Posets, Coxeter Groups, Root Systems, etc.

Colin Defant

Harvard

University of Minnesota Pre-Talk

Posets

Posets

A *poset* (partially ordered set) is a pair $P = (X, \leq_P)$, where X is a set and $\leq_P$ is a *partial order* on X. That is, $\leq_P$ is a binary relation on X that is

- *reflective* ($x \leq_P x$),
- *antisymmetric* (if $x \leq_P y$ and $y \leq_P x$, then $x = y$),
- *transitive* (if $x \leq_P y$ and $y \leq_P z$, then $x \leq_P z$).

Posets

A *poset* (partially ordered set) is a pair $P = (X, \leq_P)$, where X is a set and $\leq_P$ is a *partial order* on X. That is, $\leq_P$ is a binary relation on X that is

- *reflective* ($x \leq_P x$),
- *antisymmetric* (if $x \leq_P y$ and $y \leq_P x$, then $x = y$),
- *transitive* (if $x \leq_P y$ and $y \leq_P z$, then $x \leq_P z$).

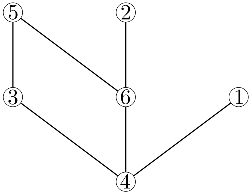
Let's always just assume that $X = [n] = \{1, \ldots, n\}$.

Posets

A *poset* (partially ordered set) is a pair $P = (X, \leq_P)$, where X is a set and $\leq_P$ is a *partial order* on X. That is, $\leq_P$ is a binary relation on X that is

- *reflective* ($x \leq_P x$),
- *antisymmetric* (if $x \leq_P y$ and $y \leq_P x$, then $x = y$),
- *transitive* (if $x \leq_P y$ and $y \leq_P z$, then $x \leq_P z$).

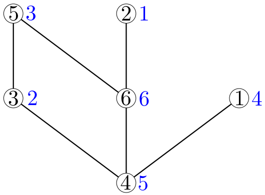
Let's always just assume that $X = [n] = \{1, \dots, n\}$.
We can represent a poset via its *Hasse diagram*.



Linear Extensions

Linear Extensions

Elements of the symmetric group $\mathfrak{S}_n$ can be seen as labelings of P. Say $u \in \mathfrak{S}_n$ is a *linear extension* of P if
$i <_P j \implies u(i) < u(j)$.



The Braid Arrangement

The Braid Arrangement

Let $V^* = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\} \cong \mathbb{R}^{n-1}$. Let $\mathrm{H}_{i,j} = \{(x_1, \dots, x_n) \in V^* : x_i = x_j\}$. The n-th *braid arrangement* is $\mathcal{H}_{\mathfrak{S}_n} = \{\mathrm{H}_{i,j} : 1 \leq i < j \leq n\}$.

The Braid Arrangement

Let $V^* = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\} \cong \mathbb{R}^{n-1}$. Let $\mathrm{H}_{i,j} = \{(x_1, \ldots, x_n) \in V^* : x_i = x_j\}$. The n-th *braid arrangement* is $\mathcal{H}_{\mathfrak{S}_n} = \{\mathrm{H}_{i,j} : 1 \leq i < j \leq n\}$.

$\mathfrak{S}_n$ acts on V^* by permuting coordinates.

The Braid Arrangement

Let $V^* = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\} \cong \mathbb{R}^{n-1}$. Let $\mathrm{H}_{i,j} = \{(x_1, \ldots, x_n) \in V^* : x_i = x_j\}$. The n-th *braid arrangement* is $\mathcal{H}_{\mathfrak{S}_n} = \{\mathrm{H}_{i,j} : 1 \leq i < j \leq n\}$.

$\mathfrak{S}_n$ acts on V^* by permuting coordinates.

Identify elements of $\mathfrak{S}_n$ with regions of $\mathcal{H}_{\mathfrak{S}_n}$ by

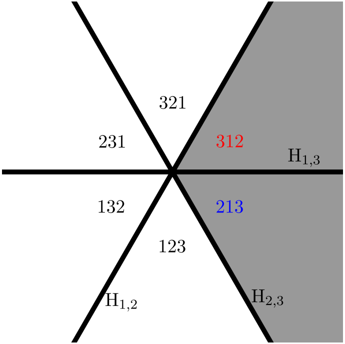
$$w \longleftrightarrow \{(x_1, \ldots, x_n) \in V^* : x_{w^{-1}(1)} \leq \cdots \leq x_{w^{-1}(n)}\}.$$

Posets

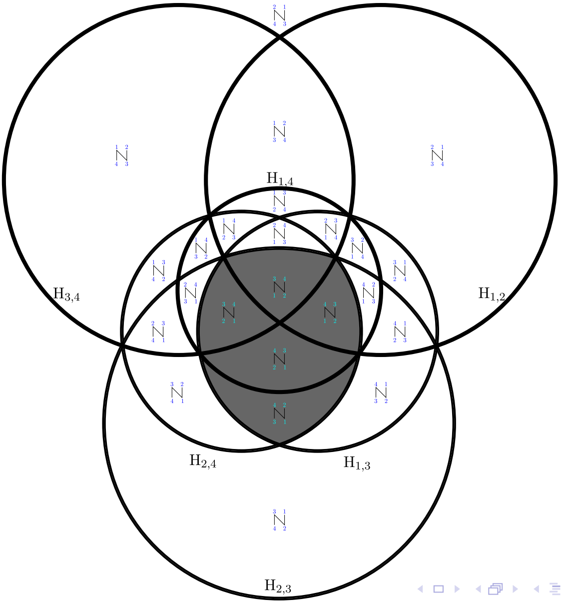
Posets

Let $P = ([n], \leq_P)$ be a poset. $\mathcal{L}(P)$ corresponds to the set of points satisfying $x_i \leq x_j$ whenever $i \leq_P j$.

Let $P = ([n], \leq_P)$ be a poset. $\mathcal{L}(P)$ corresponds to the set of points satisfying $x_i \leq x_j$ whenever $i \leq_P j$.



$\mathcal{H}_{\mathfrak{S}_4}$



Coxeter Groups

Coxeter Groups

Let I be a finite index set. For distinct $i, i' \in I$, let $m(i,i) = 1$, and choose $m(i,i') = m(i',i) \in \{2,3,\ldots\} \cup \{\infty\}$. Let $S = \{s_i : i \in I\}$.

Coxeter Groups

Let I be a finite index set. For distinct $i, i' \in I$, let $m(i,i) = 1$, and choose $m(i,i') = m(i',i) \in \{2,3,\ldots\} \cup \{\infty\}$. Let $S = \{s_i : i \in I\}$.

Let W be the *Coxeter group* with presentation

$$W = \langle S : (s_i s_{i'})^{m(i,i')} = \mathbb{1} \text{ for all } i, i' \in I \rangle,$$

where $\mathbb{1}$ is the identity.

Coxeter Groups

Let I be a finite index set. For distinct $i, i' \in I$, let $m(i,i) = 1$, and choose $m(i,i') = m(i',i) \in \{2,3,\ldots\} \cup \{\infty\}$. Let $S = \{s_i : i \in I\}$.

Let W be the *Coxeter group* with presentation

$$W = \langle S : (s_i s_{i'})^{m(i,i')} = \mathbb{1} \text{ for all } i, i' \in I \rangle,$$

where $\mathbb{1}$ is the identity.

The *Coxeter graph* of W has vertex set I. Two vertices $i, i' \in I$ are connected by an edge labeled $m(i,i')$ whenever $m(i,i') \geq 3$. We do not draw "3" labels.

Coxeter Groups

Let I be a finite index set. For distinct $i, i' \in I$, let $m(i,i) = 1$, and choose $m(i,i') = m(i',i) \in \{2, 3, \ldots\} \cup \{\infty\}$. Let $S = \{s_i : i \in I\}$.

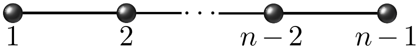
Let W be the *Coxeter group* with presentation

$$W = \langle S : (s_i s_{i'})^{m(i,i')} = \mathbb{1} \text{ for all } i, i' \in I \rangle,$$

where $\mathbb{1}$ is the identity.

The *Coxeter graph* of W has vertex set I. Two vertices $i, i' \in I$ are connected by an edge labeled $m(i,i')$ whenever $m(i,i') \geq 3$. We do not draw "3" labels.

The Coxeter graph of $\mathfrak{S}_n$ (with $s_i = (i\ i+1)$) is



Roots

Roots

Let V be a real vector space of dimension $|I|$. Fix a basis $\{\alpha_i : i \in I\}$ of V; the elements of this basis are called *simple roots*.

Roots

Let V be a real vector space of dimension $|I|$. Fix a basis $\{\alpha_i : i \in I\}$ of V; the elements of this basis are called *simple roots*.

Define a bilinear form $B\colon V \times V \to \mathbb{R}$ by $B(\alpha_i, \alpha_{i'}) = -\cos(\pi/m(i,i'))$ (where $\pi/\infty = 0$).

Roots

Let V be a real vector space of dimension $|I|$. Fix a basis $\{\alpha_i : i \in I\}$ of V; the elements of this basis are called *simple roots*.

Define a bilinear form $B\colon V \times V \to \mathbb{R}$ by $B(\alpha_i, \alpha_{i'}) = -\cos(\pi/m(i,i'))$ (where $\pi/\infty = 0$).

There is a well defined action of W on V such that $s_i\beta = \beta - 2B(\beta, \alpha_i)\alpha_i$. The *root system* of W is

$$\Phi = \{w\alpha_i : w \in W,\, i \in I\}.$$

Roots for $\mathfrak{S}_n$

Roots for $\mathfrak{S}_n$

Let $I = [n-1]$. Let e_i be the i-th standard basis vector of $\mathbb{R}^n$. Let $\alpha_i = e_i - e_{i+1}$.

Roots for $\mathfrak{S}_n$

Let $I = [n-1]$. Let e_i be the i-th standard basis vector of $\mathbb{R}^n$.
Let $\alpha_i = e_i - e_{i+1}$.
We have

$$\begin{aligned} s_i(e_i - e_{i+1}) &= \alpha_i - 2B(\alpha_i, \alpha_i)\alpha_i = \alpha_i - 2(-\cos(\pi/1))\alpha_i \\ &= -\alpha_i = e_{s_i(i)} - e_{s_i(i+1)}; \end{aligned}$$

Roots for $\mathfrak{S}_n$

Let $I = [n-1]$. Let e_i be the i-th standard basis vector of $\mathbb{R}^n$.
Let $\alpha_i = e_i - e_{i+1}$.
We have

$$\begin{aligned} s_i(e_i - e_{i+1}) &= \alpha_i - 2B(\alpha_i, \alpha_i)\alpha_i = \alpha_i - 2(-\cos(\pi/1))\alpha_i \\ &= -\alpha_i = e_{s_i(i)} - e_{s_i(i+1)}; \end{aligned}$$

$$\begin{aligned} s_i(e_{i+1} - e_{i+2}) &= \alpha_{i+1} - 2B(\alpha_{i+1}, \alpha_i)\alpha_i = \alpha_{i+1} - 2(-\cos(\pi/3))\alpha_i \\ &= \alpha_i + \alpha_{i+1} = e_i - e_{i+2} = e_{s_i(i+1)} - e_{s_i(i+2)}. \end{aligned}$$

Roots for $\mathfrak{S}_n$

Let $I = [n-1]$. Let e_i be the i-th standard basis vector of $\mathbb{R}^n$.
Let $\alpha_i = e_i - e_{i+1}$.
We have

$$s_i(e_i - e_{i+1}) = \alpha_i - 2B(\alpha_i, \alpha_i)\alpha_i = \alpha_i - 2(-\cos(\pi/1))\alpha_i$$
$$= -\alpha_i = e_{s_i(i)} - e_{s_i(i+1)};$$

$$s_i(e_{i+1} - e_{i+2}) = \alpha_{i+1} - 2B(\alpha_{i+1}, \alpha_i)\alpha_i = \alpha_{i+1} - 2(-\cos(\pi/3))\alpha_i$$
$$= \alpha_i + \alpha_{i+1} = e_i - e_{i+2} = e_{s_i(i+1)} - e_{s_i(i+2)}.$$

For $|i-j| \geq 2$,

$$s_i(e_j - e_{j+1}) = \alpha_j - 2B(\alpha_j, \alpha_i)\alpha_i = \alpha_j - 2(-\cos(\pi/2))\alpha_i$$
$$= \alpha_j = e_{s_i(j)} - e_{s_i(j+1)}.$$

Roots for $\mathfrak{S}_n$

Let $I = [n-1]$. Let e_i be the i-th standard basis vector of $\mathbb{R}^n$.
Let $\alpha_i = e_i - e_{i+1}$.
We have

$$s_i(e_i - e_{i+1}) = \alpha_i - 2B(\alpha_i, \alpha_i)\alpha_i = \alpha_i - 2(-\cos(\pi/1))\alpha_i$$
$$= -\alpha_i = e_{s_i(i)} - e_{s_i(i+1)};$$

$$s_i(e_{i+1} - e_{i+2}) = \alpha_{i+1} - 2B(\alpha_{i+1}, \alpha_i)\alpha_i = \alpha_{i+1} - 2(-\cos(\pi/3))\alpha_i$$
$$= \alpha_i + \alpha_{i+1} = e_i - e_{i+2} = e_{s_i(i+1)} - e_{s_i(i+2)}.$$

For $|i-j| \geq 2$,

$$s_i(e_j - e_{j+1}) = \alpha_j - 2B(\alpha_j, \alpha_i)\alpha_i = \alpha_j - 2(-\cos(\pi/2))\alpha_i$$
$$= \alpha_j = e_{s_i(j)} - e_{s_i(j+1)}.$$

In general, $w(e_i - e_{i+1}) = e_{w(i)} - e_{w(i+1)}$. So
$\Phi = \{e_i - e_j : i, j \in [n], i \neq j\}$.

The Coxeter Arrangement

The Coxeter Arrangement

Let V^* be the dual space of V. For $\beta \in \Phi$, consider the hyperplane $\mathrm{H}_\beta = \{f \in V^* : f(\beta) = 0\}$ in V^*.

The Coxeter Arrangement

Let V^* be the dual space of V. For $\beta \in \Phi$, consider the hyperplane $\mathrm{H}_\beta = \{f \in V^* : f(\beta) = 0\}$ in V^*.
The *Coxeter arrangement* of W is $\mathcal{H}_W = \{\mathrm{H}_\beta : \beta \in \Phi\}$. A *region* of $\mathcal{H}_W$ is the closure of a connected component of $V^* \setminus \bigcup_{\beta \in \Phi} \mathrm{H}_\beta$.
The *base region* is $\mathbb{B} = \{f \in V^* : f(\alpha_i) \geq 0 \text{ for all } i \in I\}$.

The Coxeter Arrangement

Let V^* be the dual space of V. For $\beta \in \Phi$, consider the hyperplane $\mathrm{H}_\beta = \{f \in V^* : f(\beta) = 0\}$ in V^*.
The *Coxeter arrangement* of W is $\mathcal{H}_W = \{\mathrm{H}_\beta : \beta \in \Phi\}$. A *region* of $\mathcal{H}_W$ is the closure of a connected component of $V^* \setminus \bigcup_{\beta \in \Phi} \mathrm{H}_\beta$.
The *base region* is $\mathbb{B} = \{f \in V^* : f(\alpha_i) \geq 0 \text{ for all } i \in I\}$.

There is a right action of W on V^* determined by the condition that $(fw)(\beta) = f(w\beta)$ for all $w \in W$, $\beta \in V$, and $f \in V^*$.

The Coxeter Arrangement

Let V^* be the dual space of V. For $\beta \in \Phi$, consider the hyperplane $\mathrm{H}_\beta = \{f \in V^* : f(\beta) = 0\}$ in V^*.
The *Coxeter arrangement* of W is $\mathcal{H}_W = \{\mathrm{H}_\beta : \beta \in \Phi\}$. A *region* of $\mathcal{H}_W$ is the closure of a connected component of $V^* \setminus \bigcup_{\beta \in \Phi} \mathrm{H}_\beta$.
The *base region* is $\mathbb{B} = \{f \in V^* : f(\alpha_i) \geq 0 \text{ for all } i \in I\}$.

There is a right action of W on V^* determined by the condition that $(fw)(\beta) = f(w\beta)$ for all $w \in W$, $\beta \in V$, and $f \in V^*$.

This yields a right action of W on the set of regions of $\mathcal{H}_W$.

The Coxeter Arrangement

Let V^* be the dual space of V. For $\beta \in \Phi$, consider the hyperplane $\mathrm{H}_\beta = \{f \in V^* : f(\beta) = 0\}$ in V^*.
The *Coxeter arrangement* of W is $\mathcal{H}_W = \{\mathrm{H}_\beta : \beta \in \Phi\}$. A *region* of $\mathcal{H}_W$ is the closure of a connected component of $V^* \setminus \bigcup_{\beta \in \Phi} \mathrm{H}_\beta$.
The *base region* is $\mathbb{B} = \{f \in V^* : f(\alpha_i) \geq 0 \text{ for all } i \in I\}$.

There is a right action of W on V^* determined by the condition that $(fw)(\beta) = f(w\beta)$ for all $w \in W$, $\beta \in V$, and $f \in V^*$.

This yields a right action of W on the set of regions of $\mathcal{H}_W$.

The *Tits cone* is $\mathbb{B}W$. The action of W on the regions of $\mathcal{H}_W$ in the Tits cone is free and transitive. Thus, we can identify each element $u \in W$ with the region $\mathbb{B}u$.

The Coxeter Arrangement

Let V^* be the dual space of V. For $\beta \in \Phi$, consider the hyperplane $\mathrm{H}_\beta = \{f \in V^* : f(\beta) = 0\}$ in V^*.
The *Coxeter arrangement* of W is $\mathcal{H}_W = \{\mathrm{H}_\beta : \beta \in \Phi\}$. A *region* of $\mathcal{H}_W$ is the closure of a connected component of $V^* \setminus \bigcup_{\beta \in \Phi} \mathrm{H}_\beta$.
The *base region* is $\mathbb{B} = \{f \in V^* : f(\alpha_i) \geq 0 \text{ for all } i \in I\}$.

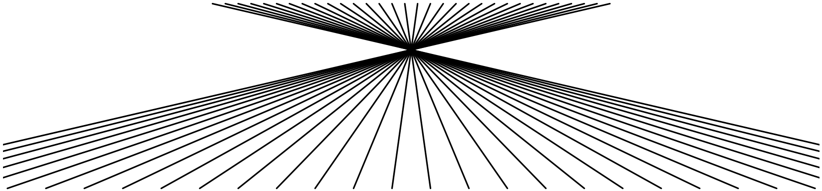
There is a right action of W on V^* determined by the condition that $(fw)(\beta) = f(w\beta)$ for all $w \in W$, $\beta \in V$, and $f \in V^*$.

This yields a right action of W on the set of regions of $\mathcal{H}_W$.

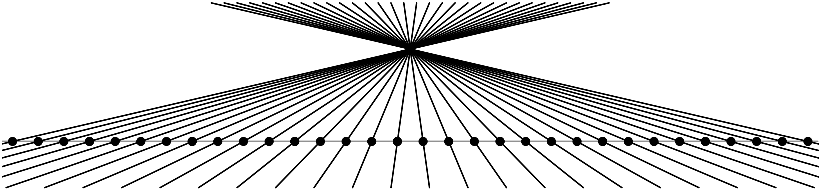
The *Tits cone* is $\mathbb{B}W$. The action of W on the regions of $\mathcal{H}_W$ in the Tits cone is free and transitive. Thus, we can identify each element $u \in W$ with the region $\mathbb{B}u$.

The regions adjacent to $\mathbb{B}u$ are $\mathbb{B}s_iu$ for $i \in I$.

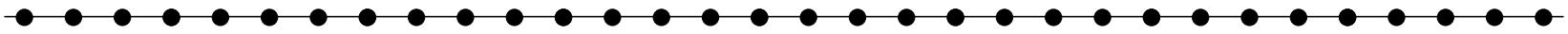
$\widetilde{A}_1$



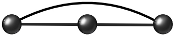
$\widetilde{A}_1$

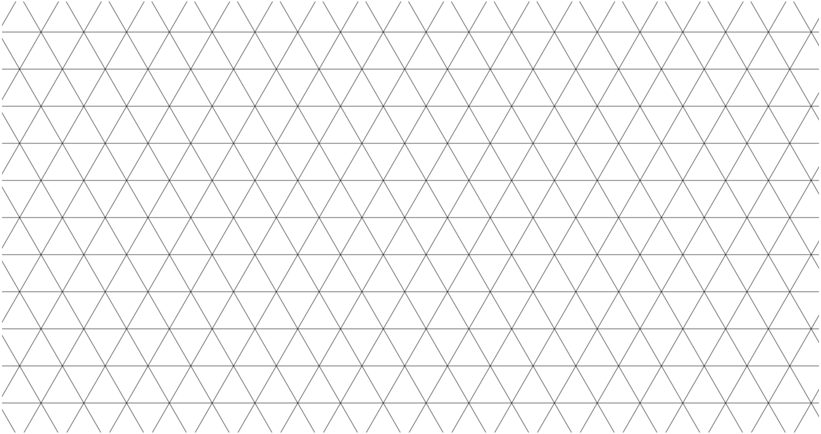


$\widetilde{A}_1$

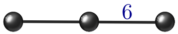


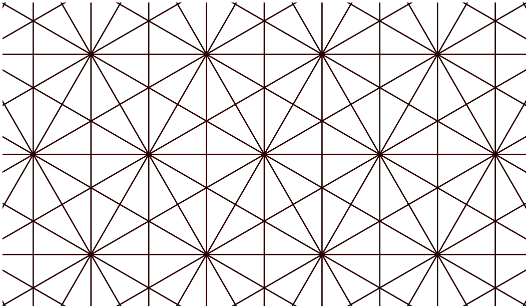
$\widetilde{A}_2$



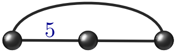


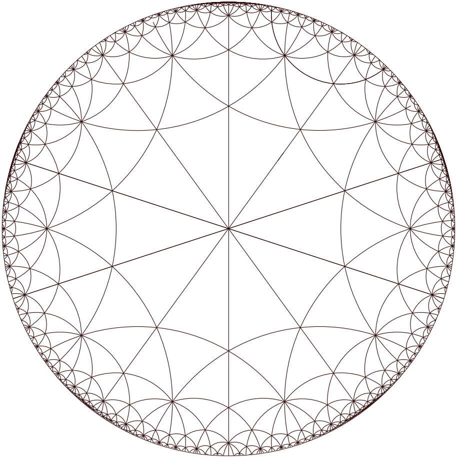
$\widetilde{G}_2$





The $(3, 3, 5)$ Triangle Group





Classes of Coxeter Groups

Classes of Coxeter Groups

A Coxeter group is *irreducible* if its Coxeter graph is connected.

Classes of Coxeter Groups

A Coxeter group is *irreducible* if its Coxeter graph is connected.

There are 3 main classes of Coxeter groups:

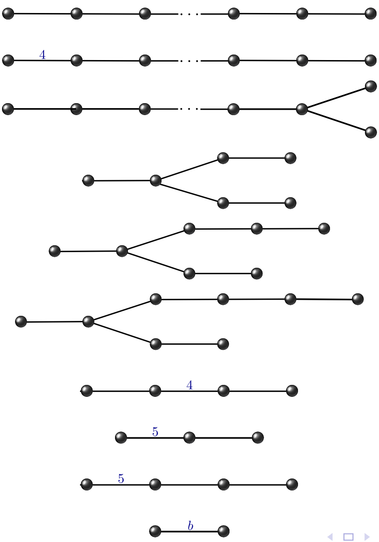
Finite. The Tits cone is all of V^*, and the bilinear form B makes the Tits cone into a spherical geometry.

Affine. The bilinear form B makes the Tits cone into a Euclidean geometry.

Everything Else.

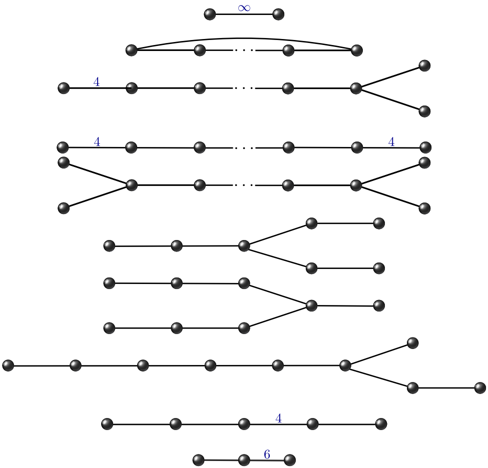
Finite Coxeter Groups

Finite irreducible Coxeter groups have been classified. They are



Affine Coxeter Groups

Affine irreducible Coxeter groups have been classified. They are



Reduced Words and Coxeter Elements

Reduced Words and Coxeter Elements

A *reduced word* for an element $w \in W$ is a word $s_{i_k} \cdots s_{i_1}$ of minimum length that represents w. The length of a reduced word for w is the *length* of w.

Reduced Words and Coxeter Elements

A *reduced word* for an element $w \in W$ is a word $s_{i_k} \cdots s_{i_1}$ of minimum length that represents w. The length of a reduced word for w is the *length* of w.

If W is finite, then there is a unique element $w_\circ \in W$ of maximum length called the *long element*. For example, in $\mathfrak{S}_n$, the long element is $n(n-1)\cdots 321$.

Reduced Words and Coxeter Elements

A *reduced word* for an element $w \in W$ is a word $s_{i_k} \cdots s_{i_1}$ of minimum length that represents w. The length of a reduced word for w is the *length* of w.

If W is finite, then there is a unique element $w_\circ \in W$ of maximum length called the *long element*. For example, in $\mathfrak{S}_n$, the long element is $n(n-1)\cdots 321$.

A *Coxeter element* is an element $c = s_{i_n} \cdots s_{i_1}$ obtained by multiplying all of the simple reflections together in some order. Any two reduced words for c are related by *commutation moves*.

Standard Parabolic Subgroups

Standard Parabolic Subgroups

Let Γ_W be the Coxeter graph of W. Let $J \subseteq I$, and let W_J be the Coxeter group whose Coxeter graph is the subgraph of Γ_W induced by J. Equivalently, W_J is the subgroup of W generated by $\{s_i : i \in J\}$. The subgroup W_J is called a *standard parabolic subgroup*.

THANK YOU!