LECTURE 23 Wed 10/30

Arnold-Orlik-Solomon Alg. A_n

Anti-commutative generators $e_{ij} = e_{ji}$ $i \neq j \in [n]$

relations: $e_{ij}e_{ik} - e_{ij}e_{jk} + e_{ik}e_{jk} = 0$ $\forall\, i<j<k$

NBC basis: (w.r.t. lex order) $e_F := \prod_{(i,j) \text{ edge of } F}$ ← assume terms are in lex order.

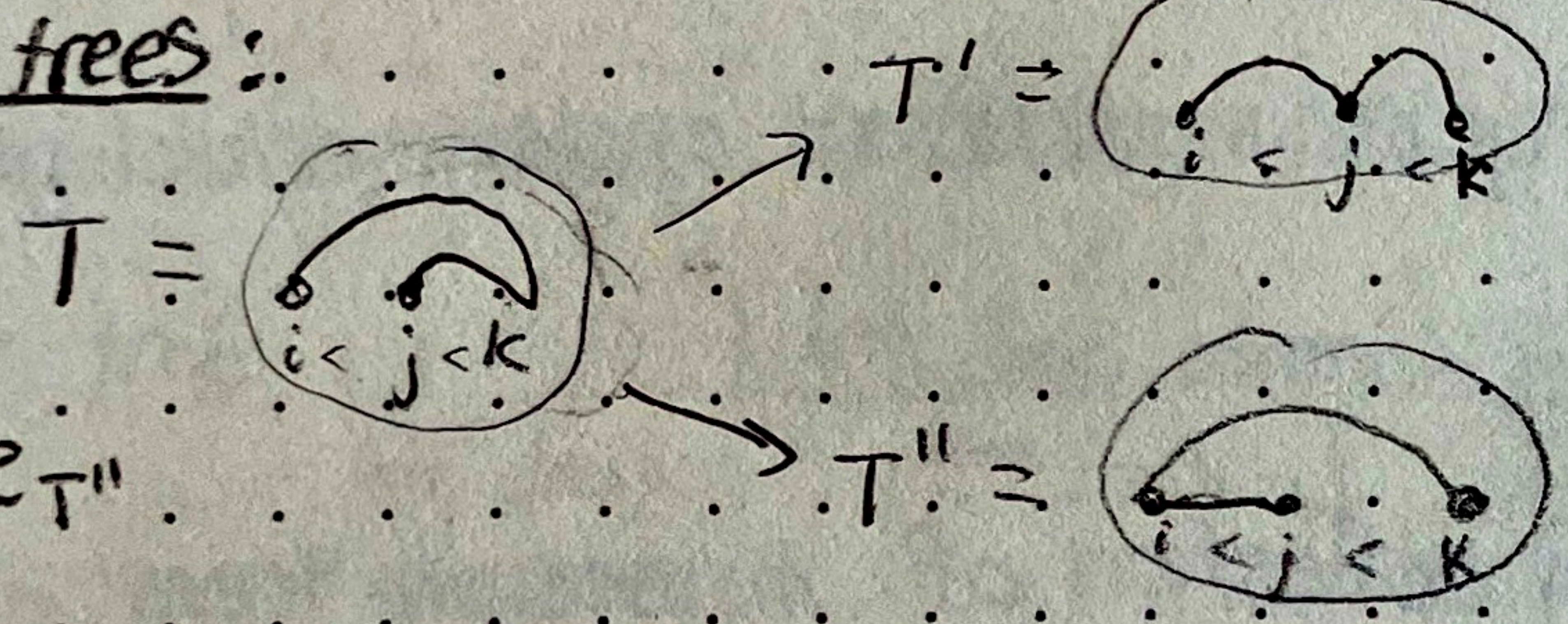
$\forall$ increasing forests $F \subset K_n$.

How to write any monomial in e_{ij}'s in the NBS-basis?

Lemma: F is increasing if it avoids the pattern.

To write elts. in terms of basis, play

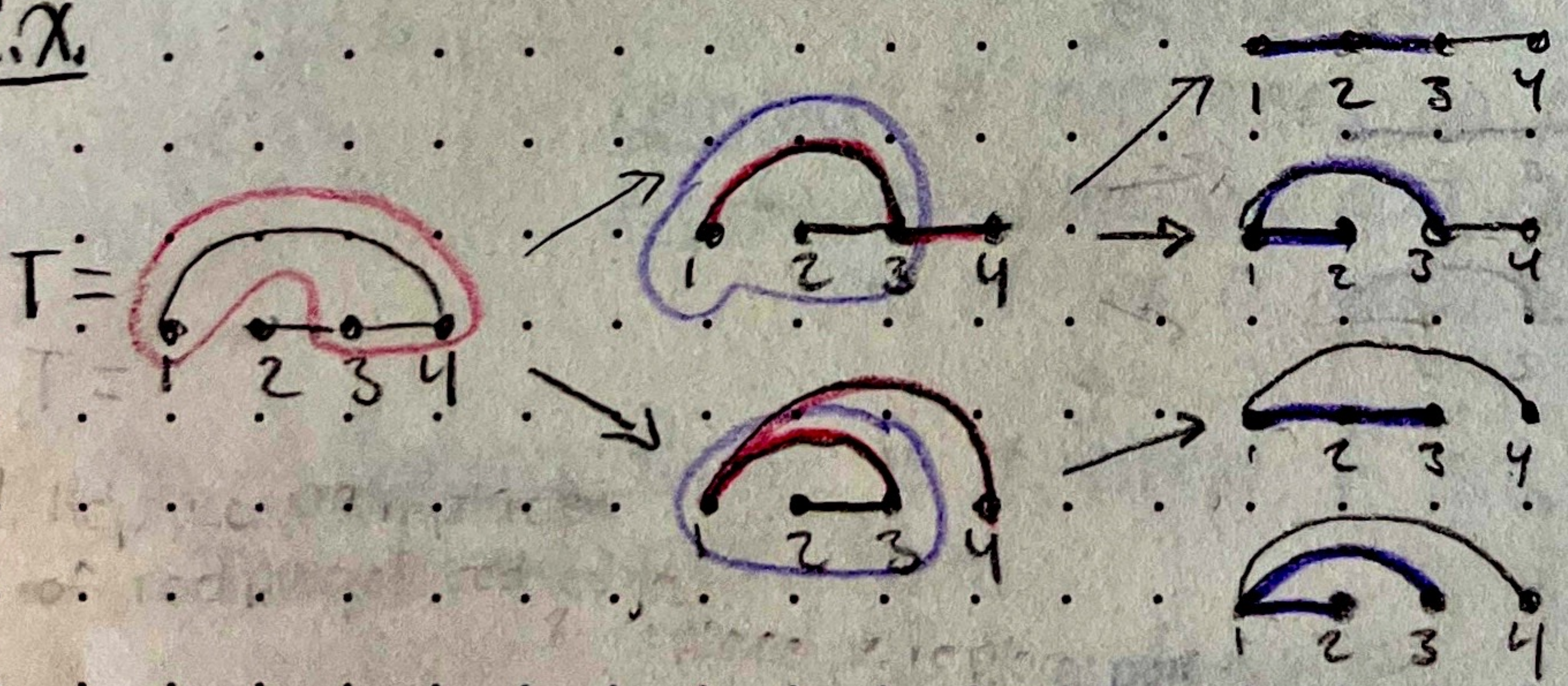
A game on trees:



$e_T = e_{T'} - e_{T''}$

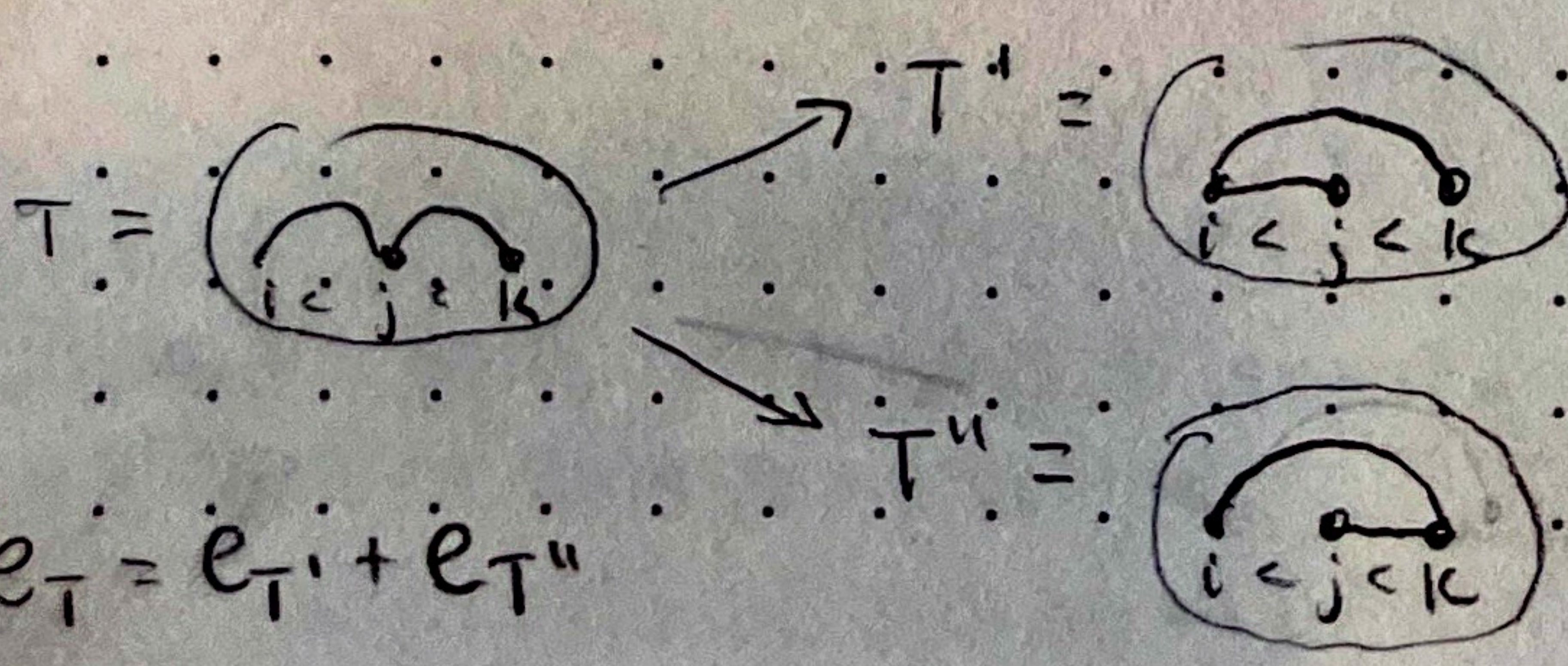
Game corresponds to replacing $e_{ik}e_{jk}$ with $e_{ij}e_{jk} - e_{ij}e_{ik}$

Ex.



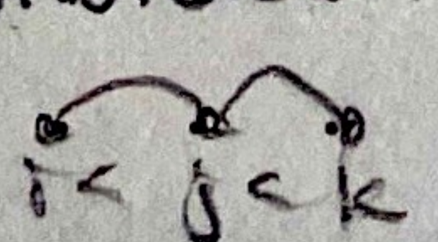
Another game on trees:

Now replace $e_{ij}e_{jk} = e_{ij}e_{ik} + e_{ik}e_{jk}$ $i<j<k$



$e_T = e_{T'} + e_{T''}$

A certain set of alternating trees (forbidden pattern



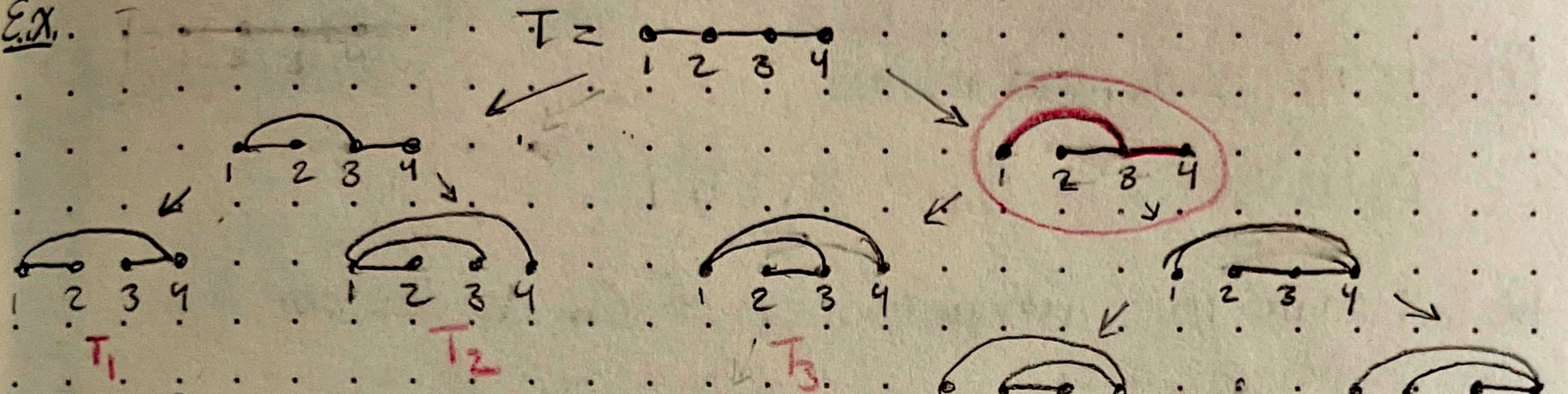
)

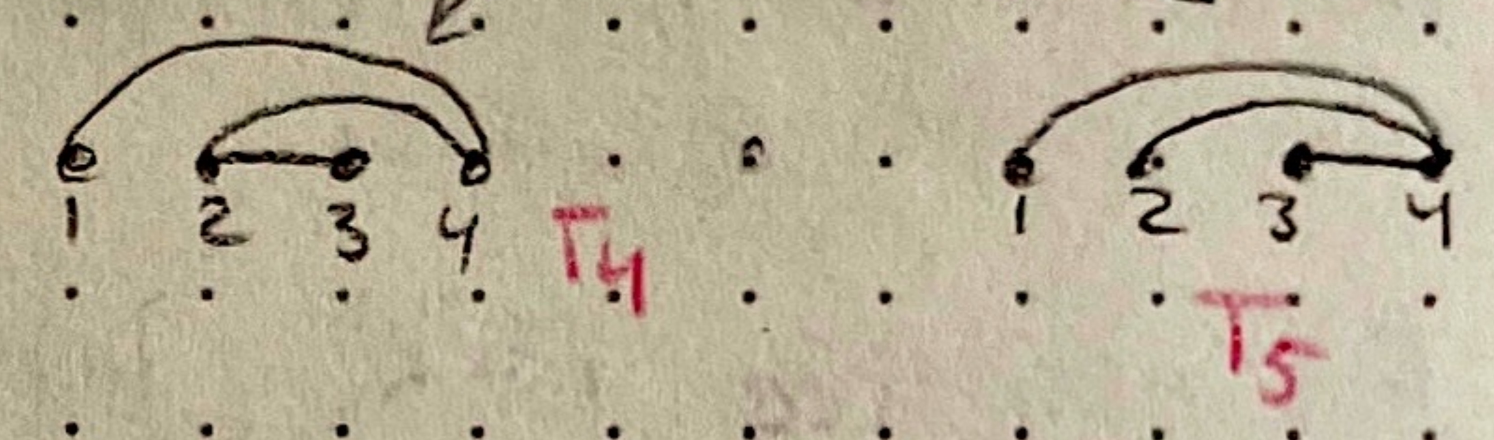
This game will end after finitely many steps. $\Rightarrow$

Prop: The set $\{e_F \mid F \text{ is an alternating forest}\}$ linearly spans A_n.

Q: Is this also a basis?

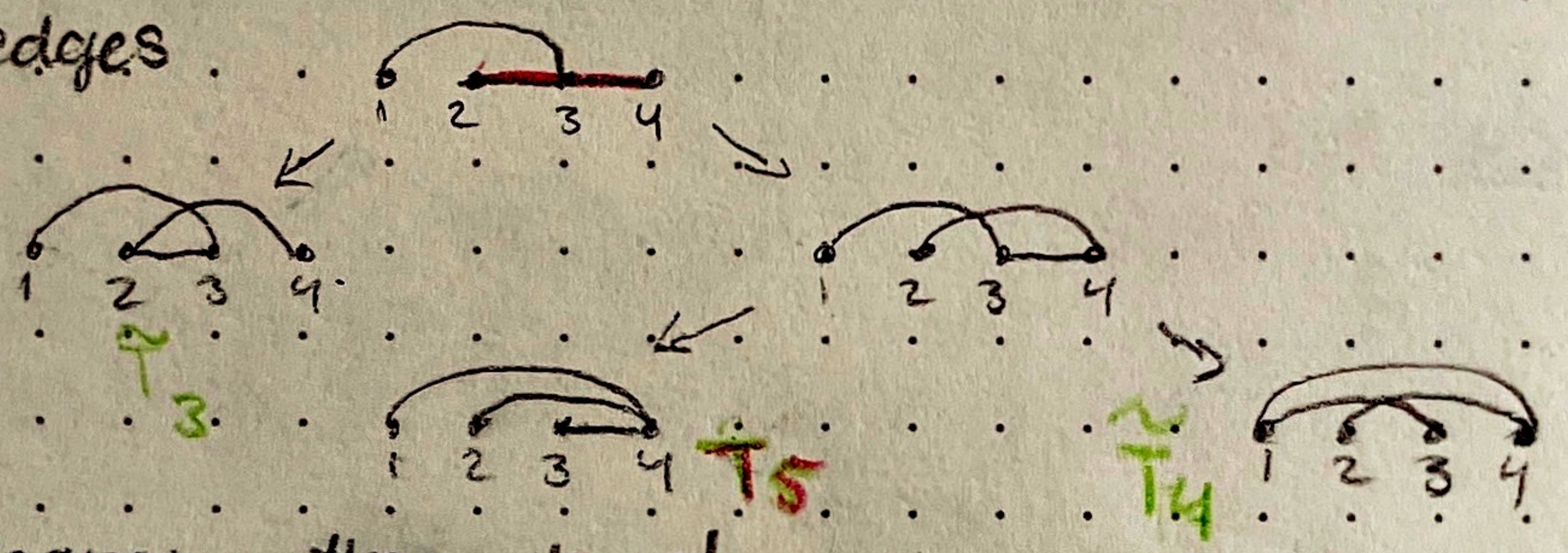
Ex. $T =$





$e_T = e_{T_1} + \cdots + e_{T_5}$

For circled tree, if instead we played game on edges



$e_T = e_{T_1} + e_{T_2} + e_{\tilde{T}_3} + e_{\tilde{T}_4}$
$+ e_{T_5}$

This is a different expansion than above!
$\Rightarrow$ Alternating forests span, but are not a basis

Prop:

$$e_{\underset{1\,2\,3\,\cdots\,n}{\bullet\!-\!\bullet\!-\!\bullet\cdots\bullet}} = \sum_{T^{NCA} \subset K_n} e_{T^{NCA}} = \sum_{T^{NNA} \subset K_n} e_{T^{NNA}}$$

non-crossing alternating trees — non-nesting alternating trees

$\#$ terms $= \frac{1}{n+1}\binom{2n}{n} =$ Catalan $\#$ C_n

Can we also see geometrically why these #'s are the same?

__Root polytopes__

$\vec{e}_1, \ldots, \vec{e}_n$ std coord vectors in $\mathbb{R}^n$

$R_n = \text{conv}(\vec{0}, \vec{e}_i - e_j \mid 1 \le i < j \le n)$

Ex. $n=3$ $\vec{e}_1-\vec{e}_2$ $\vec{e}_1-\vec{e}_3$ (These are positive roots of type A rt. system)

$R_3 =$  $\vec{e}_2-\vec{e}_3$

Let $Vol(R_n)$ = the volume of its projection to $\mathbb{R}^{n-1}$ under the map $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$

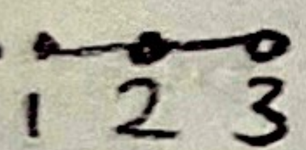
Thrm: The normalized volume

$$(n-1)!\,Vol(R_n) = C_n = \frac{1}{n+1}\binom{2n}{n}$$

We will relate this interperetation of C_n to the one from our game on trees

For any tree $T \subset K_n$, let $R_T = R_n \cap \left(\begin{array}{c}\text{the cone generated by vectors} \\ \vec{e}_i-\vec{e}_j \text{ for edge } (i,j) \text{ of } T\end{array}\right)$

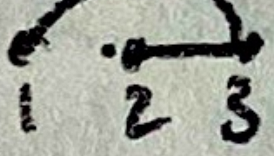
Ex. $n=3$. $\vec{e}_1-\vec{e}_2$

R 

$= R_3 =$  $\vec{e}_2-\vec{e}_3$ $\cap$  $=$ 

e_1-e_2 e_1-e_3 e_1-e_3

R 

$=$  R 

$=$  e_2-e_3

Lemma: If $T \begin{smallmatrix}\nearrow T' \\ \searrow T''\end{smallmatrix}$ (in the second game) then

$R_T = R_{T'} \cup R_{T''}$ and $R_{T'} \cap R_{T''}$ = the common facet of $R_{T'}$ & $R_{T''}$

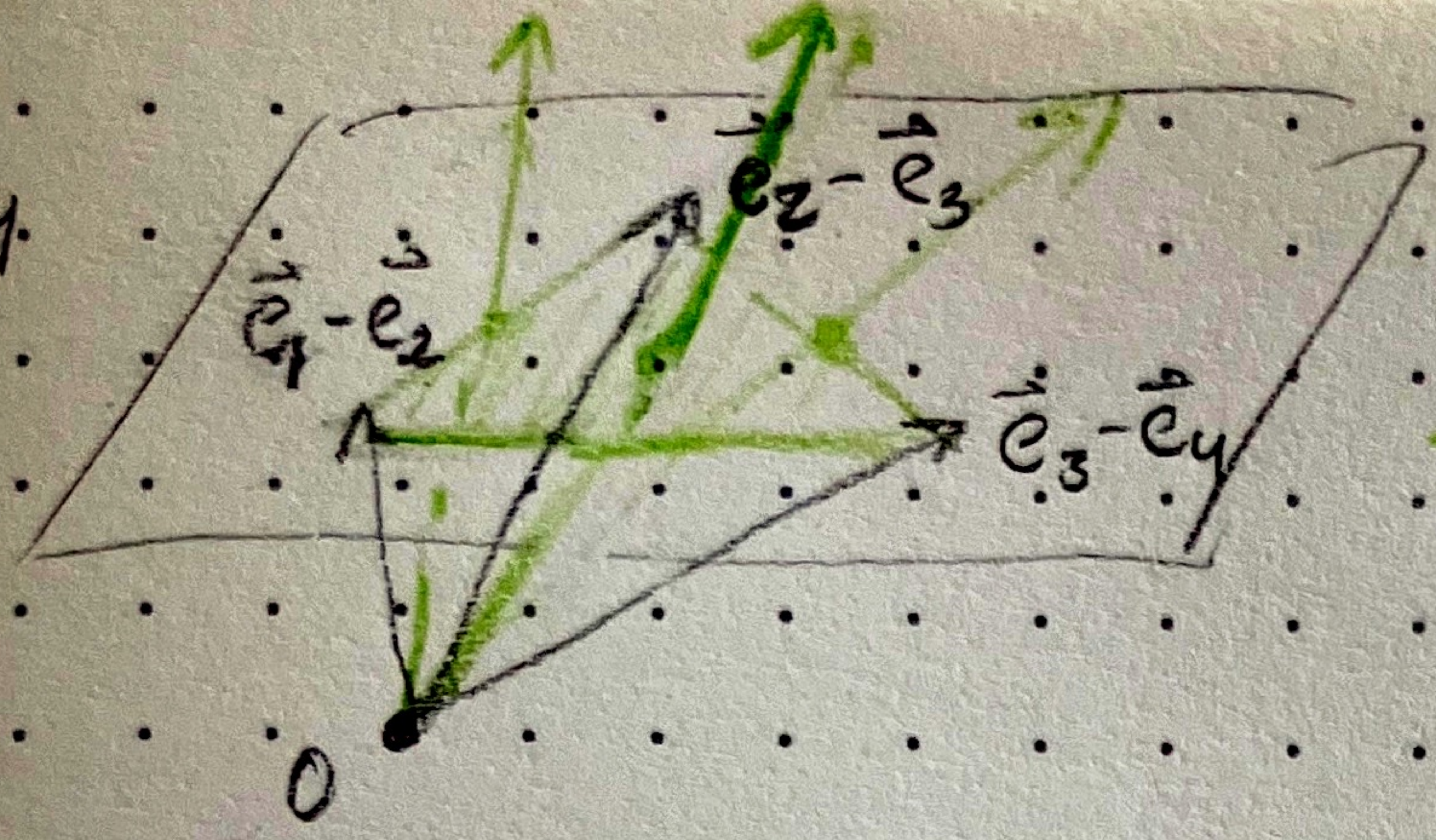
$Vol(R_T) = Vol(R_{T'}) + Vol(R_{T''})$

Lemma: If T is an alternating tree, then R_T is unit simplex (i.e simplex of volume 1)

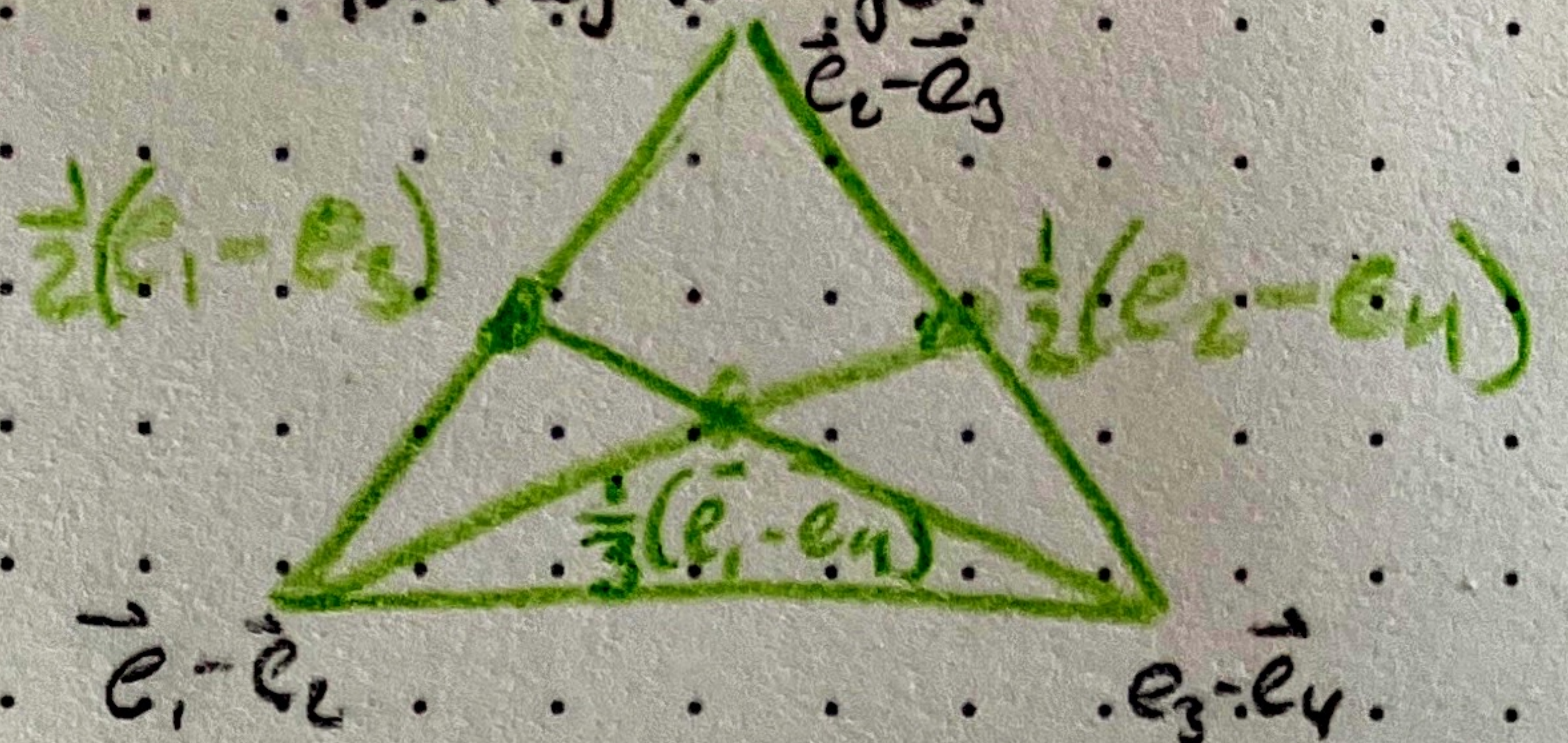
Cor: # endpts in the game $= (n-1)!\,Vol(R_T) = C_n$

Ex R_4.

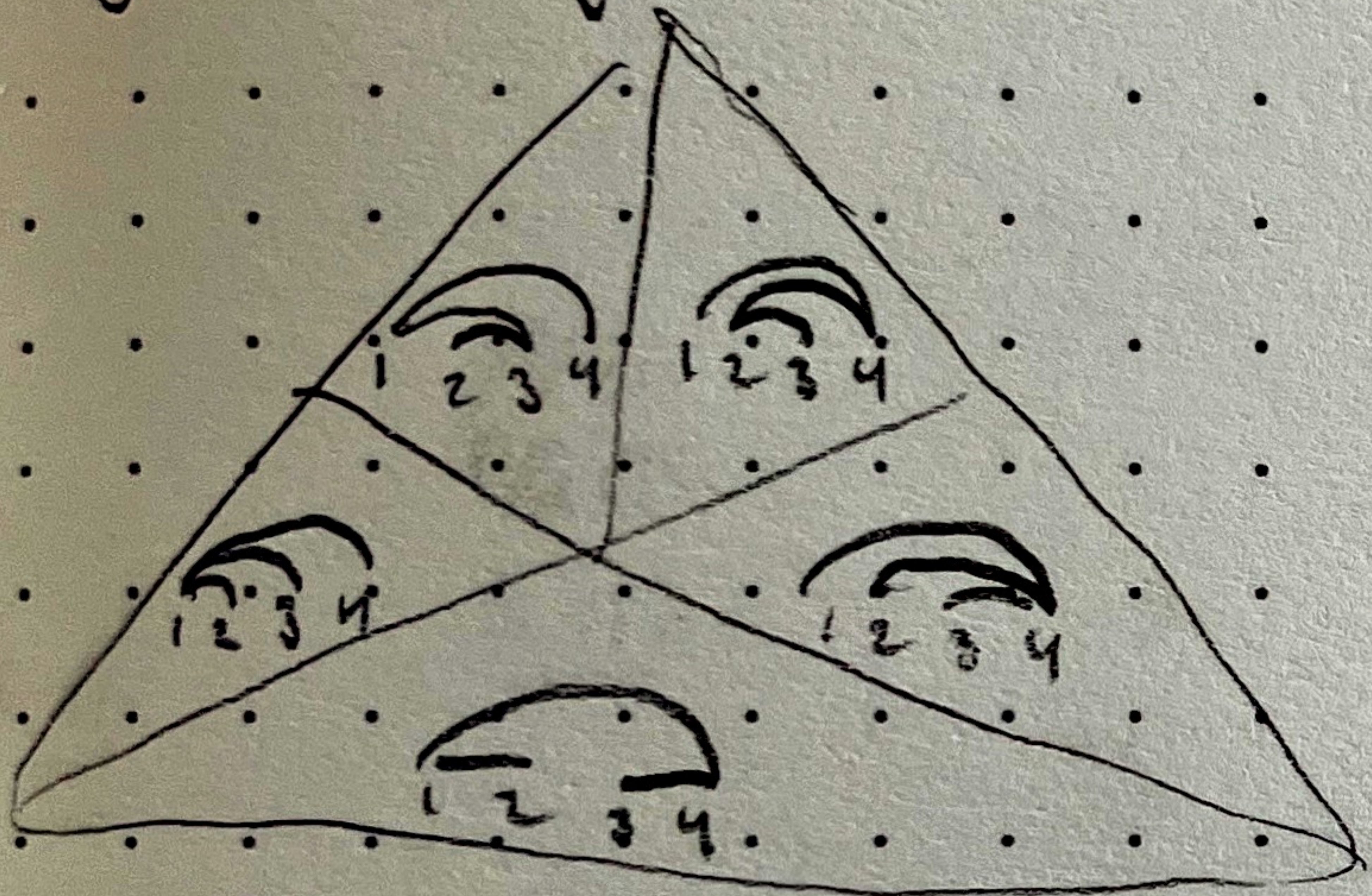
$\mathbb{R}^3$



Just looking at green triangle on plane, we get

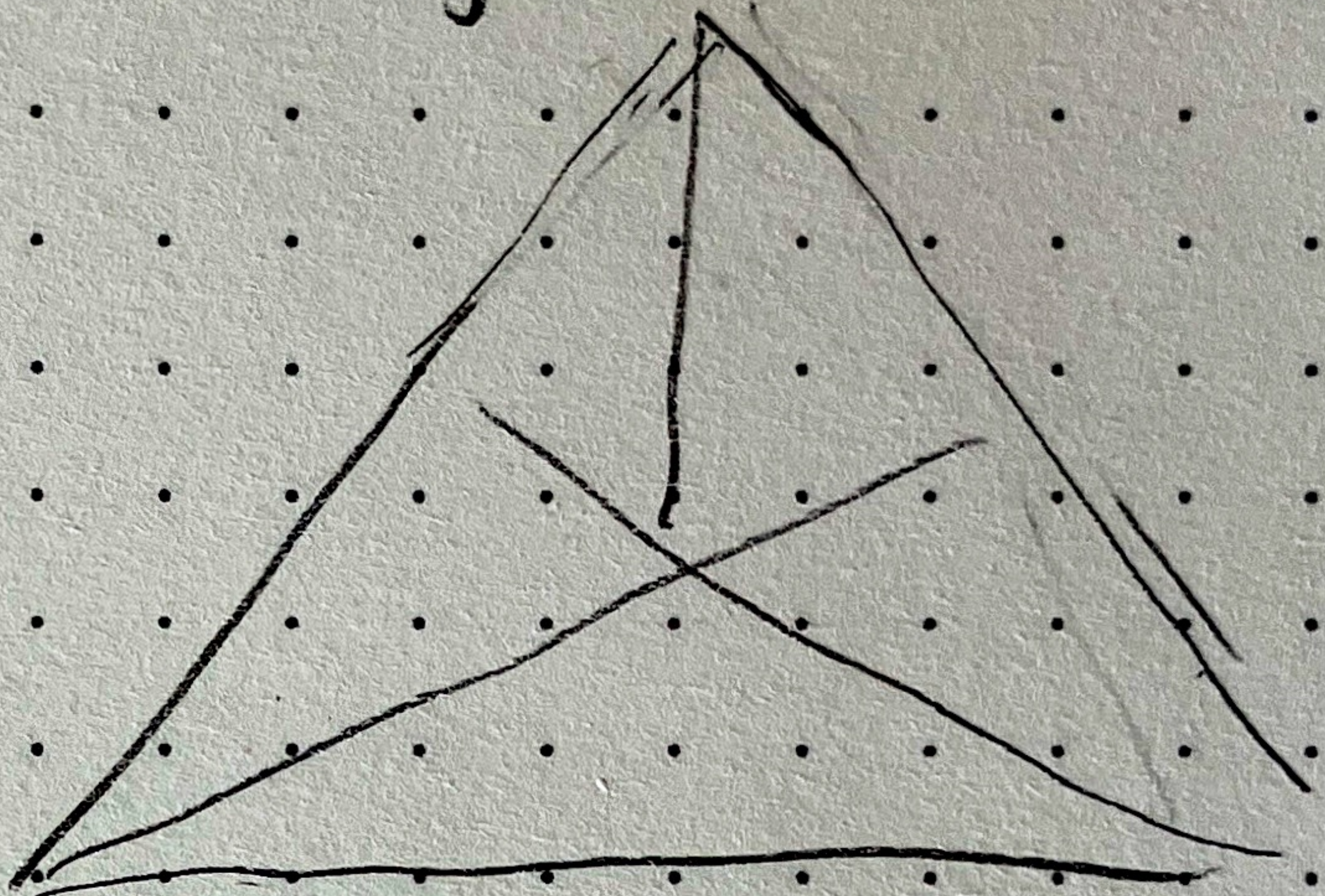


One way to play the game gives triangulation



non-crossing

Another way to play gives



non-nesting.

Somehow if you look closely enough at it it's related to the associahedron.

There might be something on this in the next pset.