**PROBLEM SET 3** (due on Tuesday 04/05/05)

Problem 1. Let $F_n = \sum_P 2^{h(P)}$ be the sum over lattice paths $P$ from $(0,0)$ to $(n,0)$ with the steps $(1,0)$, $(1,-1)$, and $(1,1)$ that are contained in the upper half-plane $\{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$, where $h(P)$ is the number of horizontal steps $(1,0)$ in the path $P$. Prove bijectively that $F_n$ equals the $(n+1)$-st Catalan number $C_{n+1}$. In other words, construct a bijection between weighted paths as above and Catalan paths of length $2(n+1)$.

Problem 2. Deduce the following recurrence relation for the Catalan numbers:

$$C_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} \binom{n}{2k} C_k \,.$$

Problem 3. Prove the following identity for formal power series:

$$\cfrac{1}{1 - x - \cfrac{x^2}{1 - 3\,x - \cfrac{2^2\,x^2}{1 - 5\,x - \cfrac{3^2\,x^2}{1 - 7\,x - \cfrac{4^2 x^2}{\cdots}}}}} = \sum_{n \geq 0} n!\, x^n \,.$$

Problem 4. The *descent set* of a permutation $w$ is $Des(w) := \{i \mid w_i > w_{i+1}\}$. Let $G_n$ be the number of permutations $w \in S_n$ that do not have 2 consecutive descents, i.e., there is no $i$ such that $i, i+1 \in Des(w)$. Show that

$$1 + \sum_{n \geq 1} G_n x^n = \cfrac{1}{1 - x - \cfrac{x^2}{1 - 2x - \cfrac{2^2 x^2}{1 - 3x - \cfrac{3^2 x^2}{1 - 4x - \cfrac{4^2 x^2}{1 - \cdots}}}}}$$

Problem 5. Calculate the following two determinants, where $C_i$ are the Catalan numbers:

$$\begin{vmatrix} C_{2n} & C_{2n-1} & \cdots & C_n \\ C_{2n-1} & C_{2n-2} & \cdots & C_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ C_n & C_{n-1} & \cdots & C_0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} C_{2n-1} & C_{2n-2} & \cdots & C_n \\ C_{2n-2} & C_{2n-3} & \cdots & C_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ C_n & C_{n-1} & \cdots & C_1 \end{vmatrix} .$$

Problem 6. The *Euler number* $E_n$ is defined as the number of *alternating permutations* of size $n$: $E_n = \#\{w \in S_n \mid w_1 < w_2 > w_3 < w_4 > w_5 < \dots\}$. (These numbers are also known as the *secant and tangent numbers*, cf. the problem below, the *zig and zag numbers*, the *André numbers*. They are also related to the *Bernoulli numbers*.)

Prove that the numbers $E_n$ can be calculated using the Euler-Bernoulli triangle, as was described in the lecture.

Problem 7. (a) Show that the Euler numbers $E_{2k+1}$ satisfy the recurrence relation

$$E_{2k+1} = \sum_{i=0}^{k-1} \binom{2k}{2i+1} E_{2i+1} E_{2(k-i)-1}\,,$$

for $k \geq 0$, and $E_1 = 1$.

(b) Show that the exponential generating function

$$T(x) = \sum_{k\geq 0} E_{2k+1} x^{2k+1}/(2k+1)!$$

satisfies the differential equation $T'(x) = 1 + T(x)^2$, $T(0) = 0$.

(c) Prove that $\sum_{k\geq 0} E_{2k+1} x^{2k+1}/(2k+1)! = \tan(x)$.

(d) Prove that $\sum_{k\geq 0} E_{2k} x^{2k}/(2k)! = \sec(x)$.

Problem 8. Let $K_{m,n,k}$ be the complete tripartite graph, i.e., the graph on the vertex set subdivided into 3 parts $\{1,\dots,m\}$, $\{m+1,\dots,m+n\}$, and $\{m+n+1,\dots,m+n+k\}$ and edges $(i,j)$ for all pairs of vertices $i$ and $j$ in different parts. Find a formula for the number of spanning trees of $K_{m,n,k}$. Can you give a combinatorial proof?

Problem 9. Let $G = (V,E)$ be a graph on the vertex set $V = \{1,\dots,n\}$ without loops or multiple edges. Let $\tilde{G}$ be the graph obtained from $G$ by adding the vertex 0 connected with all other vertices in $V$ by edges. Define the polynomial $F_G(x, x_1, \dots, x_n)$ by

$$F_G(x; x_1, \dots, x_n) := \sum_T x^{d_T(0)-1} x_1^{d_T(1)-1} \cdots x_n^{d_T(n)-1},$$

where the sum is over spanning trees $T$ of the extended graph $\tilde{G}$ and $d_T(i)$ denotes the degree of the vertex $i$ in $T$.

Let $\overline{G} = (V, \overline{E})$ denotes the *complement graph* on the same vertex set $V = \{1,\dots,n\}$ such that, for any edge $e$ of $K_n$, $e \in \overline{E}$ if and only if $e \notin E$.

Prove the following *reciprocity* of the polynomials $F_G$:

$$F_{\overline{G}}(x; x_1, \dots, x_n) = (-1)^{n-1} F_G(-x - x_1 - \cdots - x_n; x_1, \dots, x_n).$$

Problem 10. Calculate the polynomials $F_G(x; x_1, \dots, x_n)$ for the complete graph $K_n$, the complete bipartite graph $K_{m,n}$, and the complete tripartite graph $K_{m,n,k}$.

Problem 11. Let $T_{m,n}$ be the $m \times n$ torus graph. Its vertex set is $V = \{(i,j) \mid i = 1,\dots,m;\ j = 1,\dots,n\}$. Two vertices $(i,j)$ and $(i',j')$ are connected by an edge if $i = i'$ and $j - j' = \pm 1 \pmod n$, or $j = j'$ and $i - i' = \pm 1 \pmod m$. In other words, $T_{m,n}$ is the direct product of two cycles.

Find a formula for the number of spanning trees in the torus graph $T_{m,n}$.

Problem 12. Fix nonnegative integers $a_1, \dots, a_n$ such that $a_1 + 2\,a_2 + \cdots + n\,a_n = n$. Show that the number of noncrossing set-partitions of $[n]$ with $a_i$ parts of size $i$, for $i = 1, 2, \dots, n$, equals

$$\frac{n!}{a_1!\,a_2! \cdots a_n!\,(n+1-a_1-\cdots-a_n)!}\,.$$