18.217 Problem Set 1 (due Friday, October 18, 2024)

Problem 1. Define the *hypersimplex* Δ_{nk}, for integers $n > k > 0$, as

$$\Delta_{nk} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = k \text{ and } 0 \le x_i \le 1 \text{ for } i \in [n]\}$$

(1) Describe vertices, edges, and facets of the hypersimplex Δ_{nk}.

(2) Calculate the f-vector $(f_0, f_1, \dots, f_{n-1})$ of Δ_{nk}. Your answer for f_i might involve a single summation.

Problem 2. Show that the Minkowski sum of hypersimplices $\sum_{k=1}^{n-1} \Delta_{nk}$ is the standard permutohedron

$$P_n := \mathrm{conv}\{(w_1, \dots, w_n) \mid w_1, \dots, w_n \text{ is permutation of } 1, \dots, n\}$$

translated by the vector $-(1, \dots, 1)$.

Problem 3. True or false: Let P be a convex 3-dimensional polytope such that P is centrally symmetric and all facets of P are a centrally symmetric polygons. Then P is a zonotope. (Prove or find a counterexample.)

Problem 4. Let $G = (V, E)$ be a simple connected graph on the vertex set $V = \{1, \dots, n\}$. The *graphical zonotope* Z_G is defined as the Minkowski sum of line segments

$$Z_G := \sum_{\{i,j\} \in E} [\mathbf{e}_i, \mathbf{e}_j],$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the standard coordinate vectors in $\mathbb{R}^n$.

Consider a fine zonotopal tiling τ of Z_G. We say that $F \subset Z_G$ is a *face* the tiling τ if F is a face of (at least one) tile in τ. Let $f_i = f_i(\tau)$ be the number of i-dimensional faces in τ. Define the f-vector of τ as $(f_0, f_1, \dots, f_{n-1})$. For example, the zonotopal tiling of a regular hexagon into 3 rhombuses has f-vector $(7, 9, 3)$.

Show that all fine zonotopal tilings of Z_G have the same f-vector.

Problem 5. For a graph $G = (V, E)$ as in the previous problem, a *G-tournament* T is an orientation of all edges of G. (Here vertices of G represent teams; an edge $\{i, j\} \in E$ represents a game between teams i and j; and an orientation $i \to j$ of an edge means that team i won the game with team j in a tournament T.) The *score vector* $(s_1, \dots, s_n)$ of a tournament T is defined by $s_i = \text{outdegree}_T(i)$, for $i = 1, \dots, n$. In other words, s_i is the number of games that team i won.

(1) Prove that score vectors of all G-tournaments are exactly all integer lattice points of the graphical zonotope Z_G.

(2) Prove that $\mathbf{v}$ is a vertex of the graphical zonotope Z_G if and only if there exists a unique G-tournament T such that $\mathbf{v}$ is the score vector of T.

Problem 6. In class, we defined the nth associahedron A_n as the Minkowski sum of simplices

$$A_n := \sum_{1 \le i \le j \le n} \text{conv}\{\mathbf{e}_i, \mathbf{e}_{i+1}, \dots, \mathbf{e}_j\} \subset \mathbb{R}^n.$$

We explained how to describe vertices of A_n in terms of binary trees.

Describe all faces of the polytope A_n. Show that the number $f_k(A_n)$ of k-dimensional faces of A_n equals the number of subdivisions of an $(n+2)$-gon by $n-k-1$ nonintersecting diagonals. (Here the diagonals are not allowed to intersect in the interior of the $(n+2)$-gon, but they are allowed to have common vertices.)

Problem 7. The *Narayana number* $N(n, k)$, for $1 \le k \le n$, is defined as the number of Dyck paths with $2n$ steps (n "up" steps and n "down" steps) and exactly k peaks. (A *peak* in a Dyck path is an "up" step followed by a "down" step.)

In class, we showed that the kth component $h_k(A_n)$ of the h-vector of the associahedron A_n equals the number binary trees on n nodes with exactly k left edges.

Prove that $h_k(A_n) = N(n, k+1)$. In other words, show that the number of binary trees on n nodes with k left edges equals the number of Dyck paths with $2n$ steps and $k+1$ peaks. Preferably, construct a bijection between these sets.

Problem 8. Recall the relation $f(x) = h(x+1)$ between the f-polynomial and the h-polynomial of a simple polytope. For the associahedron A_n, this gives the combinatorial identity

$$\sum_{k=0}^{n-1} f_k(A_n)\, x^k = \sum_{k=0}^{n-1} h_k(A_n)\, (x+1)^k,$$

where $f_k(A_n)$ is the number of subdivisions of an $(n+2)$-gon by $n-k-1$ nonintersecting diagonals and $h_k(A_n)$ is the Narayana number $N(n, k+1)$ (see the previous two problems). Give a direct combinatorial proof of this identity.

Problem 9. (cf. previous problem) Similarly, for the standard permutohedron P_n we obtain the combinatorial identity

$$\sum_{k=0}^{n-1} f_k(P_n)\, x^k = \sum_{k=0}^{n-1} h_k(P_n)\, (x+1)^k.$$

We showed in class that $f_k(P_n) = (n-k)!\; S(n, n-k)$, where $S(n, n-k)$ is the Stirling number of the second kind, i.e., the number of set partitions of $[n]$ with $n-k$ nonempty blocks. On the other hand, $h_k(P_n)$ is the Eulerian number $A(n,k)$, i.e., the number of permutations in S_n with exactly k descents.

Give a direct combinatorial proof of this identity relating Stirling numbers of the second kind with Eulerian numbers.

Problem 10. Construct a bijection between integer lattice points of the standard permutohedron P_n and forests on n labelled vertices.

Problem 11. Construct a decomposition of the permutohedron P_n into a disjoint union $P_n = \pi_1 \dot\cup \pi_2 \dot\cup \cdots \dot\cup \pi_M$ of half-open parallelotopes π_i (of various dimensions d) such that each d-dimensional parallelotope π_i is isomorphic to the half-open cube $[1, 0[^d$ via an integer invertible (over $\mathbb{Z}$) linear map. (In class, we showed such a decomposition for P_3.)

Problem 12. We say that $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is a *quasipolynomial function* if there exist a positive integer k and polynomials $f_0(t), f_1(t), \ldots, f_{k-1}(t)$ such that $f(t) = f_i(t)$, for any $t \in \mathbb{Z}_{\geq 0}$ and $i \in \{0, \ldots, k-1\}$ such that $t \equiv i \pmod{k}$. For example, $\lfloor t/2 \rfloor$ is a quasipolynomial with $k = 2$, $f_0(t) = t/2$, and $f_1(t) = (t-1)/2$.

Let $P \subset \mathbb{R}^d$ be a *rational* polytope, i.e., all vertices of P belong to $\mathbb{Q}^d$. Show that the function

$$i_P(t) = \#(tP \cap \mathbb{Z}^d), \text{ for } t \in \mathbb{Z}_{\geq 0}$$

is quasipolynomial. (Here you may assume that we already know polynomiality of Ehrhart polynomials of integer lattice polytopes.)

Problem 13. For a simple connected graph G, let $\mathcal{F}_{\mathrm{odd}}(G)$ (resp., $\mathcal{F}_{\mathrm{even}}(G)$) and be the number of forests $F \subset G$ containing all vertices of G with odd (resp., even) number of connected components.

(1) Show that $\mathcal{F}_{\mathrm{odd}}(G) \geq \mathcal{F}_{\mathrm{even}}(G)$ for any G.

(2) Give a simple combinatorial characterization of graphs G for which $\mathcal{F}_{\mathrm{odd}}(G) = \mathcal{F}_{\mathrm{even}}(G)$.