PROBLEM SET 2 (due on Tuesday 03/15/05)

Problem 1. Let B_n^{even} be the poset of all subsets in $\{1, \dots, n\}$ of even cardinalities ordered by inclusion. Is the poset B_n^{even} Sperner or not?

Problem 2. Present a base 3 de Bruijn sequence of length $3^3 = 27$. In other words, find a sequence of length 27 in the alphabet $\{0, 1, 2\}$ arranged on a circle such that the 27 segments of 3 consecutive numbers (on the circle) form all possible ternary numbers 000,…,222.

Problem 3. Find a bijection between the set of partitions of an integer n with odd parts and partitions of n with distinct parts.

Problem 4. For any $k \geq 1$ and $n \geq 0$, prove that the number of partitions of n into parts not divisible by $k+1$ is equal to the number of partitions of n such that parts can be repeated at most k times.

Problem 5. Prove the identity of power series:

$$1 + \sum_{n \geq 1} \frac{q^n}{(1-q)(1-q^2)\dots(1-q^n)} \quad = \quad \prod_{k \geq 1} \frac{1}{(1-q^k)} \; .$$

Problem 6. For $n \geq 2$, find the number f_n of closed walks on $\mathbb{Z} \times \mathbb{Z}$ that start (and finish) at the origin $(0, 0)$, have n steps $(0, 1)$ or $(1, 0)$ followed by n steps $(0, -1)$ or $(-1, 0)$, and do not have any self-intersections (except the origin). For example, $f_2 = 2$ (up-right-down-left and right-up-left-down).

Problem 7. Let $b_k = \binom{2k}{k}$ for $k \geq 0$, and $b_k = 0$, for $k < 0$. Show that the determinant of the $(n+1) \times (n+1)$-matrix

$$(b_{i-j+1})_{1 \leq i,j \leq n+1} = \begin{pmatrix} b_1 & b_0 & 0 & \cdots & 0 \\ b_2 & b_1 & b_0 & \cdots & 0 \\ b_3 & b_2 & b_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n+1} & b_n & b_{n-1} & \cdots & b_1 \end{pmatrix}$$

equals $(-1)^n \, 2 \, C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Problem 8. Let f_n be the number of graphs (undirected, no loops, no multiple edges) on n labelled vertices such that degrees of all vertices are 0, 1, or 2. Find the exponential generating function $f(x) = 1 + \sum_{n \geq 1} f_n \frac{x^n}{n!}$.

Problem 9. Let $\binom{[n]}{k}$ denote the set of all k-element subsets in $[n] := \{1, \dots, n\}$. For integers k, n such that $1 \leq k < n/2$, construct an injective map f from $\binom{[n]}{k}$ to $\binom{[n]}{k+1}$ such that $f(I) \supset I$ for any $I \in \binom{[n]}{k}$. (Recall that an *injective map* f is a map such that $f(I) \neq f(J)$ for $I \neq J$.)

Problem 10. Let $J(P)$ be the lattice of order ideals in a poset P. Show that the lattice $J(J([2] \times [n]))$ is unimodal. In other words,

$$a_0 \leq a_1 \leq \cdots \leq a_r \geq a_{r+1} \geq a_{r+1} \geq \dots, \qquad \text{for some } r,$$

where a_k is the number of rank k elements in $J(J([2] \times [n]))$.

Problem 11. Let Π_n be the lattice of all set-partitions of $\{1,\dots,n\}$. Find the number of maximal saturated chains in the partition lattice Π_n.

Problem 12. Prove the following identity:

$$\frac{\displaystyle\sum_{n\geq 0} q^{-\binom{n+1}{2}} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q x^n}{\displaystyle\sum_{n\geq 0} q^{-\binom{n+1}{2}} \begin{bmatrix} 2n \\ n \end{bmatrix}_q x^n} = \cfrac{1}{1-\cfrac{q\,x}{1-\cfrac{q^2\,x}{1-\cfrac{q^3 x}{1-\dots}}}}.$$

Problem 13. Show that the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix},$$

for real $x_1 < x_2 < \cdots < x_n$, is *totally positive*, i.e., all minors of this matrix are strictly positive.

Problem 14. Let G a planar acyclic digraph drawn inside of a disk with the vertices $A_1,\dots,A_n,B_n,B_{n-1},\dots,B_1$ on the boundary of the disk (written in the counter-clockwise order). Assign a nonnegative weight x_e to each edge e of the graph G. Let $c_{ij} = \sum_{P:A_i\to B_j} \prod_{e\in P} x_e$ be the sum over all directed paths P from the vertex A_i to the vertex B_j. It was shown in the lecture that the matrix $C = (c_{ij})$ is totally nonnegative matrix, i.e., all it minors are nonnegative.

Show that any totally nonnegative matrix can be obtained from some planar acyclic digraph with nonnegative weights on the edges in this way.