18.318 Problem Set 1 (due Friday, March 23, 2012)

Turn in as many problems as you want.

Problem 1. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a Young diagram that fits inside the $k \times n$ rectangle. Consider the subset S_λ of the Grassmannian $Gr(k, n)$ over a finite field $\mathbb{F}_q$ that consists of the elements that can be represented by $k \times n$ matrices A with 0's outside the shape λ. For example, for $n = 4$ and $k = 2$, $S_{(4,1)}$ is the subset of elements of $Gr(2, 4)$ representable by matrices of the form $\begin{pmatrix} * & * & * & * \\ * & 0 & 0 & 0 \end{pmatrix}$

In parts 1,2,3 assume $n = 2k$ and $\lambda = (2k, 2k-2, 2k-4, \dots, 2)$.

1. Find a combinatorial expression for the number of elements of $S_{(2k,2k-2,\dots,2)}$ (over $\mathbb{F}_q$). Show that it is a polynomial in q.

2. Let $f_k(q)$ be the polynomial from part 1. Calculate $f_k(1)$, $f_k(0)$, and $f_k(-1)$.

3. Let $g_k(q) = q^d\, f_k(q^{-1})$, where d is the degree of the polynomial $f_k(q)$. Find the maximal power of 2 that divides the number $g_k(5)$.

4. Generalize (some of) the above to other Young diagrams λ.

5. What about skew shapes λ/μ?

Problem 2. Let $\mathbb{C}[\Delta_I]$ be the polynomial ring in $\binom{n}{k}$ (independent) variables Δ_I, $I \in \binom{[n]}{k}$. Let $I_{kn} = \left\langle \Delta_{i_1 \dots i_k} \Delta_{j_1 \dots j_k} - \sum \Delta_{i'_1 \dots i'_k} \Delta_{j'_1 \dots j'_k} \right\rangle$ be the ideal in $\mathbb{C}[\Delta_I]$ whose generators correspond to the Plücker relations (for all r).

Show that I_{kn} is a prime ideal. Deduce that I_{kn} consists of all polynomials in $\mathbb{C}[\Delta_I]$ that vanish on the image of the Grassmannian $Gr(k, n, \mathbb{C})$ in the projective space $\mathbb{CP}^{\binom{n}{k}-1}$ under the Plücker embedding.

Problem 3. Let $M \subseteq \binom{[n]}{k}$. Is it true that the following three properties are equivalent?

<u>Exchange Property</u>: For any $I, J \in M$ and any $i \in I$, there exists $j \in J$ such that $(I \setminus \{i\}) \cup \{j\} \in M$.

<u>Stronger Exchange Property</u>: For any $I, J \in M$ and any $i \in I$, there exists $j \in J$ such that both $(I \setminus \{i\}) \cup \{j\}$ and $(J \setminus \{j\}) \cup \{i\}$ are in M.

<u>Even Stronger Exchange Property</u>: For any $I, J \in M$, any $r \geq 1$, and any $i_1, \dots, i_r \in I$, there exist $j_1, \dots, j_r \in J$ such that both $(I \setminus \{i_1, \dots, i_r\}) \cup \{j_1, \dots, j_r\}$ and $(J \setminus \{j_1, \dots, j_r\}) \cup \{i_1, \dots, i_r\}$ are in M.

Prove the equivalence of (some of) these properties or construct counterexamples.

Problem 4. Check that the Fano plane satisfies the exchange axiom and show that this matroid is not realizable over $\mathbb{R}$.

Problem 5. Prove the equivalence of the 3 definitions of matroids: the definition in terms of exchange axiom, the definition in terms of Gale minimal elements, and the definition in terms of matroid polytopes.

Problem 6. 1. Prove that image of the Grassmannian $Gr(k, n, \mathbb{C})$ under the moment map is a convex polytope.

2. Describe the moment map image of (the closure of) the Schubert cell $\overline{\Omega_{(2,1)}} \subset Gr(2, 4, \mathbb{C})$.

3. Calculate the normalized volume of the moment map image of $\overline{\Omega_\lambda} \subset Gr(k, n, \mathbb{C})$ for any λ.

Problem 7. Prove that the $(n-1)$ dimensional volume of the permutohedron $P_n = \mathrm{ConvexHull}\{(w_1, \ldots, w_n) \mid w \in S_n\}$ is n^{n-2}.

Problem 8. Find an expression for the Ehrhart polynomial $i(P, t) := \#(tP \cap \mathbb{Z}^n)$, $t \in \mathbb{Z}_{\geq 0}$, of the hypersimplex $P = \Delta_{kn}$ using inclusion-exclusion.

Problem 9. 1. Prove geometrically Pieri's formula for the Schubert classes by intersecting Schubert varieties.

2. Prove geometrically the duality formula by intersecting Schubert varieties.

Problem 10. Show that the ideal in the ring of symmetric *polynomials* $\Lambda_k := \mathbb{C}[x_1, \ldots, x_k]^{S_k}$ generated by all Schur polynomials $s_\lambda(x_1, \ldots, x_k)$ for shapes λ that don't fit inside the $k \times (n-k)$-rectangle coincides with the ideal

$$I = \langle h_{n-k+1}, h_{n-k+2}, \ldots, h_n \rangle ,$$

where $h_i = h_i(x_1, \ldots, x_k)$ are the complete homogeneous symmetric polynomials.

Problem 11. Prove the equivalence of the following versions of the Littlewood-Richardson rules for $c_{\lambda\mu}^\nu$: the classical LR-rule, the honeycomb version of LR-rule, the web diagram version of LR-rule.

Problem 12. Explicitly show (without using LR-rule) that

$$s_{(r)} \cdot s_{(s)} = \sum_{c \geq \max(r-s, 0)} s_{(s+c,\, r-c)},$$

where we assume that $s_\lambda = 0$ unless λ is a partition with nonnegative parts.

Problem 13. Let V be the (infinite dimensional) linear space with the basis $e_0, e_1, e_2, \ldots$. For $c \in \mathbb{Z}_{\geq 0}$, let $R(c)$ be the operator on $V \otimes V$ given by

$$R(c) : e_r \otimes e_s \mapsto \begin{cases} e_{s+c} \otimes e_{r-c} & \text{if } c \geq r - s \\ 0 & \text{otherwise} \end{cases}$$

(We assume that $e_i = 0$ for $i < 0$.) Let $R_{ij}(c)$ denote the operator that acts as $R(c)$ on the i-th and j-th copies of V in the tensor power $V \otimes V \otimes V$. Show that the operator $R(c)$ satisfies the generalized Yang-Baxter equation:

$$R_{23}(c_{23}) R_{13}(c_{13}) R_{12}(c_{12}) = R_{12}(c'_{12}) R_{13}(c'_{13}) R_{23}(c'_{23}), \qquad \text{where}$$

$$\begin{cases} c'_{12} = \min(c_{12}, c_{13} - c_{23}) \\ c'_{13} = c_{12} + c_{23} \\ c'_{23} = \max(c_{23}, c_{13} - c_{12}) \end{cases}$$

Problem 14. Let A be a generic upper-triangular $n \times n$ matrix. Find the number of non-zero minors of A of all sizes (including the empty minor of size 0×0).

Problem 15. 1. Find the bijective map $(x, y) \mapsto (x', y')$ from $\mathbb{R}^2_{>0}$ to $\mathbb{R}^2_{>0}$ such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y' & 1 \end{pmatrix} \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

for some $t_1, t_2 \in \mathbb{R}_{>0}$.

2. Show that the double Bruhat cell $B_{u,w}$, $u, w \in S_n$, defined in terms of a double wiring diagram for u and w depends only on the permutations u and w (and not on a choice of a double wiring diagram).

Problem 16. Calculate the number of d-dimensional cells in the totally nonnegative Grassmannian $Gr_{\geq 0}(2, n)$.

Problem 17. Prove the equivalence of the 3 definitions of the strong Bruhat order on S_n:

A. Covering relations: $u < w$ iff $w = u \cdot (i, j)$ and $\ell(w) = \ell(u) + 1$.

B. $u \leq w$ if any reduced decomposition of w has a subword which is a reduced decomposition of u.

C. $u \leq w$ if some reduced decomposition of w has a subword which is a reduced decomposition of u.

Problem 18. Let P be a path in any directed graph. Let us start erasing loops (i.e., closed directed paths without self-intersections) in P until we get a path P' without self-intersections. Is it true that the parity of the number of erased loops is a well-defined invariant of path P and it does not depend on the order of erasing loops?