Problem 11. Let \( X = (x_{ij}) \) be the \( m \times n \) matrix filled with the variables \( x_{ij} \). A minor of a matrix is the determinant of a square submatrix. Let \( D(X) \) be the product of all minors (of all sizes) of the matrix \( X \). Let \( SP_{m,n} \) be the Newton polytope of \( D(X) \).

In class, we showed that the vertices of the polytope \( SP_{m,n} \) are in bijection with regular triangulations of the product of two simplices \( \Delta^{m-1} \times \Delta^{n-1} \).

(a) Show that \( SP_{2,n} \) is the standard permutohedron \( P_n \).

(b) Find the number of vertices of \( SP_{3,3} \).

Problem 12. Let \( G \) be a bipartite graph. Fix a total ordering of edges of \( G \). For a spanning tree \( T \subset G \) and an edge \( e \in E(G) \setminus E(T) \), we say that \( e \) is externally semi-active with respect to \( T \) if, in the unique cycle \( C \subset T \cup \{e\} \), the maximal edge of \( C \) and the edge \( e \) are in the same parity class. Let \( esa(T) \) be the number of externally semi-active edges.

Let \( B_G(x) := \sum_{T \subset G} x^{esa(T)} \) and \( V_G := B_G(0) \).

Prove that
\[
B_G(x) = \sum_{H \subset G} V_H (x - 1)^{c(H)},
\]
where the sum is over connected subgraphs \( H \) of \( G \) (that include all vertices of \( G \)), and \( c(H) \) equals the number of edges of \( H \) minus the number of vertices of \( H \) plus 1.

Problem 13. In this problem, you’ll prove a recurrence relation for the number \( V_G \).

A subset \( S \) of vertices of a bipartite graph \( G \) is called non-expanding if it belongs to one part of \( G \) and the number of vertices that are neighbors of a vertex in \( S \) is less than or equal to \( |S| \). For a subset \( J \) of vertices of \( G \), let \( G \setminus J \) denote the graph \( G \) with vertices \( J \) and all adjacent edges removed. In class, we proved the following lemma.

Lemma 1. For a non-expanding set \( S \) in \( G \) and any spanning tree \( T \subset G \), at least one of the vertices \( i \in S \) is a leaf of \( T \).

(a) Prove the following claim.
Lemma 2. For a vertex $i$ of $G$, let $G' = G \setminus \{i\}$. Let $T'$ be a spanning tree of $G'$ with $esa(T') = 0$. Then there is a unique edge $(i, j)$ of $G$ such that the tree $T$ obtained from $T'$ by adding the edge $(i, j)$ has $esa(T) = 0$.

(b) Use Lemmas 1 and 2, and the inclusion-exclusion principle to show that, for any non-expanding set $S$ in $G$, we have

$$V_G = \sum_{J \subseteq S, J \neq \emptyset} (-1)^{|J|-1} V_{G \setminus J}.$$