# LECTURE 25 Mon 11/4

## Brion's Formula (cont'd)

last time, "key identity"

$$\sum_{i\in\mathbb{Z}} x^i \overset{?}{=} 0$$

$$\sum_{i\geq 0} x^i + \sum_{i<0} x^i = \frac{1}{1-x} + \frac{x^{-1}}{1-x^{-1}}$$

BUT The power series $\sum_{i\geq 0} x^i$ & $\sum_{i<0} x_i$ don't have a common area of convergence. Formal power series don't really help.

Goal for today: Justify this identity. To do that, define

## Space of Rational Polyhedra

Def. Fix $n$. A rat. polyhedra $P\subset\mathbb{R}^n$ is given by a system of lin. inequalities w/ rational (equiv. integer) coeffs.

$$[P] : \mathbb{Z}^{[n]} \to \mathbb{R}$$
$$z \mapsto \begin{cases} 1 & z\in P \\ 0 & z\notin P \end{cases}$$

char. fcn. of $P$ restricted to $\mathbb{Z}^n$.

$A$ = the linear space of fcns $\varphi: \mathbb{Z}^n \to \mathbb{R}$ spanned by $[P]$ for all rational polyhedra $P$.

Some elts. of $A$:

- $\delta = [\{(0,\dots,0)\}] : z \mapsto \begin{cases} 1 & \text{if } z=(0,\dots,0) \\ 0 & \text{otherwise} \end{cases}$
- $\delta(z-a) : z \mapsto \begin{cases} 1 & \text{if } z=a \\ 0 & \text{otherwise} \end{cases}$ $\quad a\in\mathbb{Z}^n$
- Any function $\varphi: \mathbb{Z}^n \to \mathbb{R}$ with finite support

The field of rational functions.

$$\mathbb{R}(x_1,\dots,x_n) := \left\{ \frac{f(x_1,\dots,x_n)}{g(x_1,\dots,x_n)} \,\middle|\, f,g \text{ polynomials, } g\neq 0 \right\}$$

Thm: $\exists!$ lin. map $S: A \to \mathbb{R}(x_1,\dots,x_n)$ s.t.

(1) $S(\delta) = 1$

(2) $S(\varphi(z-a)) = x^a S(\varphi(z)) \quad \forall \varphi\in A,\ a\in\mathbb{Z}^n$

where $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$

Let's believe this theorem for a moment & consider some <u>Properties of S</u>

<u>Cor</u>: (1) $S(\delta(z-\alpha)) = x^a \quad \forall a \in \mathbb{Z}^n$

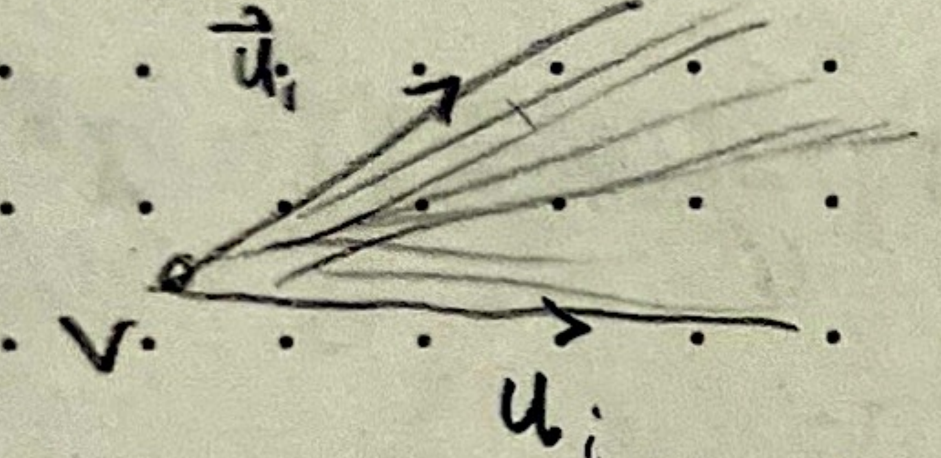
(2) For any polytope $P \subset \mathbb{R}^n$ (bounded polyhedron) has fin. many pts $\Rightarrow$

$$S([P]) = \sum_{a \in P \cap \mathbb{Z}^n} x^a$$

This is exactly the object we were studying last week when we discussed Brion's formula, so this is good.

What about infinite things?
If it's not a rational cone, won't get a rational expression
(Two different meanings of "rational" here, but this ↑ relates them)

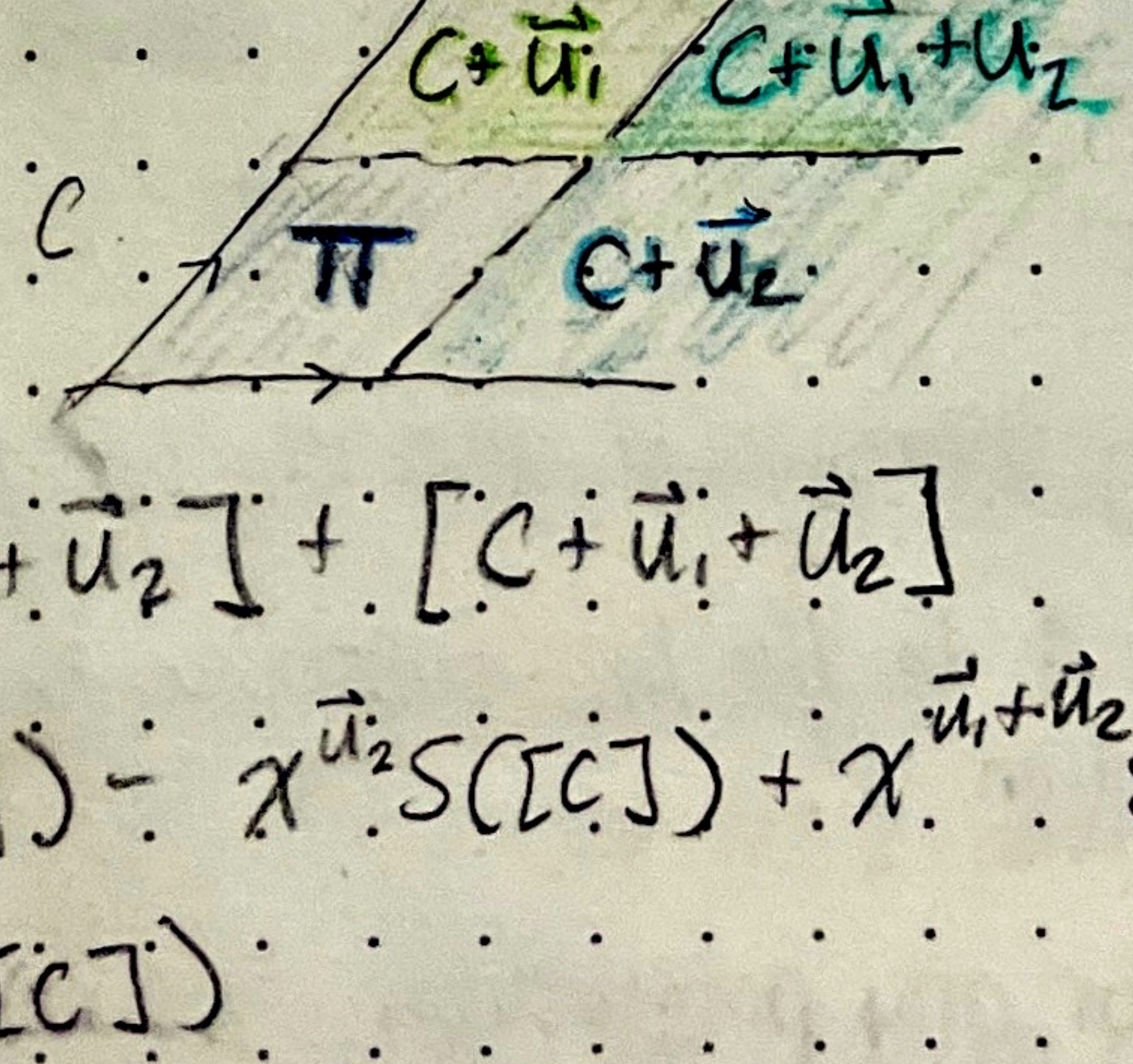
<u>Cor</u>: (3) For a simple cone $C \subset \mathbb{R}^n$ with vertex $v$ generated by vectors $\vec{u}_1, \ldots, \vec{u}_d$ $(d \leq n)$.

$C =$ 

$C = \{v + \alpha_1 \vec{u}_1 + \ldots + \alpha_d \vec{u}_d \mid \alpha_i \geq 0\}$

(*) $\boxed{S([C]) = \frac{\sum_{a \in \Pi \cap \mathbb{Z}^n} x^a}{\prod_{i=1}^{d} (1 - x^{\vec{u}_i})}}$ where $\Pi = \{v + \alpha_1 \vec{u}_1 + \ldots + \alpha_d \vec{u}_d \mid 0 \leq \alpha_i \leq 1\}$

<u>Proof</u>: (inclusion-exclusion)
case $d = 2$



$$[\Pi] = [C] - [C + \vec{u}_1] - [C + \vec{u}_2] + [C + \vec{u}_1 + \vec{u}_2]$$

$$\Rightarrow S([\Pi]) = S([C]) - x^{\vec{u}_1} S([C]) - x^{\vec{u}_2} S([C]) + x^{\vec{u}_1 + \vec{u}_2} S([C])$$

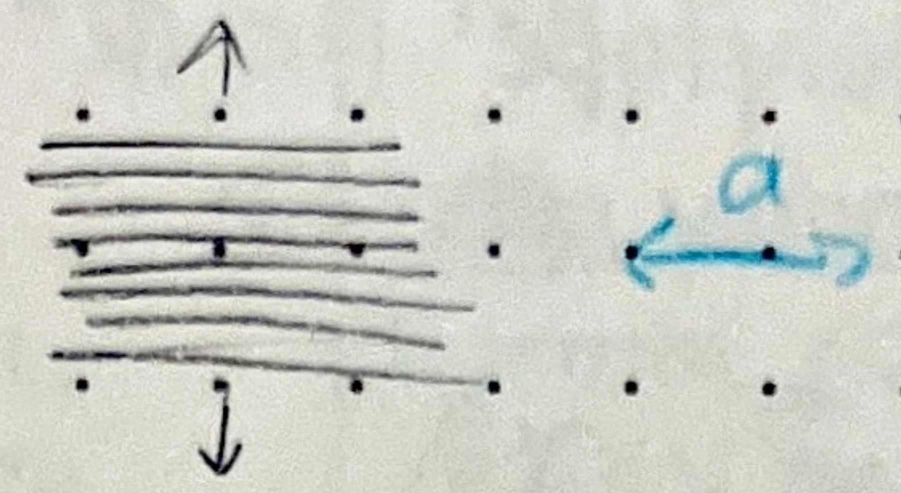
$$= (1 - x^{u_1})(1 - x^{u_2}) S([C])$$

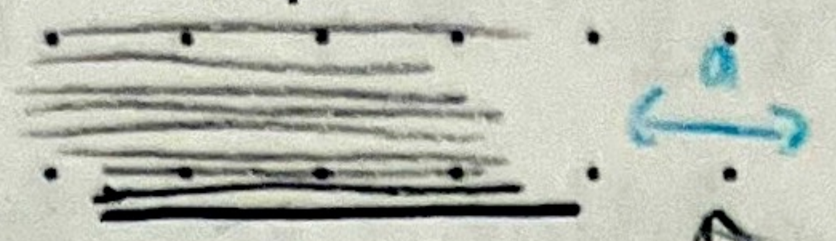
$\Rightarrow$ The identity of Cor (3)

**Def**: A rat. polyhedra $P$ is called <u>ruled</u> if $P$ is a union of parallel lines w/ rational coeff's

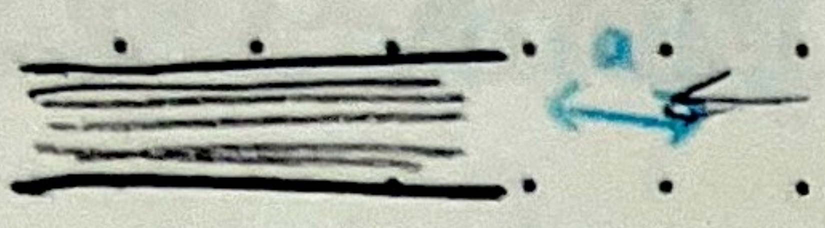
or, equivalently, if $\exists$ non-zero integer vector $a \in \mathbb{Z}^n$ s.t. $P + \vec{a} = P$.

**Ex.** $\mathbb{R}^2$ is ruled
Cover by parallel lines




$\mathbb{R} \times \mathbb{R}_{\geq 0}$ 


can shift by a

$\mathbb{R} \times [0,1]$ 


All ruled

But a pointed cone , polytope  not ruled.

A ruled polyhedron cannot have vertices.

**Cor.** (4): $\forall$ ruled rat. polyhedron $P$,
$$S([P]) = 0$$

**Proof**: By property (2) of thrm $S([P]) = S([P - \vec{a}]) = x^{\vec{a}} S([P])$.
$$\Rightarrow (1 - x^a) S([P]) = 0$$
$$\Rightarrow S([P]) = 0$$

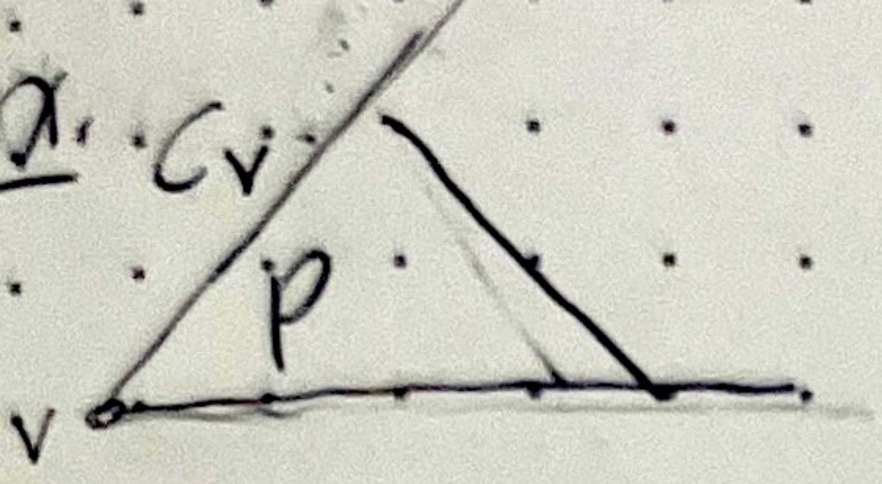
Key identity
$\mathbb{R}$ is a ruled polyhedron. $\Rightarrow$
$$S([\mathbb{R}]) = 0$$
$$\text{“} \sum_{i \in \mathbb{Z}} x^i \text{”}$$

So proving thrm would give id. we want via this corollary.

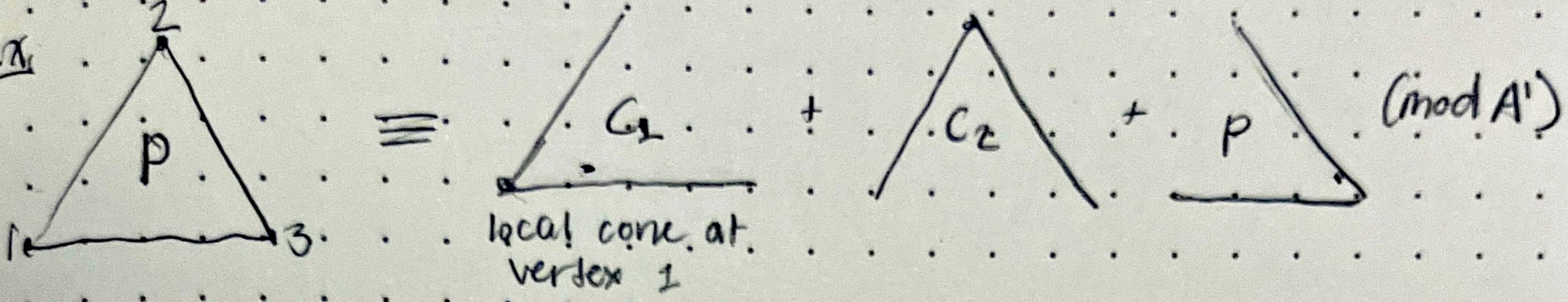
Let $A' \subset A$ be the lin. subspace spanned by $[P]$ for all ruled polyhedron $P$,

**Thrm2**: For any polyhedron,
$$[P] \equiv \sum_{v \text{ vertex of } P} [C_v] \mod A'$$
where $C_v$ is local cone of $P$ at vertex $v$.

Ex.

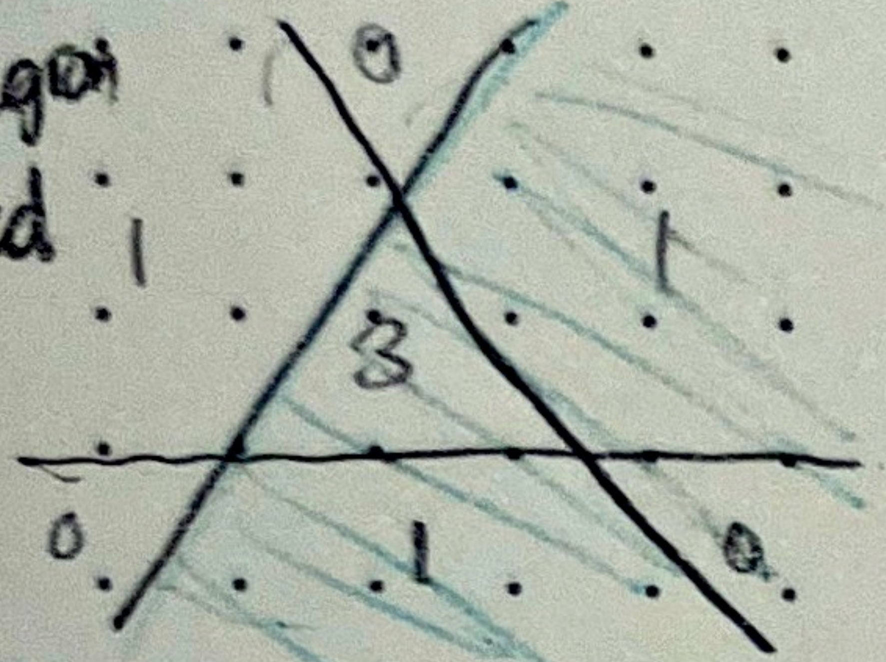



Ex.



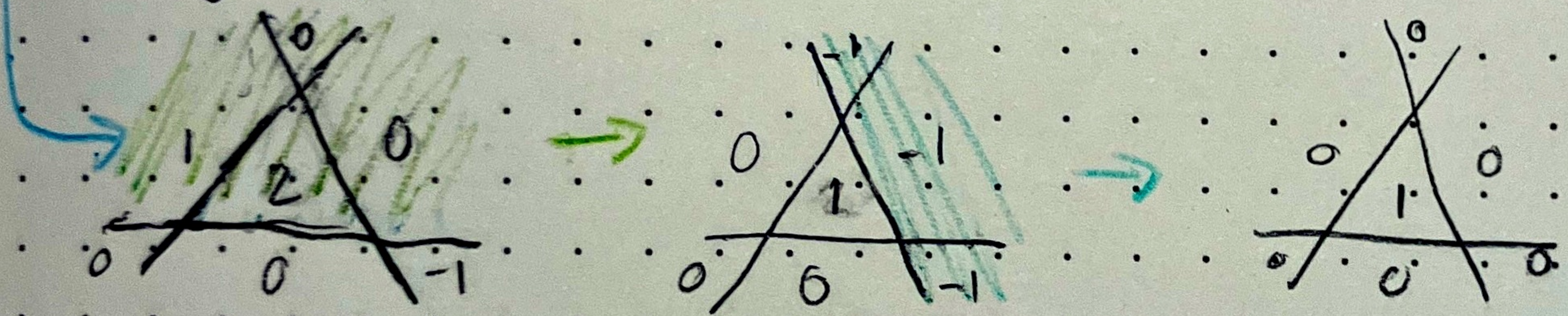

$[C_1] + [C_2] + [C_3]$.

Each region is counted




times.

Can subtract half space (which are ruled polyhedra) to get equivalent spaces.




Claim: Brion's formula follows from these two thrms.

Remains to prove these thrms.

Proof sketch: Use (*)

show

Lemma: You can express everything in terms of simple polyhedra

Harder part is showing there are no contradictions.
Can write same polyhedron as sum of cones in 2 different ways, and NTS they give the same thing.

Main idea If break down as cones in 2 ways, take common refinement to show they are the same.

More details in next lecture & some parts of proof on next PSET.