LECTURE 13 Fri 10/4

Last Time: Ehrhat & chromatic polys.

$P \subset \mathbb{R}^d$ a (closed) lattice polytope of dim. d.

$P^\circ := P \setminus \partial P$ open polytope

$i_P(t) := \#(tP \cap \mathbb{Z}^d)$ for $t \in \mathbb{Z}_{\geq 0}$.

$i_{P^\circ}(t) := \#(tP^\circ \cap \mathbb{Z}^d)$ for $t \in \mathbb{Z}_{>0}$ $\leftarrow t \neq 0$

Thrm: (1) $i_P(t)$ & $i_{P^\circ}(t)$ are polynomials in t

(2) $i_P(t) = \mathrm{Vol}(P)t^d + c_1 t^{d-1} + \dots + c_{d-1}t + 1$
where $c_1, \dots, c_{d-1} \in \mathbb{Q}$ (in fact in $\frac{1}{d!}\mathbb{Z}$)

(3) reciprocity: $\boxed{i_P(-t) = (-1)^d i_{P^\circ}(t)}$ later we'll develope theory to prove this in more general setting

(3') In particular, at $t=0$, $i_P(0)=1$, $i_{P^\circ}(0) = (-1)^d$

For a graph $G=(V,E)$, $V=[n]$

Chromatic polynomial $\chi_G(t)$ is polynomial in t s.t.

$\chi_G(q) = \#\{q\text{-colorings of } G\} := \{c: V \to \{1,\dots,q\} \text{ s.t. } c(i) \neq c(j) \text{ if } \{i,j\} \in E\}$

Stanley's Thrm: $\chi_G(-1) = (-1)^{|V|} \#\{\text{acyclic orientations of } G\}$

Q: What about $\chi_G(-2)$, $\chi_G(-3)$, etc?

Prop: Coeffs of $\chi_G(q)$ are integer & alternating

$\chi_G = q^n - c_1 q^{n-1} + c_2 q^{n-2} - \dots \pm c_{n-1} q$

where $c_1, \dots, c_{n-1} \in \mathbb{Z}_{>0}$ (Assuming G is connected graph)

Proof sketch: Use induction and deletion contraction

$\underset{\uparrow n \text{ vertices}}{\chi_G(q) = \chi_{G \setminus e}(q)} - \underset{\uparrow n-1 \text{ vertices}}{\chi_{G/e}(q)}$ use this to induct

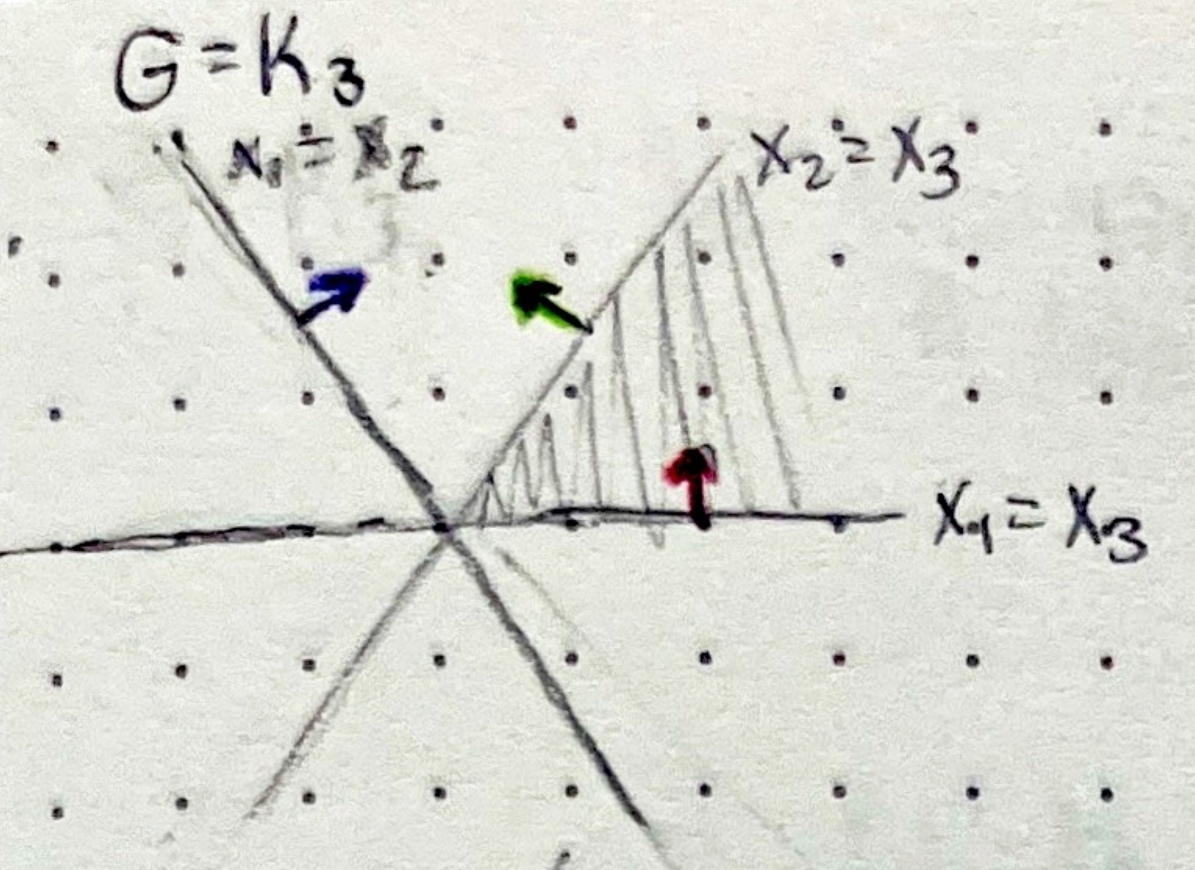
$\Rightarrow (-1)^n \chi_G(-k)$ is a positive integer for $k \in \mathbb{Z}_{>0}$.

$\mathcal{A}_G$ graphical hyperplane arrangement $\in \mathbb{R}^n$

$\mathcal{A}_G = \{H_e \mid e \in E\}$ where $H_{\{i,j\}} = \{x_i - x_j = 0\}$

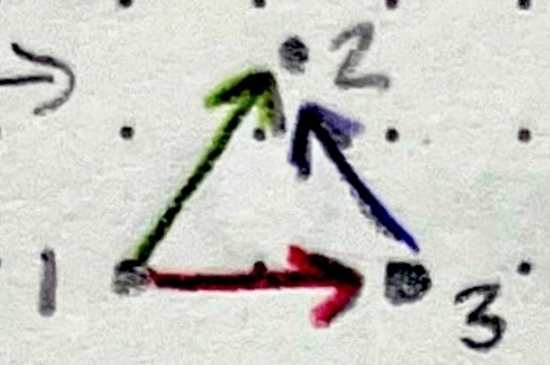
regions of $\mathcal{A}_G \longleftrightarrow$ vertices of $Z_G \longleftrightarrow$ acyclic orientations of G.

Ex.



$\{x_1 > x_2,\ x_1 > x_3,\ x_2 < x_3\}$

Acyclic orientation:



$$\chi_G(q) = \#\Big(\big(\mathbb{R}^n \setminus \bigcup_{e \in H} H_e\big) \cap \{(x_1,\dots,x_n) \in \mathbb{Z}^n \mid 0 < x_i < q+1\}\Big)$$

$$\chi_G(q) = \sum_{\mathcal{O} \text{ acyclic orient. of } G} \#\,(q-1)\cdot\left\{(x_1,\dots,x_n) \,\middle|\, \begin{array}{l} 0 < x_i < 1 \\ x_i > x_j \text{ for any directed edge } i \xrightarrow{\mathcal{O}} j \end{array}\right\}$$

Certain open polytope

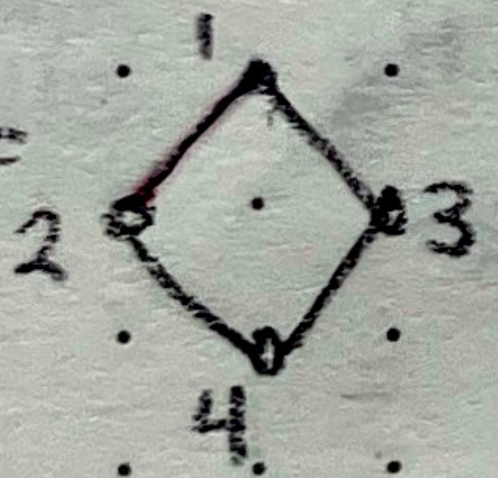
Def: The order polytope of (the poset associated w/) $\mathcal{O}$ is

$$P_{\mathcal{O}} := \left\{(x_1,\dots,x_n) \in \mathbb{R}^n \,\middle|\, \begin{array}{l} 0 \le x_i \le 1 \\ x_i \le x_j \ \forall\ i \xrightarrow{\mathcal{O}} j \end{array}\right\}$$

$$P^{\circ}_{\mathcal{O}} = P_{\mathcal{O}} \setminus \partial P_{\mathcal{O}}$$

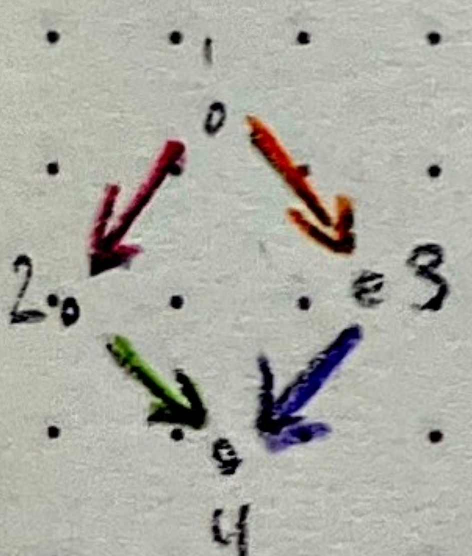
Lemma: $P_{\mathcal{O}}$ is a lattice polytope.
Its vertices are all 01-vectors $\in P_{\mathcal{O}}$

Ex. $G =$



$P_{\mathcal{O}} = \{x_1 \ge x_2,\ x_1 \ge x_3,\ x_2 \ge x_4,\ x_3 \ge x_4\}$

$= \mathrm{conv}\{(0000), (1,0,0,0), (1,1,0,0), (1,0,1,0), (1,1,1,0), (1,1,1,1)\}$



To get vertex, can have no degrees of freedom
$\Rightarrow$ No blocks between 0 & 1
$\Rightarrow$ All vertices are 01
(Can also argue why all 01 pts in $P_{\mathcal{O}}$ are vertices)

$$\chi_G(q) = \sum_{\mathcal{O} \text{ acyclic orient.}} i_{P^{\circ}_{\mathcal{O}}}(q+1) \overset{\text{reciprocity}}{=} (-1)^n \sum_{\mathcal{O}} i_{P_{\mathcal{O}}}(-q-1)$$

Thrm (Stanley): $\forall q \in \mathbb{Z}_{>0}$

$$\chi_G(-q) = (-1)^n \cdot \sum_{\mathcal{O}} i_{P_{\mathcal{O}}}(q-1)$$

$$= (-1)^n\, \#\left\{(\mathcal{O}, (c_1,\dots,c_n)) \,\middle|\, \begin{array}{l} \mathcal{O} \text{ acyclic} \\ c \in \{0,1,\dots,q-1\} \\ c_i \ge c_j \ \forall\ i \xrightarrow{\mathcal{O}} j \end{array}\right\}$$

$$= (-1)^n \cdot \#\left\{(c, \mathcal{O}_0, \dots, \mathcal{O}_{q-1}) \,\middle|\, \begin{array}{l} c: V \to \{1,\dots,q-1\} \text{ any function} \\ \mathcal{O}_i \text{ is acyclic orient. of the induced} \\ \text{subgraph on the set } \{v \in V \mid c(v) = i\} \end{array}\right\}$$

$\mathcal{A} = \{H_1, \ldots, H_N\}$ any finite arrangement of hyperplanes in $\mathbb{F}^n$ for some field $\mathbb{F}$.

Def: The <u>intersection poset</u> of $\mathcal{A}$

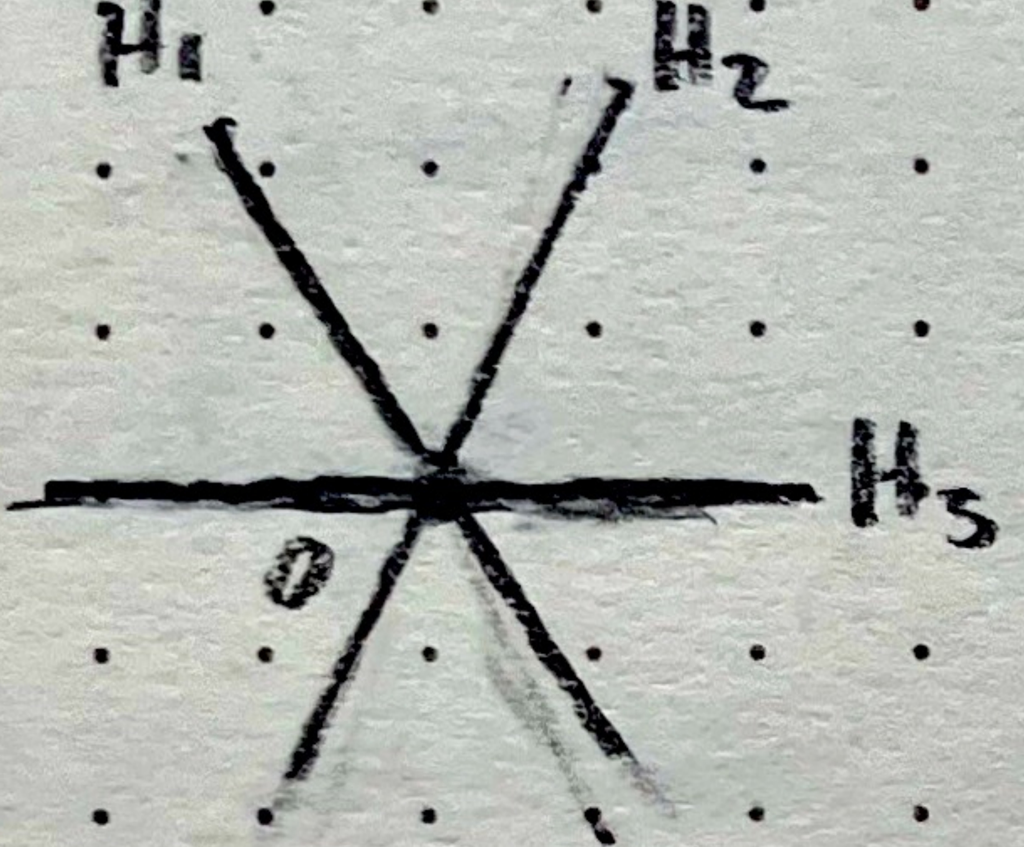
$L_{\mathcal{A}}$ = all non-empty intersections of H_i's (including the empty intersection) ordered by reverse inclusion

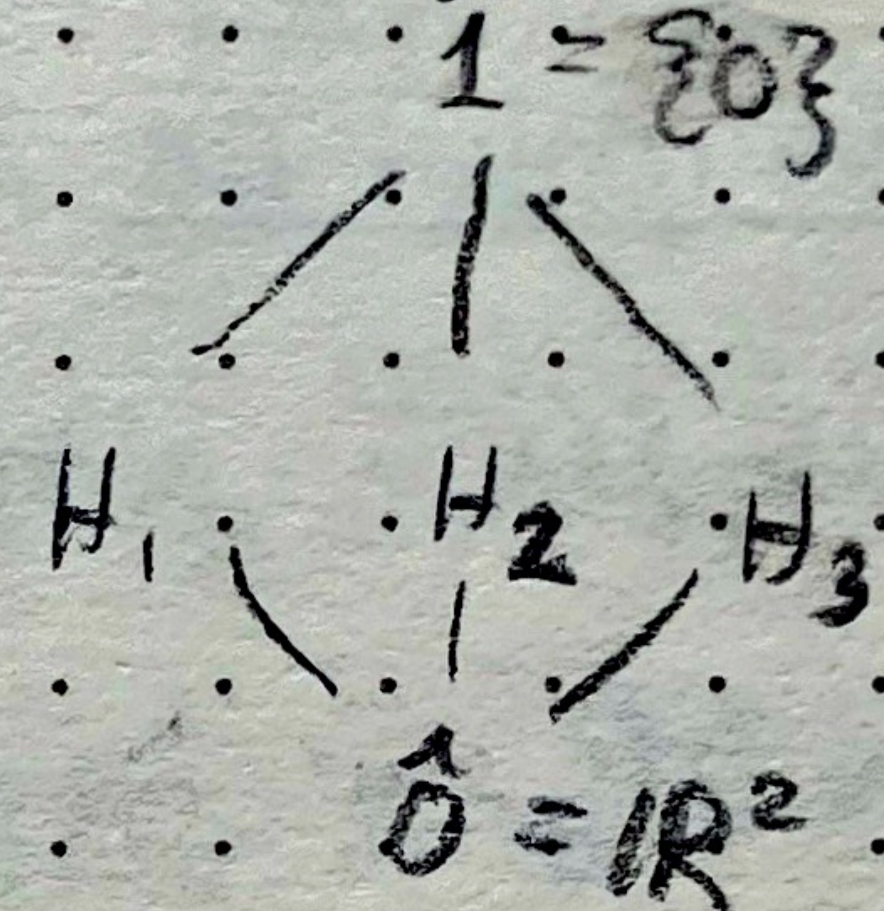
elts $H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_k} \neq \emptyset$

BUT we allow case $k=0 \rightsquigarrow$ intersection of NO hyperplanes

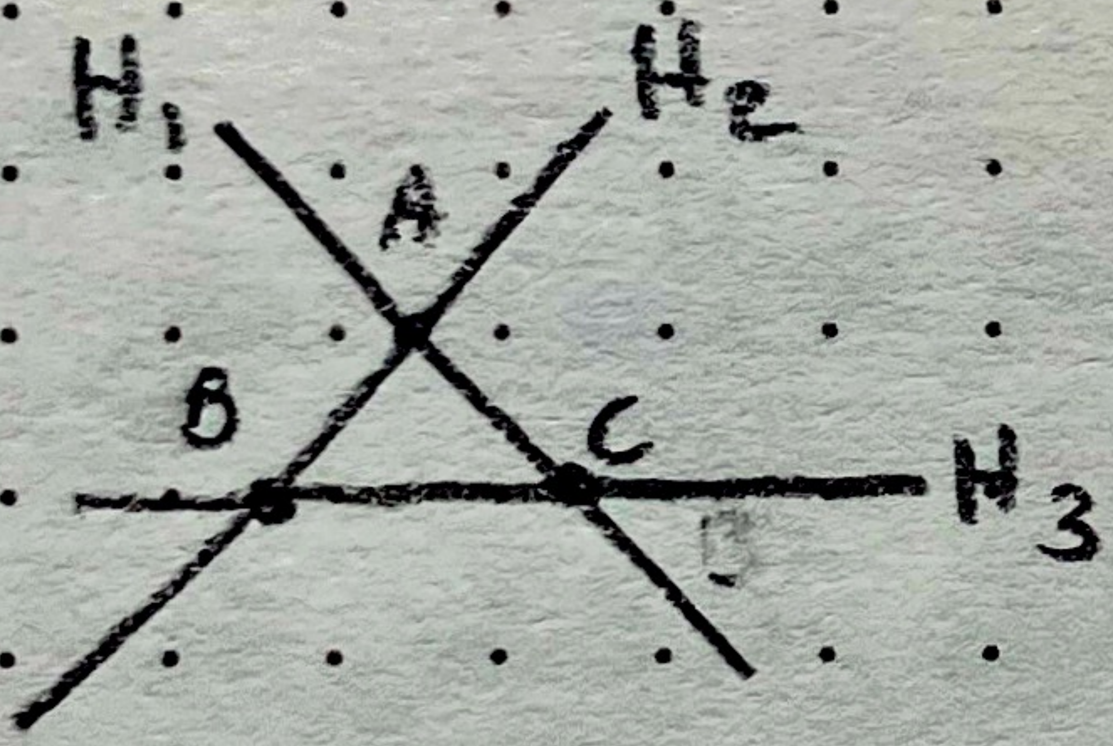
$\bigcap \emptyset = \mathbb{F}^n$

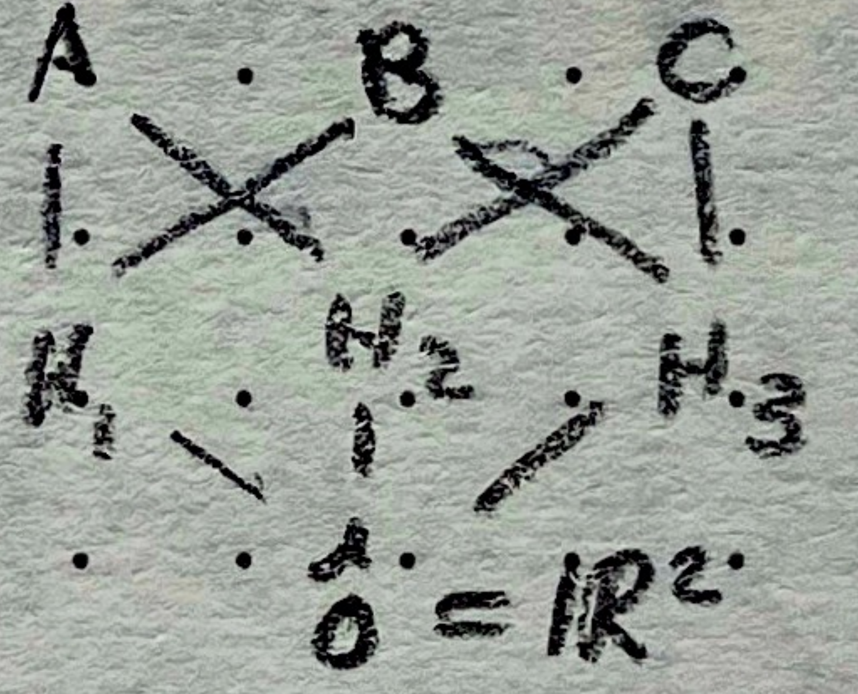
Ex. 1)

$\mathcal{A} =$ 

$L_{\mathcal{A}} =$ 

Ex. 2)

$\mathcal{A} =$ 

$L_{\mathcal{A}} =$ 

No unique maximal elt.

Alway unique minimal elt.

$\Rightarrow$ Any two elts have a join.

If central hyperplane arrangement (as in Ex. 1) then any two elts in poset have a meet.