PROBLEM SET 2  (due on Tuesday 04/09/2013)

Solve as many problems as you like. Hand in at least 5 problems.

**Problem 1.** (a) Prove that the (signless) Stirling numbers of the first kind \(c(n, k)\) can be computed using the Stirling triangle of the first kind given by the recurrence relation:

\[ c(n + 1, k) = n c(n, k) + c(n, k - 1). \]

(b) Prove that the Stirling numbers of the second kind \(S(n, k)\) can be computed using the Stirling triangle of the second kind given by the recurrence relation:

\[ S(n + 1, k) = k S(n, k) + S(n, k - 1). \]

**Problem 2.** The Bell number \(B(n)\) is the number of all set-partitions of \([n]\). Prove that the Bell numbers can be calculated using the following Bell triangle:

\[
\begin{align*}
1 & \\
1 & 2 & \\
2 & 3 & 5 & \\
5 & 7 & 10 & 15 & \\
15 & 20 & 27 & 37 & 52 \\
\vdots & \ldots & \ldots & \ldots & \ldots & \\
\end{align*}
\]

In this triangle, the last entry in a row equals the first entry of the next row, and each entry is the sum of two entries to the left (West) and to the North-West of it. (For example, \(27 = 20 + 7\).) The Bell numbers 1, 2, 5, 15, 52, \ldots appear on the main diagonal.

Can you find a combinatorial interpretation of the other entries of this triangle?

**Problem 3.** A set-partition \(\pi\) of \([n]\) is non-crossing if there are no numbers \(i < j < k < l\) such that \(i, k\) are in one block of \(\pi\), and \(j, l\) are in another block of \(\pi\). Prove bijectively that the number is non-crossing set-partitions of \([n]\) equals the Catalan number \(C_n\).

**Problem 4.** In class we described the following procedure. Pick \(n\) integer numbers between 0 and \(m - 1\), and write them in the first row of an \(m \times n\) matrix. Fill in the other entries of the matrix by cyclically shifting the first row modulo \(m\), that is, if \(a\) is an entry of the first row, then below \(a\) we should write \(a + 1, a + 2, \ldots, m - 1, 0, 1, \ldots, a - 1\). Find the median entry in each row of this matrix. Let \(S\) be the sum of the median entries. Express \(S\) in terms of \(m\) and \(n\), and prove this formula.

**Problem 5.** Check (using the axioms of lattices) that the structure of a lattice is equivalent to a poset \(P\) such that, for any two elements \(x, y \in P\), there is a unique minimal element \(u\) such that \(u \geq x\) and \(u \geq y\), and there is a unique maximal element \(v\) such that \(v \leq x\) and \(v \leq y\).

**Problem 6.** Prove Minsky’s theorem that says that in any finite poset \(P\) the maximal number of elements in a chain equals the minimal number of anti-chains needed to cover all elements of \(P\).

**Problem 7.** Find an example of graded, rank symmetric, rank unimodular poset that has Sperner property, but does not have a symmetric chain decomposition (SCD).
Problem 8. Prove that the lattice $J(J([2] \times [n]))$ is rank unimodular.

Problem 9. In class we constructed the Fibonacci lattice $F$. Since $F$ is a differential poset, we have
\[ \sum_{x \text{ element of rank } n} (f_x)^2 = n!, \]
where $f_x$ is the number of saturated chains in $F$ from $\hat{0}$ to $x$.

(a) Prove that $F$ is a lattice.

(b) Find a non-inductive combinatorial construction for $F$. In other words, label elements of $F$ by some kind of combinatorial objects and describe the covering relation in terms of these objects.

(c) Find an analog of Schensted correspondence for the Fibonacci lattice, which would provide a combinatorial proof of the above identity.

Problem 10. Let $P$ be a differential poset (such as the Young lattice $Y$ or the Fibonacci lattice $F$). Show that the number of all paths of length $2n$ from $\hat{0}$ to $\hat{0}$ that go along the edges of the Hasse diagram of $P$ (up or down in any order) equals $(2n - 1)!! = (2n - 1)(2n - 3) \cdots 3 \cdot 1$.

Problem 11. A perfect matching on $[2n]$ is a graph with vertices labeled $1, 2, \ldots, 2n$ such that each vertex belongs to exactly one edge.

(a) Show that the number of perfect matchings on $[2n]$ equals $(2n - 1)!!$.

(b) Find a bijection between paths in the Young lattice $Y$ as in the previous problem and perfect matchings on $[2n]$.

Problem 12. Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be two sequences of positive integers such that $a_1 + \cdots + a_k = b_1 + \cdots + b_k = n$.

Let $\lambda$ be the Young diagram such that the path from the bottom left corner to the upper right corner of $\lambda$ has the form: $a_1$ right steps, $b_1$ up steps, $a_2$ right steps, $b_2$ up steps, etc.

(a) Prove that the number of paths in a differential poset $P$ that start and end at $\hat{0}$ and have the following form: $a_1$ steps up, $b_1$ steps down, $a_2$ steps up, $b_2$ steps down, etc., equals the number of ways to place $n$ non-attacking rook of the board of shape $\lambda$.

(b) For the Young lattice $Y$, construct a bijection between paths and rook placements as in part (a).

(c) Find an explicit formula for the number of rook placements as in part (a).

Problem 13. Let $p$ be a prime number, and let $f(x) \in \mathbb{F}_p[x]$ be an irreducible polynomial of degree $n$ over the finite field $\mathbb{F}_p$. One can reduce any monomial $x^i$ modulo the ideal $(f(x))$ and write it as a polynomial of degree $n - 1$. Let $d_i$ be the leading coefficient (the coefficient of $x^{n-1}$) in the reduction of $x^i$.

Prove that $d_0, d_1, d_2, \ldots, d_{p^n-1}$ is a (base $p$) de Bruijn sequence, that is, its cyclically consecutive $n$-tuples contain all $p^n$ $p$-ary words of length $n$.

You can use the following facts about the finite field $\mathbb{F}_q$, where $q = p^n$. Let $\alpha \in \mathbb{F}_q$ be a root of the polynomial $f(x)$. Then

1. $\mathbb{F}_q = \{c_0 + c_1\alpha + \cdots + c_{n-1}\alpha^{n-1} | c_i \in \mathbb{F}_p\}$, and
2. $\mathbb{F}_q \setminus \{0\} = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-2}\}$.

Problem 14. Construct a base $p = 3$ de Bruijn sequence of length 27.
Problem 15. Prove that the binary word obtained by concatenation of all Lyndon words whose length divides \( n \) written in the lexicographical order is a binary de Bruijn sequence of length \( 2^n \).

Problem 16. (a) Construct a bijection between partitions of \( n \) with odd parts and partitions of \( n \) with distinct parts.

(b) Generalize (a) as follows. Prove bijectively that, for any positive integers \( n \) and \( k \), the number of partitions of \( n \) into parts not divisible by \( k + 1 \) equals the number of partitions of \( n \) such that parts can be repeated at most \( k \) times.

Problem 17. Prove that the number of self-conjugate partitions of \( n \) (that is, partitions such that \( \lambda' = \lambda \)) equals the number of partitions of \( n \) with distinct and odd parts.

Problem 18. Prove the following identity for the \( q \)-binomial coefficients

\[
\left[ \begin{array}{c} m+n \\ m \end{array} \right]_q = \sum_{k=0}^{\min(m,n)} q^k \left[ \begin{array}{c} m \\ k \end{array} \right]_q \left[ \begin{array}{c} n \\ k \end{array} \right]_q.
\]

Problem 19. For a partition \( \lambda \), let \( \ell(\lambda) \) be the total number of parts in \( \lambda \), and \( d(\lambda) \) be the number of distinct parts in \( \lambda \). For example, for \( \lambda = (5,5,5,4,2,2,1) \), \( \ell(\lambda) = 7 \), and \( d(\lambda) = 4 \). Let

\[
A = \sum_{\lambda \text{ partition}} (-1)^{\ell(\lambda)} 2^{d(\lambda)} q^{\ell(\lambda)}, \quad B = \prod_{n \geq 1} \frac{1 - q^n}{1 + q^n}, \quad C = \sum_{r=-\infty}^{+\infty} (-1)^r q^{r^2}.
\]

(a) Prove that \( A = B \).

(b) Prove that \( B = C \).

(c) Prove that \( A = C \) combinatorially using the involution principle.

Problem 20. Consider the directed graph given by the \( m \times n \) grid where all edges are directed to the right and up. This graph has a unique source \( A \) (the lower left corner) and a unique sink \( B \) (the upper right corner). An integer \( N \)-flow is a way to assign non-negative integer numbers to the edges of the graph (flows over the edges) such that for each vertex, excepts \( A \) and \( B \), the total in-flow to the vertex equals the total out-flow from the vertex; the out-flow from the source \( A \) equals \( N \), and the in-flow to the sink \( B \) equals \( N \).

(a) Show that the number of \( N \)-flows for the grid graph is a polynomial in \( N \). What is the degree of this polynomial?

(b) Find the leading coefficient of this polynomials.

You can start solving this problem by considering \( m \times n \) grids for small values of \( m \) or \( n \).