18.218 Fall 2016 — Problem Set 1 — Additional problems

due Wednesday, March 2, 2016

Problem 13. Consider the $2n$ variables $z_1, \dots, z_n$ and $a_1, \dots, a_n$. The symmetric group S_n acts by permutations of the z-variables (and does not act on the a_i): $w(z_i) = z_{w(i)}$, for $w \in S_n$.

Prove that the expression

$$\sum_{w \in S_n} w \left(\frac{(a_1\, z_1 + \cdots + a_n\, z_n)^{n-1}}{(z_1 - z_2)(z_2 - z_3) \cdots (z_{n-1} - z_n)} \right)$$

is a polynomial in $a_1, \dots, a_n$ that does not depend on the variables $z_1, \dots, z_n$.

Problem 14. Consider the $2n$ variables $x_1, \dots, x_n$ and $z_1, \dots, z_n$. The symmetric group S_n acts by permutations of the z-variables (as in the previous problem), and the cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts by cylic shifts of the x-variables. (The generator of the cyclic group acts as $x_i \mapsto x_{i+1}$, where the indices are taken modulo n.)

Prove that

$$\frac{1}{(n-1)!} \sum_{w \in S_n} \sum_{c \in \mathbb{Z}/n\mathbb{Z}} c\, w \left(\frac{\left(\sum_{1 \le i \le j \le n} x_j\, z_i \right)^{n-1}}{(z_1 - z_2)(z_2 - z_3) \cdots (z_{n-1} - z_n)} \right)$$

equals $(x_1 + \cdots + x_n)^{n-1}$.

Problem 15. The *Bernoulli numbers* B_n, $n \ge 0$, are defined as the coefficients in the following Taylor series

$$\frac{q}{1 - e^{-q}} = \sum_{n \ge 0} B_n \frac{q^n}{n!}.$$

We have $B_0, B_1, B_2, \dots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, \dots$

Show that, for $n \ge 1$, $B_{2n+1} = 0$ and that the number

$$(-1)^{n-1} \frac{2^{2n}(2^{2n} - 1)}{2n} B_{2n}$$

equals the number of *alternating permutations* of size $2n - 1$, that is, the permutations $w = w_1, \dots, w_{2n-1}$ with alternating values

$$w_1 < w_2 > w_3 < w_4 > \cdots < w_{2n-2} > w_{2n-1}.$$