Exam 1

October 07, 2004

You have one hour to solve the following problems. The problems worth 10 points each. You can use your notes, books, calculators, slide rulers, abaci, or any other computing devices (without Internet access).

Make sure to give the answer for *any* positive integer n in each problem. The answers should not involve summation. Your formulas should be simple enough so that you can easily evaluate them for, say, $n = 100$. Show your reasoning.

1. Find a formula for the number of ordered pairs (A, B) of subsets in $\{1, \ldots, n\}$ whose intersection consists of a single element: $|A \cap B| = 1$.

2. Evaluate the sum $\displaystyle\sum_{k=0}^{n} (2k - n) \binom{n}{k}$.

3. Find the number of self-conjugate partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots)$ of n such that $\lambda_2 = 3$.

4. Find the number $f(n)$ of set partitions of the set $\{1, \ldots, n\}$ into 3 nonempty blocks such that no block contains two consecutive elements $i, i + 1$. For example, $f(4) = 3$, corresponding to the set partitions $13|2|4$, $14|2|3$, $1|24|3$. In this problem, partial credit will be given for correct calculation of $f(5)$, $f(6)$, and/or guessing the general answer.

   Hint: Try to express $f(n)$ in terms of $f(n-1)$.