18.218 Spring 2020 — Problem Set 1

due Friday, March 06, 2020

Hand in solutions for four (or more) of the following problems.

Problem 1. The *hypersimplex* Δ_{kn}, for $1 \le k < n$, is the polytope in $\mathbb{R}^n$ defined as the convex hull of the $\binom{n}{k}$ points $(a_1, \dots, a_n)$ such that all $a_i \in \{0,1\}$ and $a_1 + \cdots + a_n = k$. Use the definition of faces of a polytope as supporting faces of a linear function to give an explicit description of

(a) all edges of Δ_{kn},
(b) all facets of Δ_{kn}.

Problem 2. In class, we computed the f-vector and h-vector of permutohedron and deduced the following identity involving the *Stirling numbers* of the second kind $S(n,k)$ and the *Eulerian numbers* $A(n,k)$. (Recall that $S(n,k)$ is the number of set partitions of $[n]$ with k blocks, and $A(n,k)$ is the number of permutations in S_n with k descents.)

$$\sum_{i=0}^{n-1} (n-i)!\, S(n, n-i)\, x^i = \sum_{i=0}^{n-1} A(n,i)\, (x+1)^i.$$

Give a direct combinatorial proof of this identity.

Problem 3. Prove that the normal fan N_{P+Q} of the Minkowski sum $P+Q$ of two polytopes P and Q is the common refinement of the normal fans N_P and N_Q.

Problem 4. True or false: Any centrally symmetric 3-dimensional polytope is a zonotope. Prove this claim or find a counterexample (and prove that it is a counterexample).

Problem 5. Prove that each vertex of the Minkowski sum $P+Q$ of two polytopes can be *uniquely* written as a sum of a vertex of P and a vertex of Q.

Problem 6. Find a bijection between integer lattice points of the permutohedron P_n and forests on n labelled vertices.

Problem 7. Prove that the expansion of the product

$$\prod_{1\leq i<j\leq n} (x_i + x_j)$$

contains the monomials $x_1^{a_1}\cdots x_n^{a_n}$ (with nonzero coefficients) for *all* integer lattice points $a = (a_1, \ldots, a_n)$ of the (shifted) permutohedron $P_n + \{(-1, \ldots, -1)\}$. Describe all monomials in this expansion whose coefficients are equal to 1.

Problem 8. Fix $(n-1)\binom{n}{2}$ nonzero complex constants c_{ijk}, for $1 \leq i < j \leq n$ and $k = 1, \ldots, n-1$. Assume that the product of any nonempty subset of the numbers $(c_{ijk})^{\pm 1}$ is not equal to 1. Consider the following polynomials $f_k(x_1, \ldots, x_n)$, $k = 1, \ldots, n-1$, in the variables $x_1, \ldots, x_n$:

$$f_k(x_1, \ldots, x_n) = \prod_{1\leq i<j\leq n} (x_i - c_{ijk}\, x_j).$$

Find the number of solutions in $(\mathbb{C}\setminus\{0\})^n$ of the system of n equations in the n variables $x_1, \ldots, x_n$ by explicitly solving this system:

$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ f_2(x_1, \ldots, x_n) = 0 \\ \cdots \\ f_{n-1}(x_1, \ldots, x_n) = 0 \\ x_n = 1 \end{cases}$$

Compare your answer with Kushnirenko's theorem.

Problem 9. For a polytope P that belongs to the hyperplane $H := \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 0\} \subset \mathbb{R}^n$, we defined the volume $\mathrm{Vol}_H(P)$ as $\mathrm{Vol}(p(P))$, where $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection $p : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$. Also let $\mathrm{Vol}_{eucl}(P)$ be the usual $(n-1)$-dimensional Euclidian volume of P.

For any n, find the constant C such that $\mathrm{Vol}_{eucl}(P) = C \cdot \mathrm{Vol}_H(P)$.

For example, for $n = 2$ and the line segment $P = [(0, 0), (1, -1)]$, we have $\mathrm{Vol}_H(P) = 1$ and $\mathrm{Vol}_{eucl}(P) = \sqrt{2}$, so $C = \sqrt{2}$.

Problem 10. In class, we constructed a pseudoline arrangement by splitting all triple intersections in the Pappus configuration into 3 double intersections. Give a rigorous proof that this pseudoline arrangement cannot be drawn on the plane with all straight lines.

Problem 11. Let $G = (V, E)$ be a graph without loops. Pick orientations of all edges in G. Let $\mathbb{R}^E$ be the vector space of functions $f : E \to \mathbb{R}$ on edges of G, and let $\mathbb{Z}^E \subset \mathbb{R}^E$ be the lattice of all integer-valued functions on edges. For a vertex $v \in V$, $f_v \in \mathbb{R}^E$ is given by

$$f_v(e) = \begin{cases} 1 & \text{if } e \text{ is an outgoing edge from the vertex } v, \\ -1 & \text{if } e \text{ is an incoming edge to the vertex } v, \\ 0 & \text{otherwise.} \end{cases}$$

Let $C_G \simeq \mathbb{R}^m$ be the quotient space of $\mathbb{R}^E$ by the linear subspace spanned by all f_v, for $v \in V$. Let $p : \mathbb{R}^E \to C_G$ be the natural projection to C_G. Also let $L_G := p(\mathbb{Z}^E) \simeq \mathbb{Z}^m$ be the integer lattice in C_G.

Let $\mathbf{e}_e$, $e \in E$, denote the coordinate vectors in the space $\mathbb{R}^E$. (In other words, $\mathbf{e}_e$ is the function on edges of G which is equal to 1 on the edge e and 0 on all other edges.)

The *cographical vector arrangement* is the arrangement of the vectors $p(\mathbf{e}_e)$, for $e \in E$, in the vector space C_G.

(a) Prove that the cographical vector arrangement is unimodular with respect to the integer lattice L_G.

(b) Describe all bases of the cographical vector arrangement.

Problem 12. Let $v_1, \ldots, v_N \in \mathbb{Z}^d$ be a unimodular collection of vectors, and let $Z = \mathrm{Zon}(v_1, \ldots, v_N)$ be the associated zonotope. Prove that the Ehrhart polynomial $i_Z(t)$ of the zonotope Z equals

$$i_Z(t) = \sum_{I \text{ independent subset in } [N]} t^{|I|}.$$

Use the Ehrhart reciprocity (or some other method) to deduce that the number $\#(P_n \setminus \partial P_n) \cap \mathbb{Z}^n$ of integer lattice points in the *interior* of the permutohedron P_n equals $(-1)^{n-1}(F_n^{even} - F_n^{odd})$, where F_n^{even} (resp., F_n^{odd}) is the number of forests on n labelled vertices with even (resp., odd) number of edges. Can you give a direct proof of this claim?

Problem 13. For integers $n \geq 1$ and $k \geq 0$, calculate the number of regions of the extended Catalan arrangement, which consists of the hyperplanes in $\mathbb{R}^n$ given by the equations:

$$x_i - x_j = r, \text{ for } 1 \leq i < j \leq n, \text{ and } r = -k, -k+1, \ldots, k-1, k.$$

Problem 14. For integers $n \geq 1$ and $k \geq 0$, calculate the number of regions of the extended Shi arrangement, which consists of the hyperplanes in $\mathbb{R}^n$ given by the equations:

$$x_i - x_j = r, \text{ for } 1 \leq i < j \leq n, \text{ and } r = -k, -k+1, \ldots, k, k+1.$$

Problem 15. Find a bijective proof for the formula $(n+1)^{n-1}$ for the number of regions of the Shi arrangement in $\mathbb{R}^n$.