18.218 Fall 2016 — Problem Set 2

due Wednesday, May 4, 2016

Problem 1. (a) Prove Bernoulli's formula

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r \, n^{m+1-r},$$

for $m, n \geq 0$. Here B_r are the *Bernoulli numbers* defined by the power series $x/(1-e^{-x}) = \sum_{r\geq 0} B_r x^r/r!$.

(b) Prove Euler's formula

$$\sum_{k=0}^{\infty} k^m x^k = \frac{x \displaystyle\sum_{r=0}^{m-1} A(r,m)\, x^r}{(1-x)^{m+1}},$$

where $A(r,m)$ is the *Eulerian number* defined as the number of permutations in S_m with exactly r descents.

Problem 2. For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, define the function $b_\lambda(x_1, \dots, x_n)$, as follows:

$$b_\lambda(x_1, \dots, x_n) = \sum_{w \in S_n} w \left(\frac{x_1^{\lambda_1} \cdots x_n^{\lambda_n}}{\prod_{i=1}^{n-1} (1 - x_{i+1}/x_i)} \right).$$

(Here the symmetric group S_n acts on rational functions by permutations of the variables x_i.)

(a) Show that $b_\lambda(x_1, \dots, x_n)$ is a symmetric polynomial in $\mathbb{Z}[x_1, \dots, x_n]^{S_n}$.

(b) Prove that the polynomials $b_\lambda(x_1, \dots, x_n)$ form a $\mathbb{Z}$-linear basis of $\mathbb{Z}[x_1, \dots, x_n]^{S_n}$.

Problem 3. Prove the equivalence of the following two definitions of matroids in terms of bases and rank functions.

Fix positive integers $n \geq k > 0$.

Definition 1. The *base set of a matroid* is a nonempty subset $B \subseteq \binom{[n]}{k}$ that satisfies the **Exchange Axiom:**

$\forall\, I, J \in B,\ \forall\, i \in I \setminus J \quad \exists\, j \in J \setminus I$ such that $(I \setminus \{i\}) \cup \{j\} \in B$.

Definition 2. The *rank function of a matroid* is a function $rank : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ such that

(1) $rank(\emptyset) = 0$ and $rank([n]) = k$.

(2) $\forall\ I \subset [n]$ and $j \in [n] \setminus I$, $rank(I \cup \{j\}) - rank(I) \in \{0, 1\}$.

(3) The rank function *rank* satisfies the **Submodular Condition:**

$$\forall\ I, J \subset [n], \quad rank(I) + rank(J) \geq rank(I \cup J) + rank(I \cap J).$$

The correspondence between base sets B and rank functions *rank* is given by

$$rank(I) = \max_{J \in B} |I \cap J|, \qquad \text{for } I \subseteq [n].$$

Problem 4. In class, we showed that the *reduced deformation cone* of the permutohedron can be described as

$$\tilde{D}_n = \{ \text{ submodular functions } f : 2^{[n]} \to \mathbb{R}\} / \{ \text{ modular functions } g : 2^{[n]} \to \mathbb{R}\}.$$

Recall that *submodular functions* are given by the condition

$$f(I) + f(J) \geq f(I \cup J) + f(I \cap J);$$

and *modular functions* are given by the condition

$$g(I) + g(J) = g(I \cup J) + g(I \cap J);$$

for any $I, J \subset [n]$.

For $n = 2, 3, 4$, describe all generators (i.e., one-dimensional rays) of the cone $\tilde{D}_n$.

Problem 5. Let $V_1, \dots, V_n$ be a collection of linear subspaces in $\mathbb{R}^k$. For $I \subseteq [n]$, let

$$f(I) = \text{ dimension of the span of } V_i,\ i \in I.$$

Prove that $f(I)$ is a submodular function.

Problem 6. Consider a hypergraph $H = \{I_1, I_2, \dots, I_m\}$ with hyperedges $I_i \subseteq [n]$.

An *acyclic orientation* of H is a choice of a node in each hyperedge $i_s \in I_s$, $s = 1, \dots, m$, such that the directed graph with edges $i_s \to j$, $\forall\, s = 1, \dots, m$, $\forall j \in I_s$ is acyclic.

For a positive integer q, a *q-coloring* of H is a map $color : [n] \to [q]$ such that, for any hyperedge I_s of H, the map $color$ has a unique maximal value on I_s, i.e., $\exists\, i \in I_s$ such that $\forall\, j \in I_s \setminus \{i\}$, we have $color(i) > color(j)$.

(a) Prove that there exists a polynomial $\chi_H(q) \in \mathbb{Q}[q]$ (called *chromatic polynomial* of the hypergraph H) such that, for any positive integer q, the value $\chi_H(q)$ equals the number of q-colorings of H.

(b) Prove that $(-1)^n\, \chi_H(-1)$ equals the number of acyclic orientations of the hypergraph H.

(c) Find an explicit expression for the chromatic polynomial $\chi_H(q)$ of the hypergraph $H = \{I_1\}$ with a single hyperedge $I_1 = [n]$.

Problem 7. Prove the following claim

Dragon Marriage Theorem. Let $J_1, \dots, J_{n-1}$ be nonempty subsets $J_i \subseteq [n]$. The following conditions are equivalent:

(1) $\forall\, j_0 \in [n]$, $\exists\; j_1 \in J_1,\, j_2 \in J_2, \dots, j_{n-1} \in J_{n-1}$ such that the elements $j_0, \dots, j_{n-1}$ are all distict.

(2) $\forall\; k = 1, \dots, n-1$ and $\forall$ distinct $i_1, \dots, i_k \in [n-1]$ we have $|J_{i_1} \cup \cdots \cup J_{i_k}| \geq k+1$.

(3) $\exists$ 2-element subsets $e_1 \subset J_1, \dots, e_{n-1} \subset J_{n-1}$ such that $\{e_1, \dots, e_{n-1}\}$ is the edge set of a spanning tree of the complete graph K_n.

Problem 8. The *mixed volume* $V(P_1, \dots, P_d)$ of convex polytopes $P_1, \dots, P_d \in R^d$ is defined as the coefficient of $\lambda_1 \dots \lambda_d$ of the polynomial

$$f(\lambda_1, \dots, \lambda_d) = \mathrm{Vol}(\lambda_1 P_1 + \cdots + \lambda_d V_d).$$

Prove that the mixed volume $V(P_1, \dots, P_d)$ is a multilinear function of $P_1, \dots, P_d$ with respect to Minkowski sum. In other words, show that

$$V(a\, P_1 + b\, P_1', P_2, \dots, P_d) = a\, V(P_1, P_2, \dots, P_d) + b\, V(P_1', P_2, \dots, P_d),$$

for $a, b \geq 0$.

Problem 9. (a) Prove that the set of simplices

$$\Delta_T := \mathrm{conv}(0, e_i - e_j \mid (i,j) \in E(T)),$$

where T ranges over all alternating non-crossing trees on the nodes $1, \dots, n$, forms a triangulation of the root polytope

$$R_n := \mathrm{conv}(0, e_i - e_j \mid 1 \leq i < j \leq n).$$

(b) Show that the normalized volume $(n-1)!\,\mathrm{Vol}(R_n)$ of the root polytope R_n equals the Catalan number $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

(c) Prove that the set of simplices

$$\tilde{\Delta}_T := \mathrm{conv}(e_i - e_j \mid (i,j) \in E(T)),$$

where T ranges over all non-crossing bipartite trees in $K_{m,n}$, forms a triangulation of the product of two simplices $\Delta^{m-1} \times \Delta^{n-1}$.

Problem 10. (cf. Problem 10 from Problem Set 1)

Let G be a (simple undirected) graph on the vertices $1, \dots, n$. Let $Z_G = \sum_{(i,j)\in E(G)} [e_i, e_j]$ be the graphical zonotope for the graph G. In class we explained how a triangulation of the associated root polytope produces a bijection between lattice points of the zonotope Z_G and forests F in the graph G.

Present an explicit construction of bijection between lattice points of Z_G and forests $F \subset G$, and prove that it is indeed a bijection. Alternatively, prove that the following construction gives a needed bijection.

Fix a total ordering of vertices of G and also a total ordering of edges of G. For a forest $F \subset G$, construct the directed graph $D = D(F)$ as follows:

(1) For any (undirected) edge $\{i, j\}$ of F, the graph D contains two directed edges (i, j) and (j, i).

(2) For each connected component of the forest F, the graph D contains loop (m, m) at the maximal vertex m of this connected component. (Let's call the vertex m the *root* of the connected component.)

(3) For any edge e of G between two different connected components of F, orient the edge e from the component with a smaller root towards the component with larger root. The graph D contains this oriented edge e.

(4) For any edge e of G that connects two vertices in the same connected component of F (but e is not an edge of F), let C be the unique cycle in the graph $F \cup \{e\}$. Pick the orientation of the cycle C so that the maximal edge e' of C is oriented from the larger vertex of e' towards the smaller vertex of e'. The graph D contains the edge e directed as in this oriented cycle C.

Define the map

$$\phi : F \mapsto (x_1, \dots, x_n), \qquad \text{where } x_i = \mathrm{outdegree}_{D(F)}(i) - 1.$$

Prove that the map ϕ is a bijection between forests $F \subset G$ and lattice points of the grapical zonotope Z_G.