PROBLEM SET 5 (due on Thursday 12/09/2004)

The problems worth 10 points each.

Problem 1 A *system of distinct representatives* (SDR) in a collection of subsets $A_1, \dots, A_n$ is a choice of elements $x_i \in A_n$, for $i = 1, \dots, n$, such that $x_1, \dots, x_n$ are all distinct.

Find the number of different SDR's for $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, $A_{n-1} = \{n-1, n\}$, and $A_n = \{n, 1\}$. Can you reformulate this problem in terms of rook placements?

Problem 2 Find the number R_n of different SDR's for $A_1 = \{1, 2, 3\}$, $A_2 = \{2, 3, 4\}$, $A_{n-2} = \{n-2, n-1, n\}$, $A_{n-1} = \{n-1, n, 1\}$, and $A_n = \{n, 1, 2\}$. Hint: Try to express the numbers R_n in terms of the Fibonacci numbers.

Problem 3 Let G be a connected graph with positive weights on edges. We would like to find a subgraph H of G such that (1) the graph $G \setminus H$ (obtained from G by removing edges of H) is connected; (2) the sum of weights of edges in H is as big as possible for subgraphs that satisfy (1).

Let us try to use the *greedy algorithm:* Pick an edge e_1 of G of maximal weight such that $G \setminus \{e_1\}$ is connected; then pick an edge e_2 of maximal weight such that $G \setminus \{e_1, e_2\}$ is still connected, etc. Will this algorithm always produce a subgraph with the desired property? Prove that the greedy algorithm always works or find a counterexample.

Problem 4 Find the number of spanning trees in the 3×3-grid:

$$\begin{array}{ccccc}
\bullet & - & \bullet & - & \bullet \\
| & & | & & | \\
\bullet & - & \bullet & - & \bullet \\
| & & | & & | \\
\bullet & - & \bullet & - & \bullet
\end{array}$$

Problem 5 Let K_3 be the complete graph on 3 vertices. Find the number of spanning trees in $K_3 \times K_3 \times \cdots \times K_3$ (n factors). This graph can be drawn on the n-dimensional torus.

Problem 6 Prove that the number of spanning trees in the complete bipartite graph K_{mn} is $m^{n-1}\, n^{m-1}$. Hint: You can use the Matrix-Tree theorem or try to modify Prüfer coding for bipartite graphs.

Problem 7 Let $L = (l_{ij})$ be a symmetric $n \times n$-matrix with real entries such that $l_{ij} \le 0$, for $i \ne j$, and $\sum_j l_{ij} \ge 0$, for any i (all row sums are nonnegative).

(A) Prove that $\det(L) \ge 0$.

(B) Recall that a *principal minor* of a matrix is the determinant of a square submatrix located in rows and columns with the same index set. A matrix is called *positive-definite* if all its principal minors are strictly positive. Find a simple necessary and sufficient condition that describes all cases when matrix L is positive-definite.

(C)* Let $L = (l_{ij})$ be any symmetric $n \times n$-matrix with real entries such that $l_{ij} \leq 0$. (Drop the second condition.) Find an example of a positive-definite matrix of this form that does not have nonnegative row sums. Prove that such a matrix is positive-definite if and only if there are exist numbers $h_1, \ldots, h_n > 0$ such that $\sum_j h_i \, l_{ij} > 0$, for any i.

Problem 8 A sequence $(f_1, \ldots, f_n)$, $f_i \in \{1, \ldots, n\}$ is called *a parking function* if it satisfies the following condition:

$$\#\{i \mid f_i \leq k\} \geq k, \qquad \text{for } k = 1, \ldots, n.$$

(A) Show that a sequence of positive integers $(f_1, \ldots, f_n)$ is a parking function if and only its increasing rearrangement $c_1 \leq c_2 \leq \ldots \leq c_n$ satisfies $c_i \leq i$, for $i = 1, \ldots, n$.

(B) Find a bijection between parking functions and Catalan words $(\epsilon_1, \ldots, \epsilon_{2n})$, $\epsilon_i = \pm 1$ with all 1's labelled by the integers $1, \ldots, n$ such that if $\epsilon_i = \epsilon_{i+1} = \cdots = \epsilon_j = 1$ then their labels satisfy $l_i < l_{i+1} < \cdots < l_j$. For example, for $n = 2$ we have the following 3 labelled Catalan words: $(1^1, 1^2, -1, -1)$, $(1^1, -1, 1^2, -1)$, and $(1^2, -1, 1^1, -1)$. (Labels are indicated by superscripts.) Hint: Sequences $c_1 \leq c_2 \leq \cdots \leq c_n$ from part (B) are in 1-1 correspondence with Catalan words.

(C)* Find a bijection between labelled Catalan words from part (B) and spanning trees in K_{n+1}.

Problem 9 Calculate the chromatic polynomial for the $2 \times n$-grid, i.e., for the graph on $2n$ vertices connected by edges as follows:

$$\begin{array}{ccccccccc} \bullet & - & \bullet & - & \bullet & - \cdots & - & \bullet \\ | & & | & & | & & & | \\ \bullet & - & \bullet & - & \bullet & - \cdots & - & \bullet \end{array}$$

Problem 10 Fix n and k such that $k < n$. Consider the graph on the vertices $1, \ldots, n$ such that (i, j), $i < j$, is an edge whenever $j - i \leq k$. Find the number of acyclic orientations of this graph.