A decomposable graph G can be naturally described [9, 11] by its *decomposition description* $D(G) = (T, s, \mathcal{B})$ where

$(a1)$ $T = T(G)$ is a ditree,
$(a2)$ $s = s(G) : V(T) \to R_e$ is a function that assigns the label $s(u)$ to every vertex u of T,
$(a3)$ $\mathcal{B} = \mathcal{B}(G) = \{G_a : a \in L(T)\}$ where $L(T)$ is the set of leaves (or sinks) of T.

Every vertex t of $T(G)$ will correspond to a graph $\Omega(t)$.

Given a weighted digraph G, we can define $D(G) = (T, s, \mathcal{B})$ and the function Ω recursively as follows. If G is not decomposable then let T be the trivial ditree consisting of one vertex r, $s(r) = 0$, $\mathcal{B} = \{G_r = G\}$, and $\Omega(r) = G$. Suppose that G is p–decomposable, i.e. $G = G_1(p) \ldots (p)G_k$ where each G_i is not p–decomposable. We assume that $D(G_i) = (T_i, s_i, \mathcal{B}_i)$ and $\Omega(r_i) = G_i$ (where r_i is the root of T_i) are already defined for every $i \in I_1^k$. Then

$(c1)$ T is obtained from the disjoint rooted ditrees T_i, $i \in I_1^k$, by adding a new vertex r, the root of T, and by connecting each root r_i with r by the arc (r_i, r),
$(c2)$ $s(r) = p$ and $s(v) = s_i(v)$ if $v \in V(T_i)$, and,
$(c3)$ $\mathcal{B} = \cup\{\mathcal{B}_i : i \in I_1^k\}$, and
$(c4)$ $\Omega(r) = G$.

From the above description it follows that

9.2 *Let $D(G) = (T, s, \mathcal{B})$ be the decomposition description of a weighted digraph G and $L(T)$ the set of leaves of T. Then $\mathcal{B} = \{\Omega(t) : t \in L(T)\}$, and $s(u) \neq s(v)$ if $(u, v) \in E(T)$, and $s(u) = 0$ if $u \in L(T)$.*

Note that the function Ω can be easily found from the description $D(G) = (T, s, \mathcal{B})$.

If G is totally decomposable then clearly every member of $\mathcal{B}(G)$ is the one vertex graph. So a totally decomposable graph can be described by a pair $D(G) = (T, s)$.

Two decomposition descriptions $D(G_1) = (T_1, s_1, \mathcal{B}_1)$ and $D(G_2) = (T_2, s_2, \mathcal{B}_2)$ are *isomorphic* if there exists an isomorphism $\alpha : V(T_1) \to V(T_2)$ such that $s_1(v) = s_2(\alpha(v))$, $v \in V(T_1)$, and $G_a \in \mathcal{B}_1$ is isomorphic to $G_{\alpha(a)} \in \mathcal{B}_2$ for every $a \in S(T_1)$.

It is easy to see that

9.3 *Two graphs are isomorphic if and only if their decomposition descriptions are isomorphic.*

An s–ditree M is a pair (T, s) where T is a ditree and $s : V(T) \to R_e$ is a function such that $s(u) = 0$ if $u \in S(T)$, and $s(u) \neq s(v)$ if $(u, v) \in E(T)$.

Given an s–ditree $\hat{T} = (T, s)$, we say that G is the *M–aggregate* of $\mathcal{B} = \{G_a : a \in A\}$, written $G = \hat{T}(\mathcal{B}) = \hat{T}[G_a : a \in A]$, if $D(G) = (T, s, \mathcal{B})$. The s–ditree $\hat{T}$ is called the *frame* and the graphs G_a, $a \in V(G)$, are called the *bricks* of the $\hat{T}$–aggregate.

Now we will describe another important construction called G–composition of weighted digraphs.

Let $G = (A, g)$ and $G_a = (B_a, g_a)$, $a \in A$, be disjoint weighted digraphs where $g : A^2 \to R_e$ and $g_a : B_a^2 \to R_e$ are functions such that $g(a, a) = 0$ and $g_a(b, b) = 0$ for $a \in A$ and $b \in B$, and R_e is a commutative ring. As usual $V(G) = A$ and $V(G_a) = B_a$. Let $B = \cup\{B_a : a \in A\}$. Let $L_a = \{(a, b) : b \in B_a\}$ and $L = \cup\{L_a : a \in A$, and so $L = A \times B$.

The weighted digraph $\Gamma = (V, \gamma)$ is called the *G–composition of* $\{G_a : a \in V(G)\}$, written $\Gamma = G\{G_a : a \in V(G)\}$, if $V = V(\Gamma) = L$ and for two vertices $v_1 = a_1b_1$ and $v_2 = a_2b_2$ of Γ, $\gamma(v_1, v_2) = g(a_1, a_2)$ if $a_1 \neq a_2$, and $\gamma(v_1, v_2) = g_a(b_1, b_2)$ if $a_1 = a_2 = a$.

The graph G in $\Gamma = G\{G_a : a \in V(G)\}$ is called the *frame* and the graphs G_a, $a \in V(G)$, are called the *bricks* of the G–composition.

A *graph–extension* (or *G–extension*) considered in [15, 18] is a particular case of a G–composition $G\{G_a : a \in V(G)\}$ where every brick G_a is a graph having no edges.

It is easy to see that

9.4 $\hat{T}[G^i\{G^i_a : a \in V(G)\} : i \in I^k_1] = \hat{T}[G^i : i \in I^k_1]\{G^i_a : a \in V(G)\}$.

10 Tree and forest volumes of decomposable graphs

We recall the notion of the p–operation on digraphs. Let $G_1 = (V_1, g_1)$ and $G_2 = (V_1, g_1)$ be two disjoint weighted digraphs. Let $G_1(p)G_2$ denote a weighted digraph $G = (V, g)$ such that $V = V_1 \cup V_2$, and $g(u, v) = g_i(u, v)$ if $u, v \in V_i$, $i = 1, 2$, and $g(u, v) = p$ if $u \in V_1$ and $v \in V_2$ or $v \in V_1$ and $u \in V_2$.

It is easy to see that

10.1 *Let G_1 and G_2 be two disjoint weighted digraphs. Then*

$M(\lambda, G_1 + G_2, x_1 \cup x_2) = M(\lambda, G_1, x_1)M(\lambda, G_2, x_2)$*, and so*

$\check{M}(\lambda, G_1 + G_2, x_1 \cup x_2) = \lambda\check{M}(\lambda, G_1, x_1)\check{M}(\lambda, G_2, x_2)$*.*

From **9.1** and **10.1** we have:

10.2 *Let G_1 and G_2 be two disjoint weighted digraphs. Let $x = x_1 \cup x_2$. Then*

$$\check{M}(\lambda, G_1(p)G_2, x) = (\lambda - px(G_1(p)G_2))\check{M}(\lambda - px(G_2), G_1, x_1)\check{M}(\lambda - px(G_1), G_2, x_2).$$

Proof (uses **9.1** and **10.1**). Let $G = G_1(p)G_2$, $n_i = v(G_i)$, and $n = n_1 + n_2$.

$\check{M}(\lambda, G_1(p)G_2, x) = \check{M}(\lambda, \overline{\bar{G_1}^p + \bar{G_2}^p}^p, x) = (-1)^{n-1}\check{M}(-\lambda + px(G), \bar{G_1}^p + \bar{G_2}^p, x) =$

$(-1)^{n-1}(-\lambda + px(G))\check{M}(-\lambda + px(G), \bar{G_1}^p, x_1)\check{M}(-\lambda + px(G), \bar{G_2}^p, x_2) =$

$(-1)^{n-2}(\lambda - px(G))(-1)^{n_1-1}(-1)^{n_2-1}$

$\check{M}(px(G_1) - px(G) + \lambda, G_1, x_1)\check{M}(px(G_2) - px(G) + \lambda, G_2, x_2) =$

$(\lambda - px(G))\check{M}(\lambda - px(G_2), G_1, x_1)\check{M}(\lambda - px(G_1), G_2, x_2)$. □

From **7.4** and **10.1** we have:

10.3 *Let G_1 and G_2 be two disjoint weighted digraphs. Then*

$\mathcal{F}(z, G_1 + G_2, x_1 \cup x_2) = z\mathcal{F}(z, G_1, x_1)\mathcal{F}(z, G_2, x_2)$*.*

From **9.1**, **8.3**, and **10.3** we have:

10.4 *Let $G_1 = (V_1, w_1)$ and $G_2 = (V_1, w_1)$ be two disjoint weighted digraphs. Let $x = x_1 \cup x_2$. Then*

$$F(z, G_1(p)G_2, x) = (z + px(G_1(p)G_2))\mathcal{F}(z + px(G_2), G_1, x_1)\mathcal{F}(z + px(G_1), G_1, x_2).$$

Proof (uses **9.1**, **8.3**, and **10.3**).

$\mathcal{F}(z, G_1(p)G_2, x) = \mathcal{F}(z, \overline{\bar{G_1}^p(0)\bar{G_2}^p}^p, x) = (-1)^{n-1}\mathcal{F}(-z - px(G), \bar{G_1}^p + \bar{G_2}^p, x) =$

$(-1)^{n-1}(-z - px(G))\mathcal{F}(-z - px(G), \bar{G_1}^p, x)\mathcal{F}(-z - px(G), \bar{G_2}^p, x) =$

$(-1)^{n-2}(z + px(G))$

$(-1)^{n_1-1}\mathcal{F}(z + px(G) - px(G_1), \bar{G_1}^p, x)(-1)^{n_2-1}\mathcal{F}(z + px(G) - px(G_2), \bar{G_2}^p, x) =$

$(z + px(G))\mathcal{F}(z + px(G_2), G_1, x_1)\mathcal{F}(z + px(G_1), G_1, x_2)$. □

The above results allow us to give an algorithm for finding the forest volumes of decomposable graphs analogous to that for the Laplacian polynomials [9, 11].

Let $D(G) = (T, s, \mathcal{B})$ be the decomposition description of G (see Section 9).

The algorithm (described by formulas (10.1) and (10.2 below) finds the forest volume $\mathcal{F}(z, G, x)$ for a decomposable weighted digraph G if the forest volumes of the graphs in $\mathcal{B}$ are known.

As we mentioned in Section 9, we can easily find the graph $\Omega(t)$ for every vertex t of T. We need only $x(\Omega(t)) = \sum\{x(v) : v \in V(\Omega(t))\}$. Let $x[t] = x(\Omega(t))$.

For a subgraph H of G, we write $\mathcal{F}(z, H, x)$ instead of $\mathcal{F}(z, H, x|_H)$. Let $\delta(t)$ denote the indegree of a vertex t in T (i.e. the number of arcs in T of the form (y, t)). Put

$$z(t, x) = z(t) = x[r]s(r) + \sum\{x[a](s(a) - s(b)) : ((b, a) \in E(rPt)\} \qquad (10.1)$$

where rPt is a (unique) path in T from the root r to t. Let as above $L(T)$ denote the set of leafs of T and $L^*(T)$ the set of leafs l of T such that $\Omega(l)$ is not the one vertex graph (and so the graph $\Omega(l)$ is not decomposable).

The following theorem is analogous to that for the Laplacian polynomials [11]:

10.5 *Let $D(G) = (T, s, \mathcal{B})$ be the decomposition description of G. Then*

$$\mathcal{F}(z, G, x) = \prod\{(z + z(t))^{\delta(t)-1} : t \in V'(T)\}\times$$
$$\prod\{\mathcal{F}(z + z(t), \Omega(t), x) : t \in V^*(T)\}, \qquad (10.2)$$

and

$$T(G, x) =$$
$$\prod\{(z(t))^{\delta(t)-1} : t \in V(T) \setminus L(T)\} \times \prod\{\mathcal{F}(z(t), \Omega(t), x) : t \in L^*(T)\}, \qquad (10.3)$$

and so $-z(t, x)$ is a root of the polynomial $P_{G,x}(z) = \mathcal{F}(z, G, x)$ for every $t \in V(T) \setminus L(T)$.

Proof (uses **5.1** and **10.4**). We prove the theorem by induction on $|V(T)|$. If $|V(T)| = 1$ then the statement is obvious. Let r be the root of T. By the definition of the decomposition tree, T is obtained from the list of decomposition trees T_i of digraphs G_i, $i \in I_1^k$ so that $G = G_1(p)\dots(p)G_k$, $p = s(r)$, and $p \neq s(r_i)$ for every $i \in I_1^k$ where r_i is the root of T_i. Clearly $|V(T_i)| < |V(T)|$. Therefore by the induction hypothesis,

$$\mathcal{F}(z, G_i, x) = \prod\{(z + z_i(t))^{\delta(t)-1} : t \in V(T_i) \setminus L(T_i)\}\times$$
$$\prod\{\mathcal{F}(z + z_i(t), \Omega(t), x) : t \in L^*(T_i)\} \qquad (10.4)$$

for
$z_i(t) = \mathbf{x}(G_i)s(r) + \sum\{x[a])(s(a) - s(b)) : ((b, a) \in E(r_iPt)\}$ and $i \in I_1^k$.

Let $X^i = X \setminus X_i$. By using **10.4**, it is easy to prove by induction on k that

$$\mathcal{F}(z, G, x) = (z + px(G))^{k-1} \prod\{\mathcal{F}(z + px(G \setminus G_i), G_i, x) : i \in I_1^k\}. \tag{10.5}$$

By (10.4),

$$\mathcal{F}(z + px(G \setminus G_i), G_i, X_i) = \prod\{(z + px(G \setminus G_i) + z_i(t))^{\delta(t)-1} : t \in V(T_i) \setminus L(T_i)\} \times$$
$$\prod\{\mathcal{F}(z + p\sigma(X^i) + z_i(t), \Omega(t), X_i(t)) : t \in L^*(T_i)\}. \tag{10.6}$$

Since $p = s(r)$, $x(G) = x[r]$, and $x(G_i) = x[r_i]$, we have from (10.1):

$px(G \setminus G_i)) + z_i(t) = px(G) - px(G_i) + z_i(t) =$

$x[r]s(r) - x[r_i]s(r) + x[r_i]s(r_i) + \sum\{x[a])(s(a) - s(b)) : ((b, a) \in E(r_iPt)\} =$

$x[r]s(r) + \sum\{x[a])(s(a) - s(b)) : ((b, a) \in E(rPt)\}$.

Now (10.2) follows from (10.5) and (10.6) because $k = \delta(r)$, and (10.3) follows from **5.1** and (10.2). □

From **10.5** we have, in particular:

10.6 *Let G be a totally decomposable weighted digraph, and let $D(G) = (T, s)$ be the decomposition description of G. Then*

$$\mathcal{F}(z, G, x) = \prod\{(z + z(t))^{\delta(t)-1} : t \in V(T) \setminus L(T)\}$$

and

$$T(G, x) = \prod\{(z(t))^{\delta(t)-1} : t \in V(T) \setminus L(T)\}.$$

In particular, $\{-z(t, x) : t \in V(T) \setminus L(T)\}$ is the set of roots of the polynomial $P_{G,x}(z) = \mathcal{F}(z, G, x)$ implying that every root of $P_{G,x}(z)$ is of the form $\sum\{a_ib_i : i \in I_1^k\}$ where $a_i \in R_e$ and $b_i \in R_v$.

11 Tree and forest volumes of graph–compositions

In this section we show that the forest volume of a graph–composition is uniquely defined by the forest volumes of its frame and its bricks by establishing a relation between these volumes (see **11.2**).

We recall some notions and notation. Let $G = (A, g)$ and $G_a = (B_a, g_a)$, $a \in A$, be disjoint weighted digraphs where $g : A^2 \to R_e$ and $g_a : B_a^2 \to R_e$ are functions such that $g(a, a) = 0$ and $g_a(b, b) = 0$ for $a \in A$ and $b \in B$, and R_e is a commutative ring. As usual $V(G) = A$ and $V(G_a) = B_a$. Let $B = \cup\{B_a : a \in A\}$. Let $L_a = \{(a, b) : b \in B_a\}$ and $L = \cup\{L_a : a \in A$, and so $L = A \times B$.

The weighted digraph $\Gamma = (V, \gamma)$ is called the *G–composition of* $\{G_a : a \in V(G)\}$, written $\Gamma = G\{G_a : a \in V(G)\}$, if $V = V(\Gamma) = L$ and for two vertices $v_1 = a_1b_1$ and $v_2 = a_2b_2$ of Γ, $\gamma(v_1, v_2) = g(a_1, a_2)$ if $a_1 \neq a_2$, and $\gamma(v_1, v_2) = g_a(b_1, b_2)$ if $a_1 = a_2 = a$.

The graph G in $\Gamma = G\{G_a : a \in V(G)\}$ is called the *frame* and the graphs G_a, $a \in V(G)$ are called the *bricks* of the G–composition.

A G–composition $G\{G_a : a \in V(G)\}$ is called a *G–extension* if $E(\grave{G}_a) = \emptyset$ for every $a \in V(G)$.

Let $x : L \to R_v$ where R_v is a commutative ring. Put

$x_a = x|_{L_a}$,

$x(G_a) = \sum\{x(a,b) : b \in B_a\}$,

$x(\Gamma) = \sum\{x(a,b) : a \in A, b \in B\}$,

$s : A \to R_v$ such that $s(a) = x(G_a)$ for $a \in A$, and

$d_a(G,s) = \sum\{s(c)\gamma(a,c) : c \in A \setminus a\}$ for $a \in A$.

Suppose first that $G\{G_a : a \in V(G)\}$ is a G–extension. Then it turns our that the $\mathcal{T}$–volume of the composition $\Gamma = G\{G_a : a \in V(G)\}$ is uniquely defined by the $\mathcal{T}$–volumes of G and G_a's, $a \in V(G)\}$, of the composition, and by G (more precisely, by G, $\{|V(G_a)| : a \in V(G)\}$ and the $\mathcal{T}$–volumes of G). Namely

11.1 [18] *Let* $n_a = |V(G_a)|$. *Then*

$$\mathcal{T}(G\{G_a : a \in V(G)\}, x) = \mathcal{T}(G, s) \times \prod\{(d_a(G,s))^{n_a - 1} : a \in V(G)\}.$$

Now suppose that $E(\grave{G}_a) \neq \emptyset$ for some $a \in V(G)$. Then the $\mathcal{T}$–volume of the composition $\Gamma = G\{G_a : a \in V(G)\}$ is no longer defined by the $\mathcal{T}$–volumes of G and G_a, $a \in V(G)\}$, and G. But it turns out that the $\mathcal{F}$–volume of the composition $\Gamma = G\{G_a : a \in V(G)\}$ is again uniquely defined by the $\mathcal{F}$–volumes of the frame G and all the bricks G_a, $a \in V(G)$, of the composition. This relation is described by the following theorem.

11.2 $\mathcal{F}(z, G\{G_a : a \in V(G)\}, x) = \mathcal{F}(z, G, s) \times \prod\{\mathcal{F}(z + d_a(G,s)), G_a, x_a) : a \in V(G)\}$.

Proof (uses **5.2**, **6.2**, and **8.4**).
(**p1**) Let $\Gamma = G\{G_a : a \in V(G)\}$. Let

$\mathcal{R}(z, \Gamma, x) = \mathcal{F}(z, G, s) \prod\{\mathcal{F}(z + d_a(G,s), G_a, x_a) : a \in A\}$ and

$\Delta(z, \Gamma, x) = \mathcal{F}(z, G\{G_a : a \in A\}, x) - \mathcal{R}(z, \Gamma, x)$.

Put $Q_p(z, \Gamma, x) = (\mathcal{F}(z, G, s) \times \prod\{\mathcal{F}(z + d_a(G,s), G_a, x_a) : a \in A \setminus p\}$.

Then

$$\mathcal{R}(z, \Gamma, x) = Q_p(z, \Gamma, x)\mathcal{F}(z + d_p(G,s), G_p, x_p). \tag{11.1}$$

Clearly

$$Q_p(z, \Gamma, x \downarrow_0^u) = Q_p(z, \Gamma, x|^u). \tag{11.2}$$

We will prove by induction on $|V(\Gamma)|$ that $\Delta(z, \Gamma, x) \equiv 0$. For $|V(\Gamma)| = 1$ the statement is trivially true. Let us consider Γ with $|V(\Gamma)| \ge 2$. Let $u = (p, q) \in V(\Gamma)$.

(**p2**) Suppose that $|V(G_p)| \ge 2$. Then $\Gamma \setminus u = G\{G_p \setminus q, G_a : a \in A \setminus p\}$.

By **6.2**,

$$\mathcal{F}(z + d_p(G, s), G_p, x_p)|_{x(u)=0} = \mathcal{F}(d_p(G, s) + z), G_p, x_p \downarrow_0^u) =$$
$$(z + d_p(G, s) + d_q(G_p, x_p))\mathcal{F}(z + d_p(G, s), G_p \setminus q, x_p|^u). \tag{11.3}$$

Clearly

$$d_p(G, s) + d_q(G_p, x_p) =$$
$$\sum\{x(G_c)\gamma(p, c) : c \in A \setminus p\} + \sum\{x(v)w(q, v) : v \in B_p \setminus q\} = d_u(\Gamma, x). \tag{11.4}$$

Therefore by (11.1), (11.2), (11.3), and (11.4),

$$\mathcal{R}(z, \Gamma, \downarrow_0^u) = (z + d_u(\Gamma, x))\mathcal{R}(z, \Gamma \setminus u, x|^u). \tag{11.5}$$

By **6.2**,

$$\mathcal{F}(z, \Gamma, \downarrow_0^u) = (z + d_u(\Gamma, x))\mathcal{F}(z, \Gamma \setminus u, x|^u). \tag{11.6}$$

Thus by (11.5) and (11.6), $\Delta(z, \Gamma, \downarrow_0^u) = \mathcal{F}(z, \Gamma, \downarrow_0^u) - \mathcal{R}(z, \Gamma, \downarrow_0^u) =$
$(z + d_u(\Gamma, x))(\mathcal{F}(z, \Gamma \setminus u, x|^u) - \mathcal{R}(z, \Gamma \setminus u, x|^u) =$
$(z + d_u(\Gamma, x))\Delta(z, \Gamma \setminus u, x|^u)$.

(**p3**) Now suppose that $|V(G_p)| = 1$. Then $\mathcal{F}(z + d_p(G, s), G_p, x_p) \equiv 1$ and
$\Gamma \setminus u = G'\{G_a : a \in V(G')\}$ where $G' = (G \setminus p)$.
Therefore by (11.1), $R(z, \Gamma, x) = Q_p(z, \Gamma, x)$ and by (11.2), (11.5) and (11.6),
$\Delta(z, \Gamma, \downarrow_0^u) = (z + d_u(\Gamma, x))\Delta(z, \Gamma \setminus u, x|^u)$.

(**p4**) In both cases by the induction hypothesis,
$\Delta(z, \Gamma \setminus u, x|^u) \equiv 0$. Therefore $\Delta(z, \Gamma, \downarrow_0^u) = 0$ for every $u \in V(\Gamma)$. By **5.2**,
$\deg(\mathcal{F}(z, \Gamma, x)) = |V(\Gamma)| - 1$, $\deg(\mathcal{F}(z, G, s)) = |V(G)| - 1$, and
$\deg(\mathcal{F}(z + d_a(G, s)), G_a, x_a)) = |V(G_a)| - 1$. Therefore
$\deg(\mathcal{R}(z, \Gamma, x)) = |V(G)| - 1 + \sum\{|V(G_a)| - 1 : a \in A\} = |V(\Gamma)| - 1$.
Hence $\deg(\Delta(z, \Gamma, x)) \le |V(\Gamma)| - 1$. Then by **8.4**, $\Delta(z, \Gamma, x) \equiv 0$. □

From **5.1** and **11.2** we have

11.3 $\mathcal{T}(G\{G_a : a \in V(G)\}, x) = \mathcal{T}(G, s) \times \prod\{\mathcal{F}(d_a(G, s), G_a, x_a) : a \in V(G)\}$.

Clearly **11.1** is a particular case of **11.3**. Let us consider another natural particular case of **11.3** when all G_a's are weighted complete digraphs.

From **11.3** and **12.1** below we have:

11.4 *Suppose that G_a is a p_a–complete graph for $a \in V(G)$. Then*

$$\mathcal{F}(z, G\{G_a : a \in V(G)\}, x) = \mathcal{T}(G, s) \times \prod\{(z + d_a(G, s) + p_a s(a))^{n_a - 1} : a \in V(G)\}$$

where $n_a = |V(G_a)|$.

Theorem **11.4** is a particular case of Corollary 2.1 in [15].

It is easy to show that the result of the p–operation on a set of digraphs $\{G_s : s \in I_1^n\}$ (see Section 9) is the graph–composition $K^p\{G_s : s \in I_1^n\}$ where n is the number of vertices of K^p.

We know (see Section 9) that every decomposable weighted digraph can be obtained from a set of digraph–bricks $\{G_i : i \in I_1^n\}$ by a series of p–operations $(p_1), \dots, (p_k)$. Therefore every decomposable weighted digraph can be obtained by a series of K^{p_i}–compositions, $i \in I_1^k$, starting from a set of digraph–bricks $\{G_s : s \in I_1^n\}$. As to the totally decomposable weighted graphs, they can be obtained from one vertex graphs by a series of K^{p_i}–compositions, $i \in I_1^k$.

Therefore theorem **10.5** can also be proved by using **11.2**. The relation **9.4** between the aggregation and composition constructions shows that theorem **10.5** can be obtained from its specification **10.6** by using **11.4**.

12 Forest volumes of some special graphs

In this section we find the forest volumes of some special weighted digraphs. We use these results in Section 13.

It follows from **10.6** and it is easy to see directly that

12.1 *Let $|V(K^p)| = n$. Then $\mathcal{F}(z, K^p, x) = (z + px(K^p))^{n-1}$.*

It is easy to find the forest volume of an acyclic digraph. Let, as above, $S(D)$ denote the set of sources of a directed graph D.

12.2 *Let $G = (V, g)$ be a weighted digraph. Let $d_{out}(u, G, x) = \sum\{x(v)g(u, v) : v \in V \setminus u\}$ and $V(G) = \{v_1, \dots, v_n\}$. Suppose that the skeleton $\grave{G}$ of G is acyclic (i.e. G has no directed cycles) and that $v_i \in S(\grave{G} \setminus \{v_1, \dots v_{i-1}\})$ for every $i \in I_1^n$. Then*

$$\mathcal{F}(z, G, x) = z^{-1} \prod\{z + d_{out}(u, G, x) : u \in V(G)\}.$$

Proof (uses **7.4**). By **7.4**,

$$z\mathcal{F}(z, G, x) = (-1)^n M(-z, G, x) = det(zI_n + M^x(G)). \tag{12.1}$$

Let $M^x(G) = \{m_{ij}\}$. Since $\grave{G}$ is an acyclic, and $v_i \in S(\grave{G} \setminus \{v_1, \dots v_{i-1}\})$ for every $i \in I_1^n$, we have for $i, j \in I_1^n$: $m_{ij} = 0$ for $i > j$ and $m_{ii} = d_{out}(v_i, G, x)$. Therefore the required formula follows from (12.1). □

From **12.2** we have in particular:

12.3 *Let $T = (V, t)$ be a weighted ditree, $E = E(\grave{T})$, and for $e = (u, v) \in E$ let $y(e) = t(u, v)x(v)$. Then*

$$\mathcal{F}(z, T, x) = z^{-1} \prod\{z + y(e) : e \in E\}.$$

Now we can find the forest volume of a weighted cycle:

12.4 *Let $C = (V, c)$ be a weighted dicycle with n vertices, $E = E(\grave{C})$, and for $e = (u, v) \in E$ let $y(e) = c(u, v)x(v)$. Then*

$$\mathcal{F}(z, C, x) = \sum\{z^k (n-k)/(k+1) \sum\{y(B) : B \subseteq E, |B| = k\} : k \in I_O^{n-1}\}.$$

Proof (uses **6.3** and **12.3**). By **6.3**,

$$\Phi(z, C, x) = \int \sum\{\Phi(z, C \setminus e, x) : e \in E(G)\} dz. \tag{12.2}$$

Clearly $Q_a = C \setminus a$, $a \in E$, is a path and

$\Phi(z, C, x) = z\mathcal{F}(z, C, x)$ and $\Phi(z, Q_e, x) = \mathcal{F}(z, Q_e, x)$.

Therefore we can use **12.3**:

$\Phi(z, Q_a, x) = \prod\{z + y(e) : e \in E_a)\} = \sum\{z^k \sum\{y(B) : B \subseteq E_a, |B| = k\} : k \in I_O^{n-1}\}$

where $E_a = E(\grave{Q}_a)$. Now

$\sum\{\Phi(z, Q_e, x) : e \in E\} =$

$\sum\{\sum\{z^k \sum\{y(B) : B \subseteq E_a, |B| = k\} : k \in I_O^{n-1}\} : a \in E\} =$

$\{\sum\{z^k (n-k) \sum\{y(B) : B \subseteq E, |B| = k\} : k \in I_O^{n-1}\}.$

Therefore we have from (12.2):

$\Phi(z, C, x) = z\mathcal{F}(z, C, x) = \int \sum\{\Phi(z, Q_e, x) : e \in E\} dz =$

$\{\sum\{z^{k+1}(n-k)/(k+1) \sum\{y(B) : B \subseteq E, |B| = k\} : k \in I_O^{n-1}\}.$ □

Since $\mathcal{T}(z, C) = \mathcal{F}(z, C, \bar{1})$ we have from **12.4**:

12.5 *Let C be a dicycle. Then*

$$z\mathcal{T}(z, C) = (z+1)^n - 1.$$

Proof By **12.4**,

$z\mathcal{T}(z, C) = \{\sum\{z^{k+1}(n-k)/(k+1)\binom{n}{k}\} : k \in I_O^{n-1}\} =$

$\{\sum\{z^i\binom{n}{i}\} : i \in I_1^n\} = (z+1)^n - 1.$ □

Let $P\{G_i : i \in I_1^n\} = G_1(p)\dots,(p)G_n\}$ and $G^{(p)n} = P\{G_i : i \in I_1^n\}$ if every $G_i = G$.

From **10.5** we have:

12.6 *Let* $\Gamma = P\{g^{(p_i)n_i} : i \in I_1^k\}$ *and* $x : V(\Gamma) \to R_v$. *Let* $X = x(\Gamma)$, $X_i = x(g^{(p_i)n_i})$, *and* $\bar{X}_i = X - X_i$. *Then*

$$\mathcal{F}(z,\Gamma,x) = (z+pX)^{k-1}\prod\{(z+p\bar{X}_i+p_iX_i)^{n_i-1} : i \in I_1^k\}.$$

13 Combinatorial interpretation of Hurwitz's identity

In the 19-th century, N. Abel found the following surprising generalization of the binomial formula [1] (see also [21]):

13.1 $(x+y)^n = \sum\{\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k} : k \in \{0,\dots,n\}\}$.

Abel's theorem has been further generalized by A. Hurwitz as follows [8] (see also [21]):

13.2 *Let* V *be a finite set, and* $x = \{(v,x(v)) : v \in V\}$. *For a set* A, *let* $x(A) = \sum\{x(a) : a \in A\}$. *Then*

$(z+y)(z+y+x(V))^{|V|-1} = \sum\{z(z+x(A))^{|A|-1})\cdot y(y+x(B))^{|B|-1}) : A \subseteq V, B = V\setminus A\}.$

In this section we show that relation **11.2** allows to give a natural combinatorial prove and interpretation of Hurwitz's identity in terms of the forest volume of a digraph.

13.3 *Let* G *is a simple digraph with exactly two vertices* u, v *and exactly one arc* (v,u). *Let* $\Gamma = G\{G_u,G_v\}$ *where* G_u *is a graph consisting of one vertex* u *and no edges, and* G_v *is the complete digraph with the vertex set* V *(where* $u \notin V$*), and so* Γ *is a an extension of* G. *Let* $x = \{(v,x(v)) : v \in V\}$, $t = (u,t(u))$, *and* $t(u) = y$. *For* $C \subseteq V$, *let* $x(C) = \sum\{x(v) : v \in C\}$.

Then

$(z+y)(z+y+x(V))^{|V|-1} = \mathcal{F}(z,\Gamma,x\cup t) =$

$\sum\{z(z+x(A))^{|A|-1}y(z+x(B))^{|B|-1} : A \subseteq V, B = V\setminus A\}.$

Proof (uses **11.2** and **12.1**). Clearly $V(\Gamma) = \{u\}\cup V$. We will show that both sides of Hurwitz's identity are equal to $\mathcal{F}(z,\Gamma,x\cup t)$. In order to do this, we will find $\mathcal{F}(z,\Gamma,x\cup t)$ in two different ways.

(**p1**) We first find $\mathcal{F}(z,\Gamma,x\cup t)$, by using the forest volume relation **11.2**.

By **11.2**,

$$\mathcal{F}(z,\Gamma,x\cup t) = \mathcal{F}(z,G,s)\mathcal{F}(z+d_u(G,s),u,t)\mathcal{F}(z+d_v(G,s),G_v,x) \qquad (13.1)$$

where $s = \{(u, s(u), (v, s(v)),\ s(u) = t(u) = y$, and $s(v) = x(V)$. Since G has two vertices u, v and exactly one arc (v, u), we have: $\mathcal{F}(z, G, s) = z + t(u)$ and $d_v(G, s) = t(u) = y$. Therefore $\mathcal{F}(z + d_v(G, s)), G_v, x) = \mathcal{F}(z + y), G_v, x)$. Since $G_u = \{u\}$ has no arcs, clearly $\mathcal{F}(z + d_u(G, s)), u, t) = 1$. Since G_v is a complete digraph, we have from (13.1) and **12.1**:

$$\mathcal{F}(z, \Gamma, x \cup t) = (z + y)\mathcal{F}(z + y, G_v, x) = (z + y)(z + y + x(V))^{|V|-1}. \qquad (13.2)$$

(**p2**) Now let us find $\mathcal{F}(z, \Gamma, x \cup t)$ in another way. Note that Γ is a simple digraph. Let $x \cup t = h$ and $h^* = h \cup \{(*, z)\}$.

If $H \subseteq G$ and f is a function defined on $V(G)$, we write $\mathcal{F}(z, H, f)$ and $T(H, f)$ instead of $\mathcal{F}(z, H, f|_{V(H)})$ and $T(H, f|_{V(H)}$, respectively.

By the definition of the forest volume of a digraph,

$$\mathcal{F}(z, \Gamma, h) = \mathcal{T}_*(\Gamma^*, h^*) = \sum\{\mathcal{T}_*(T, h^*) : T \in \mathcal{T}_*(\Gamma^*)\} \qquad (13.3)$$

where $\mathcal{T}_*(T, h^*) = \prod\{h(v)^{d(v,T)-1} : v \in V(T)\}$ for $T \in \mathcal{T}_*(\Gamma^*)$.

Since Γ has no arc (u, v) with $v \in V$, every spanning ditree of Γ^* contains edge $(u, *)$. For a spanning ditree T of Γ^*, let $T' = T \setminus (u, *)$. Clearly T' is a spanning diforest of Γ^* consisting of exactly two components T_u and T_* such that $u \in T_u$ and $* \in T_*$. Since in Γ there is no edge going out of u, clearly u is a root of T_u. Then

$$\mathcal{T}_*(T, h^*) = zy(u)\mathcal{T}_u(T_u, h)\mathcal{T}_*(T_*, h^*). \qquad (13.4)$$

By (13.3),

$$\mathcal{F}(z, \Gamma, h) = \sum\{\mathcal{T}_*(T, h^*) : T \in \mathcal{T}_*(\Gamma^*)\} = \sum\{S(\Gamma, h^*, B) : B \subseteq V\}, \qquad (13.5)$$

where $S(\Gamma, h^*, B) = \sum\{\mathcal{T}_*(T, h^*) : T \in \mathcal{T}_*(\Gamma^*), V(T_u \setminus u) = B\}$.

By (13.4),

$S(\Gamma, h^*, B) = \sum\{\mathcal{T}_*(T^*, h^*) \prod\{zy(u)\mathcal{T}_u(T_u, h) : T \in \mathcal{T}_*(\Gamma^*), V(T_u \setminus u) = B\} =$

$(\sum\{\mathcal{T}_*(T, h^*) : T \in \mathcal{T}_*(\Gamma^* - (B \cup u))\}) \prod\{zy(u) \sum\{\mathcal{T}_u(T, h) : T \in \mathcal{T}_u(\Gamma^* - (A \cup *)\}$.

Thus by (13.3),

$$S(\Gamma, h^*, B) = z\mathcal{F}(z, \Gamma^* - (B \cup \{u, *\}), x)\ y\ \mathcal{F}(y, \Gamma^* - (A \cup \{u, *\}), x), \qquad (13.6)$$

where $A \cup B = V$.

Since G_v is a complete digraph, its induced subgraphs $\Gamma^* - (B \cup \{u, *\})$ and $\Gamma^* - (A \cup \{u, *\})$ are also complete digraphs. Therefore from (13.5), (13.6), and **12.1** it follows that

$\mathcal{F}(z, \Gamma, x \cup t) = \sum\{z(z + x(A))^{|A|-1} y(z + x(B))^{|B|-1} : A \subseteq V, B = V \setminus A\}$. □

In another paper we describe a mechanism that provides an infinite variety of identities that are generlizations of Huwitz's identity.

The results of this paper were presented at several conferences and seminars, in particular, at the Moscow Discrete Mathematics Seminar, February, 1989, and at the Eighth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms, and Applications, Kalamazoo, Michigan, June, 1996.

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