18.218 FALL 2016 — PROBLEM SET 2 — ADDITIONAL PROBLEMS

due Wednesday, May 4, 2016

Problem 11. Let $X = (x_{ij})$ be the $m \times n$ matrix filled with the variables x_{ij}. A *minor* of a matrix is the determinant of a square submatrix. Let $D(X)$ be the product of all minors (of all sizes) of the matrix X. Let $SP_{m,n}$ be the Newton polytope of $D(X)$.

In class, we showed that the vertices of the polytope $SP_{m,n}$ are in bijection with regular triangulations of the product of two simplices $\Delta^{m-1} \times \Delta^{n-1}$.

(a) Show that $SP_{2,n}$ is the standard permutohedron P_n.

(b) Find the number of vertices of $SP_{3,3}$.

Problem 12. Let G be a bipartite graph. Fix a total ordering of edges of G. For a spanning tree $T \subset G$ and an edge $e \in E(G) \setminus E(T)$, we say that e is *externally semi-active* with respect to T if, in the unique cycle $C \subset T \cup \{e\}$, the maximal edge of C and the edge e are in the same parity class. Let $\mathrm{esa}(T)$ be the number of externally semi-active edges.

Let $B_G(x) := \sum_{T \subset G} x^{\mathrm{esa}(T)}$ and $V_G := B_G(0)$.

Prove that

$$B_G(x) = \sum_{H \subset G} V_H \, (x-1)^{c(H)},$$

where the sum is over connected subgraphs H of G (that include all vertices of G), and $c(H)$ equals the number of edges of H minus the number of vertices of H plus 1.

Problem 13. In this problem, you'll prove a recurrence relation for the number V_G.

A subset S of vertices of a bipartite graph G is called *non-expanding* if it belongs to one part of G and the number of vertices that are neighbors of a vertex in S is less than or equal to $|S|$. For a subset J of vertices of G, let $G_{\setminus J}$ denote the graph G with vertices J and all adjacent edges removed. In class, we proved the following lemma.

Lemma 1. *For a non-expanding set S in G and any spanning tree $T \subset G$, at least one of the vertices $i \in S$ is a leaf of T.*

(a) Prove the following claim.

Lemma 2. *For a vertex i of G, let $G' = G_{\setminus\{i\}}$. Let T' be a spanning tree of G' with* $\mathrm{esa}(T') = 0$. *Then there is a unique edge (i, j) of G such that the tree T obtained from T' by adding the edge (i, j) has* $\mathrm{esa}(T) = 0$.

(b) Use Lemmas 1 and 2, and the inclusion-exclusion principle to show that, for any non-expanding set S in G, we have

$$V_G = \sum_{J \subseteq S,\, J \neq \emptyset} (-1)^{|J|-1}\, V_{G_{\setminus J}}.$$