18.218 Fall 2016 — Problem Set 1

due Wednesday, March 2, 2016

Turn in as many problems as you want.

Problem 1. Find a combinatorial proof of the identity:

$$\sum_{i=0}^{n-1} (n-i)!\, S(n,n-i)\, q^i = \sum_{i=0}^{n-1} A(n,i)\,(q+1)^i,$$

where $S(n,n-i)$ is the *Stirling number* of the second kind (the number of set partitions of $[n]$ with $n-i$ non-empty blocks), and $A(n,i)$ the the *Eulerian number* (the number of permutations $w \in S_n$ with i descents).

(We obtained this identity in class by comparing the f-vector and the h-vector of the permutohedron.)

Problem 2. Prove that the Eulerian numbers $A(n,i)$ satisfy *Euler's triangle* recurrence (similar to Pascal's triangle but with weights).

Problem 3. Prove that the normal fan N_{P+Q} of the Minkowski sum $P+Q$ of two (or more) polytopes P and Q is the common refinement of the normal fans N_P and N_Q of P and Q.

Problem 4. Let $a_1 \le \cdots \le a_n$ and $b_1 \le \cdots \le b_n$ be two collections of real numbers. Show that

$$P(a_1,\dots,a_n) + P(b_1,\dots,b_n) = P(a_1+b_1,\dots,a_n+b_n)$$

(the Minkwoski sum of two permutohedra is a permutohedron).

Problem 5. Let $\lambda = (\lambda_1,\dots,\lambda_n)$, $\lambda_1 \ge \cdots \lambda_n \ge 0$, be a partition. The *Schur polynomial* $s_\lambda(x_1,\dots,x_n)$ is defined as

$$s_\lambda(x_1,\dots,x_n) = \frac{1}{\prod_{1\le i<j\le n}(x_j - x_i)} \det(x_i^{\lambda_{n-j}+j})_{i,j\in[n]}.$$

Show that the Newton polytope of the Schur polynomial $s_\lambda(x_1,\dots,x_n)$ is the permutohedron $P(\lambda_1,\dots,\lambda_n)$:

$$Newton(s_\lambda(x_1,\dots,x_n)) = P(\lambda_1,\dots,\lambda_n).$$

Problem 6. Let G be a simple connected graph of the vertex set $[n]$ with the edge set $E(G)$. Fix a reference orientation of all edges of G. (For example, orient each edge as $i \to j$ if $i < j$.)

Define the *graphical collection of vectors* $\mathcal{C}_G$ as the set of vectors $e_i - e_j$ for each edge $i \to j$ of G.

Let the *flow space* F be the linear space of functions $f : E(G) \to \mathbb{R}$ on edges of G such that, for each vertex i of G, we have

$$\sum_{e=(j\to i)\in E(G)} f(e) = \sum_{e'=(i\to k)\in E(G)} f(e'),$$

that is, the in-flow of the vertex i equals the out-flow of the vertex i.

Define the *cographical collection of vectors* $\mathcal{C}_G^*$ as the set of vectors v_e, for $e \in E(G)$, in the dual flow space F^* given by the linear forms on the flow space F:

$$v_e : f \mapsto f(e), \qquad \text{for } f \in F.$$

Define the integer lattice in the flow space space F as the subset of integer-valued flows: $f(e) \in \mathbb{Z}$, for all $e \in E(G)$. Its dual lattice gives the integer lattice in the dual flow space F^*.

(a) Prove that the graphical collection of vectors $\mathcal{C}_G$ is unimodular.

(b) Prove that the cographical collection of vectors $\mathcal{C}_G^*$ is unimodular.

Problem 7. Define the *cographical zonotope* Z_G^* as the zonotope associated with the cographical collection of vectors $\mathcal{C}_G^*$ (see the previous problem):

$$Z_G^* = \sum_{e\in E(G)} [0, v_e] \qquad \text{(Minkowski sum of line segments)}$$

A *totally cyclic orientation* of G is an orientation of all edges of G such that each edge $e \in E(G)$ belongs to a directed cycle with respect to this orientation.

(a) Prove that the number of vertices of the cographical zonotope Z_G^* equals the number of totally cyclic orientations of G.

(b) Prove that the volume of the cographical zonotope Z_G^* equals the number of spanning trees of G.

(c) Prove that the number of integer lattice points of the cographical zonotope Z_G^* equals the number of connected subgraphs of G.

Problem 8. (a) In class we gave an example of non-regular triangulation of a triangle. Prove rigorously that it is indeed non-regular.

(b) We also constructed an example of pseudo-line arrangement using Pappus's hexagon theorem. Prove rigorously that this example is not equivalent to a line arrangement.

Problem 9. Let T be any regular fine zonotopal tiling of a zonotope $Z(v_1, \ldots, v_N) \in \mathbb{R}^d$. The f-vector of this tiling is $(f_0, \ldots, f_d)$, where f_i the the number of i-dimensional faces in the tiling. Let $\mathrm{Ind} \subset 2^{[N]}$ be the set of all *independent subsets* of the matroid associated with vectors $v_1, \ldots, v_N$. In class the proved that $f_0 = |\mathrm{Ind}|$ (the number of independent subsets), and f_d equals the number of bases (i.e., indepenendent subsets of maximal size d).

Prove that f_i is the number of pairs (I, J) of independent subsets $I, J \in \mathrm{Ind}$ such that $I \subseteq J$ and $|I| = i$, for any $i = 0, \ldots, d$.

Problem 10. Find a bijection between integer lattice points of the standard permutohedron $P_n = P(1, \ldots, n)$ and forests on the vertex set $[n]$.

Problem 11. Prove that the *Ehrhart polynomial* $i_Z(t) := \#(tP \cap \mathbb{Z}^d)$ (for $t \in \mathbb{Z}_{>0}$) of any unimodular zonotope $Z = Z(v_1, \ldots, v_N) \in \mathbb{R}^d$ equals the generating function for independent subsets $I \in \mathrm{Ind}$ of the associated matroid:

$$i_Z(t) = \sum_{I \in \mathrm{Ind}} t^{|I|}.$$

In particular, the Ehrhart polynomial of the standard permutohedron $P_n = P(1, \ldots, n)$ is the generating function for forests F on vertices $1, \ldots, n$ counted according to the number of edges $|E(F)|$:

$$i_{P_n}(t) = \sum_{F \text{ forest on } [n]} t^{|E(F)|}.$$

Problem 12. Show that the number $\#((P_n \setminus \partial P_n) \cap \mathbb{Z}^n)$ of *interior* lattice points of the standard permutohedron $P_n = P(1, \ldots, n)$ (where ∂P_n is the boundary of P_n) equals the number of forests on $[n]$ with even number of edges minus the number of forests on $[n]$ with odd number of edges.

(a) Prove this fact using Zaslavsky's formula for the number of bounded regions of an affine hyperplane arrangement.

(b) Prove this fact using Ehrhart's reciprocity.

(c) Give a bijective proof.