18.218 PROBLEM SET 1 (due Monday, April 02, 2018)

Solve as many problems as you want. Turn in your favorite problems. (Is is enough to turn in 3–4 problems.)

Problem 1. Let $T_{m,n}$ be the polytope (called the transportation polytope) of real $m \times n$ matrices $A = (a_{ij})$ such that (1) all $a_{ij} \geq 0$; (2) (column sums) $\sum_i a_{ij} = m$, for any j; (3) (row sums) $\sum_j a_{ij} = n$, for any i.

(a) Describe the vertices of the polytope $T_{m,m+1}$ and find their number.

(b)* Can you say anything about vertices of $T_{m,n}$ when $n \neq m-1, m, m+1$?

Problem 2. Find the f-vector $(f_0, f_1, \ldots, f_{n-1})$ of the hypersimplex Δ_{kn}. Here f_i is the number of i-dimensional faces of the hypersimplex.

Problem 3. (a) Prove that the Eulerian numbers $A(n,k)$ (the numbers of permutations in S_n with k descents) can be computed using the Euler triangle. (Recall that the Euler triangle is similar to the Pascal triangle but with weights.)

(b) Prove the formula for the Eulerian numbers:

$$A(n,k) = \sum_{i=0}^{k+1} (-1)^i \binom{n+1}{i} (k+1-i)^n.$$

Problem 4. (Multi-triangulations of n-gons)

(a) Prove that any r-triangulation of an n-gon contains exactly $r(2n-2r-1)$ edges.

(b) Prove that 2-triangulations of an n-gon are in bijection with pairs of nested Dyck paths.

(c) Prove that r-triangulations of an n-gon are in bijection with collections of r nested Dyck paths.

(d) Prove that the number of r-triangulations of an n-gon is given by the deteminant of $r \times r$ matrix $A = (a_{ij})$, where $a_{ij} = C_{m+i+j-2}$ (the Catalan numbers). (Express m in terms of n and r.)

Problem 5. Let D_{kn} be the directed graph on the vertex set $\binom{[n]}{k}$ with directed edges $I \to J$ if $j_1 = i_1, j_2 = i_2, \ldots, j_{s-1} = i_{s-1}, j_s = i_s + 1 \pmod{n}, j_{s+1} = i_{s+1}, \ldots, j_k = i_k$ for some $s \in [k]$. We proved in class that the number of cyles in the graph D_{kn} of length n is the Eulerian number $A(n-1, k-1)$. Find an expression for the number of simple cycles of length rn in the graph D_{kn}.

Problem 6. Let $M \subseteq \binom{[n]}{k}$. Is it true that the following three properties are equivalent?

<u>Exchange Property</u>: For any $I, J \in M$ and any $i \in I$, there exists $j \in J$ such that $(I \setminus \{i\}) \cup \{j\} \in M$.

<u>Stronger Exchange Property</u>: For any $I, J \in M$ and any $i \in I$, there exists $j \in J$ such that both $(I \setminus \{i\}) \cup \{j\}$ and $(J \setminus \{j\}) \cup \{i\}$ are in M.

<u>Even Stronger Exchange Property</u>: For any $I, J \in M$, any $r \geq 1$, and any $i_1, \ldots, i_r \in I$, there exist $j_1, \ldots, j_r \in J$ such that both $(I \setminus \{i_1, \ldots, i_r\}) \cup \{j_1, \ldots, j_r\}$ and $(J \setminus \{j_1, \ldots, j_r\}) \cup \{i_1, \ldots, i_r\}$ are in M.

Prove the equivalence of (some of) these properties or construct counterexamples.

Problem 7. Let M be a nonempty subset of $\binom{[n]}{k}$.

(a) Prove that M is a matroid (that is, M satisfies the above Exchange Property) if and only if the set $w(M)$ has a unique minimal element in the Gale order, for any permutation $w \in S_n$.

(b) Prove that M is a matroid if and only if the polytope $P_M := \mathrm{conv}\{e_I \mid I \in M\}$ has all edges of the form $[e_I, e_J]$ for $I, J \in \binom{[n]}{k}$ such that $|I \setminus J| = |J \setminus I| = 1$.

Problem 8. Check that the following objects satisfy the Exachange Property (that is, they are matroids) but they are not realizable over $\mathbb{R}$:

(a) The Fano plane.

(b) The non-Pappus matroid.

(b) The non-Desargues matroid.

Problem 9. (co-graphical matroids) Let G be a simple graph. Pick any orientations of all edges of G. Let F_G be the *flow space* of G, that is the space of functions on edges of the graph (flows through the edges) such that, for each vertex v, the in-flow to v equals to the out-flow from v. For an edge e of G, let f_e be the linear function on the flow space F_G that associates to any flow from F_G its value on the edge e. Then the f_e are elements of the dual space $(F_G)^*$.

Prove that the matroid given by the configuraion of vectors f_e in $(F_G)^*$ is dual to the graphical matroid of G.

Problem 10. (a) Prove the image of the Grassmannian $Gr(k, n, \mathbb{C})$ in the projective space $\mathbb{CP}^{\binom{n}{k}-1}$ under the Plücker embedding is the zero locus of the Plücker relations $\Delta_{i_1 \ldots i_k} \Delta_{j_1 \ldots j_k} = \sum \Delta_{i'_1 \ldots i'_k} \Delta_{j'_1 \ldots j'_k}$ for $r = 1$.

(b) Let $\mathbb{C}[\Delta_I]$ be the polynomial ring in $\binom{n}{k}$ (independent) variables Δ_I, $I \in \binom{[n]}{k}$. Let $I_{kn} = \left\langle \Delta_{i_1 \dots i_k} \Delta_{j_1 \dots j_k} - \sum \Delta_{i'_1 \dots i'_k} \Delta_{j'_1 \dots j'_k} \right\rangle$ be the ideal in $\mathbb{C}[\Delta_I]$ whose generators correspond to the Plücker relations (for all r). Prove that I_{kn} is the ideal of all polynomials in $\mathbb{C}[\Delta_I]$ that vanish on the image of the Grassmannian $Gr(k, n, \mathbb{C})$ in the projective space $\mathbb{CP}^{\binom{n}{k}-1}$ under the Plücker embedding.

Problem 11. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a Young diagram that fits inside the $k \times n$ rectangle. Consider the subset S_λ of the Grassmannian $Gr(k, n)$ over a finite field $\mathbb{F}_q$ that consists of the elements that can be represented by $k \times n$ matrices A with 0's outside the shape λ. For example, for $n = 4$ and $k = 2$, $S_{(4,1)}$ is the subset of elements of $Gr(2, 4)$ representable by matrices of the form $\begin{pmatrix} * & * & * & * \\ * & 0 & 0 & 0 \end{pmatrix}$

In parts a,b,c assume $n = 2k$ and $\lambda = (2k, 2k-2, 2k-4, \dots, 2)$.

(a) Find a combinatorial expression for the number of elements of $S_{(2k,2k-2,\dots,2)}$ (over $\mathbb{F}_q$). Show that it is a polynomial in q.

(b) Let $f_k(q)$ be the polynomial from part 1. Calculate $f_k(1)$, $f_k(0)$, and $f_k(-1)$.

(c) Let $g_k(q) = q^d\, f_k(q^{-1})$, where d is the degree of the polynomial $f_k(q)$. Find the maximal power of 2 that divides the number $g_k(5)$.

(d) Generalize (some of) the above to other Young diagrams λ.

(e) What about skew shapes λ/μ?

Problem 12. (a) Prove that image of the Grassmannian $Gr(k, n, \mathbb{C})$ under the moment map is a convex polytope.

(b) Describe the moment map image of (the closure of) the Schubert cell $\overline{\Omega_{(2,1)}} \subset Gr(2, 4, \mathbb{C})$.

(c) Calculate the normalized volume of the moment map image of $\overline{\Omega_\lambda} \subset Gr(k, n, \mathbb{C})$ for any λ.

Problem 13. Find an expression for the Ehrhart polynomial $i(P, t) := \#(tP \cap \mathbb{Z}^n)$, $t \in \mathbb{Z}_{\geq 0}$, of the hypersimplex $P = \Delta_{kn}$ using inclusion-exclusion.

Problem 14. Let A be a generic upper-triangular $n \times n$ matrix. Find the number of non-zero minors of A of all sizes (including the empty minor of size 0×0).

Problem 15. Find the "birational subtraction-free bijection" $(x, y, z) \mapsto (x', y', z')$ from $\mathbb{R}^3_{>0}$ to itself such that

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z' \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Find the bijective map $(x, y) \mapsto (\tilde{x}, \tilde{y})$ from $\mathbb{R}^2_{>0}$ to $\mathbb{R}^2_{>0}$ such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tilde{y} & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

for some $t_1, t_2 \in \mathbb{R}_{>0}$.

Problem 16. Calculate the number of d-dimensional cells in the totally nonnegative Grassmannian $Gr_{\geq 0}(2, n)$.

Problem 17. Let P be a path in any directed 3-valent graph. Let us start erasing loops (i.e., closed directed paths without self-intersections) in P until we get a path P' without self-intersections. Is it true that the parity of the number of erased loops is a well-defined invariant of path P and it does not depend on the order of erasing loops? (In class, we proved this for planar graphs embedded into a disk and paths P between two boundary vertices.)