LECTURE 7 Wed 9/18

Zonotopes $\xleftrightarrow{\text{dual to}}$ central hyperplane arrangements

regular zonotopal tilings $\longleftrightarrow$ affine hyperplane arrangements

A vector configuration $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^d$

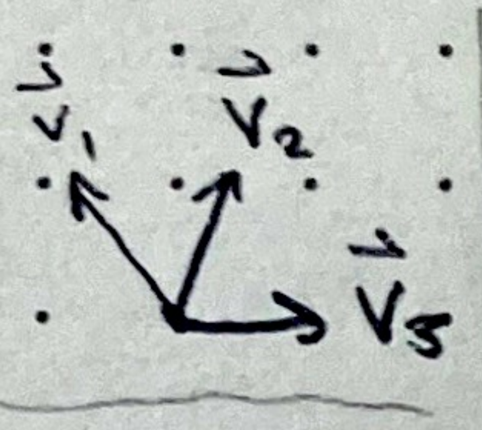
Assume v_i's span $\mathbb{R}^d$

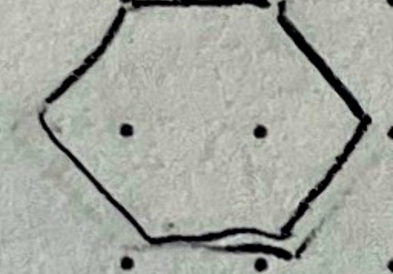
Zonotope. $Z = Z(\vec{v}_1, \dots, \vec{v}_N) = \sum_i [0, \vec{v}_i]$.

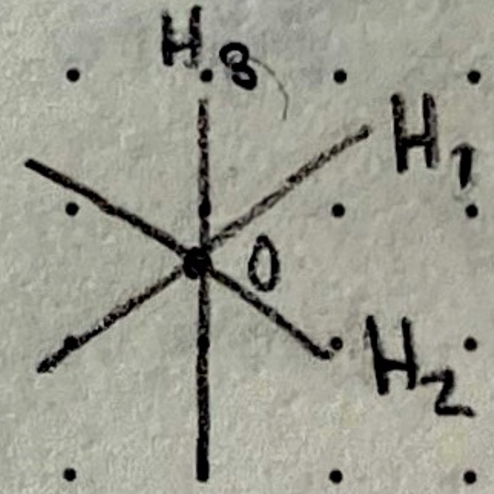
central hyp. arr. $A = \{H_1, \dots, H_N\}$, $H_i = \{\vec{x} \in \mathbb{R}^d \mid (\vec{v}_i, \vec{x}) = 0\}$.

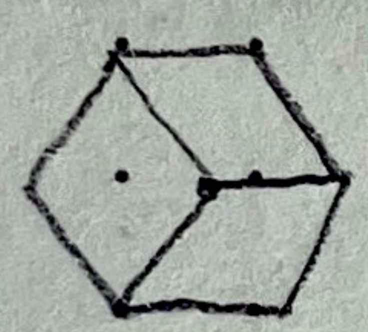
affine hyp. arr. $A' = \{H_1', \dots, H_N'\}$, $H_i' = \{\vec{x} \in \mathbb{R}^d \mid (\vec{v}_i, \vec{x}) = h\}$

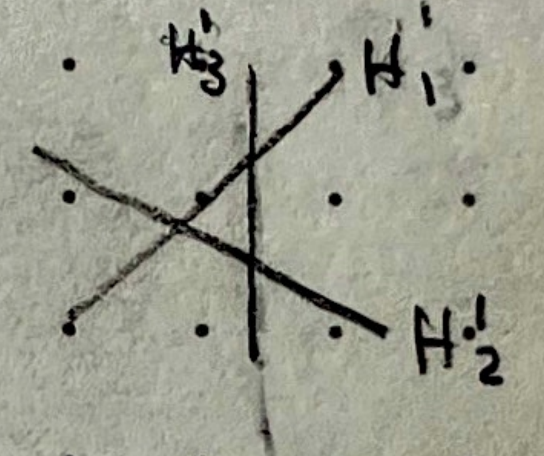
(here we are identifying $\mathbb{R}^d$ w/ $(\mathbb{R}^d)^*$ — originally H_i is actually the map)

Ex. 

$Z =$ 

$A =$ 

a tiling 

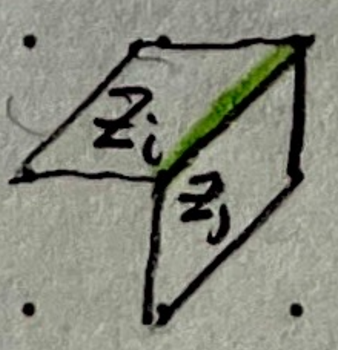
$A' =$ 

Def: A zonotopal tiling τ of Z is a collection of zonotopes Z_i (called tiles) s.t.

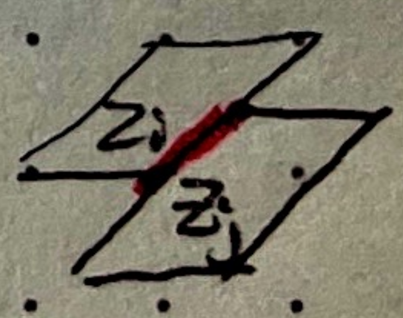
(0) $\dim Z_i = d$

(1) $Z = \bigcup Z_i$

(2) $Z_i \cap Z_j$ is the common face of Z_i & Z_j



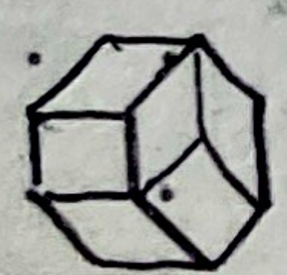
✓ propper intersection



NOT propper ✗ intersection

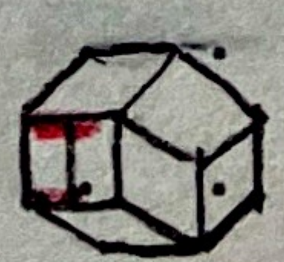
(3) Each Z_i is a parallel translation of $Z(\vec{v}_i \mid i \in B)$ for some $B \subseteq [N]$

A tiling is fine if it cannot be refined to a finer tiling.



fine tiling

Tiling, but not fine



NOT a tiling (fails (2) & (3))

Matroidal Terminology

Def: $B \subseteq [N]$ is a <u>base</u> if $\{\vec{v}_i, i \in B\}$ is a lin. basis of $\mathbb{R}^d$

$I \subseteq [N]$ is an <u>independent set</u> if $\{\vec{v}_i, i \in I\}$ are lin. indep.

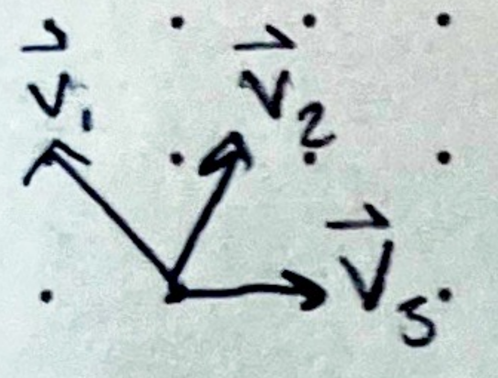
<u>Thm</u>: For a zonotopal tiling,

(1) # (top-dim'l) tiles in τ = # basis

Moreover, $\forall$ base B, $\exists!$ tile which is a parallel translation of the parallelotope $Z(\vec{v}_i \mid i \in B)$

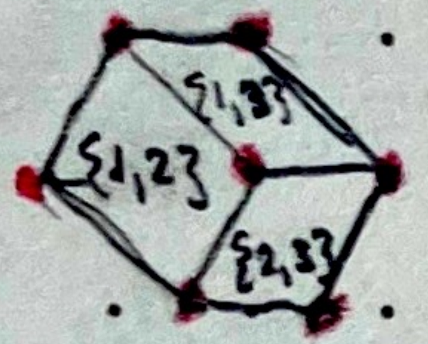
(2) # vertices in τ = # independent sets

<u>Ex.</u>



3 bases $\{1,2\}$, $\{2,3\}$, $\{1,3\}$

7 indep. sets ———— " ———— $+ \{1\}, \{2\}, \{3\}, \emptyset$



7 vertices

counter <u>Ex.</u> $d=1$, $\vec{v}_1 = (1)$, $\vec{v}_2 = 2$.

$Z =$

2 tiles, 3 vertices

3 tiles, 4 vertices BAD

Any easy fix: Require that all v_i & v_j not parallel for $i \neq j$.
(but unsatisfying)

Instead, modify def of zonotopal tilings to work for parallel vectors

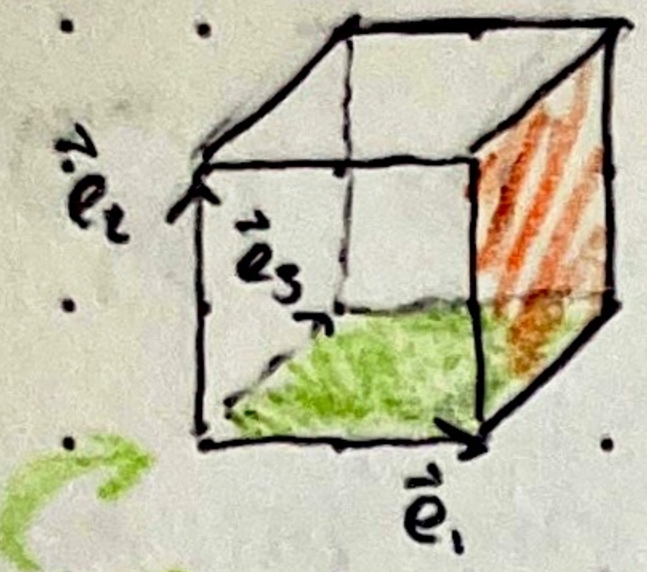
std N-dim cube $\square_N := [0,1]^N \subset \mathbb{R}^N$

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^d$$
$$e_i \longmapsto \vec{v}_i \quad \text{for } i = 1, \dots, N$$

Then $Z = \pi(\square_N)$

$\square_N$ has 3^N faces (not including the empty face)

Faces of $\square_N$: $\square_{A,B} := \sum_{i \in A} \vec{e}_i + \sum_{j \in B} [0, \vec{e}_j]$ for $A, B \subset N$ disjoint

E.x. 

$\square_{\{1\},\{2,3\}}$

$\square_{\emptyset,\{1,3\}}$

B gives tile shape
A tells us how it's translated

In our zonotope, have <u>labelled tiles</u>

$Z_{A,B} := \pi(\square_{A,B})$ (labelled by A,B)

<u>Note</u>: Could have 2 faces project to same tile, but now have distinct labellings

<u>**New Def**</u>: A <u>zonotopal tiling</u> $\mathcal{T}$ of Z is a collection of labelled tiles $Z_{A,B}$ s.t.

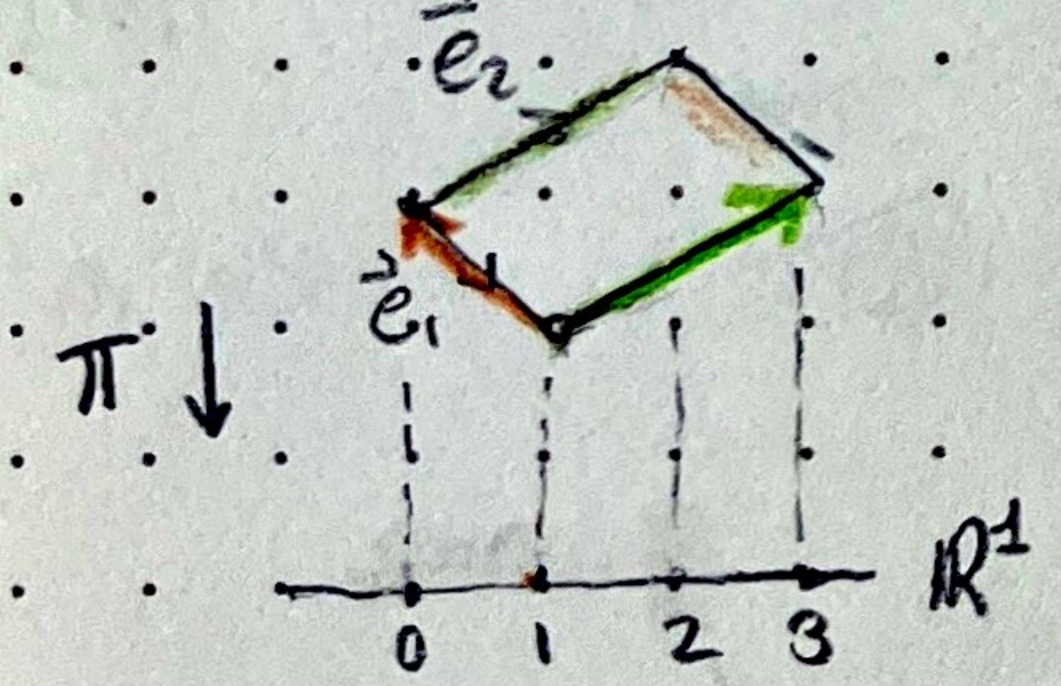
(0) $\dim Z_{A,B} = d$

(1) $Z = \bigcup Z_{A,B}$

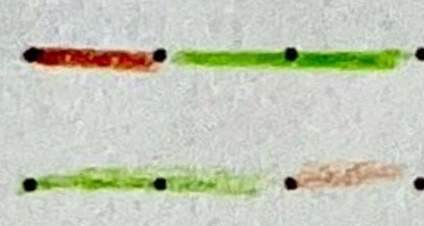
(2) $\forall$ pairs of tiles, $Z_{A,B} \cap Z_{A',B'}$ is the common faces of $Z_{A,B}$ & $Z_{A',B'}$ (or $\emptyset$) and it is equal to $\pi(\square_{A,B} \cap \square_{A',B'})$

is <u>fine</u> if $\forall$ tiles $Z_{A,B}$, B is a base

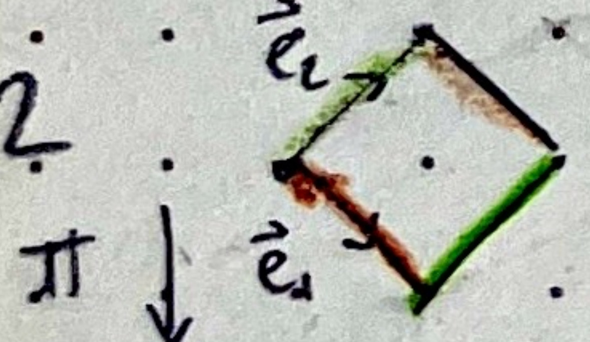
<u>Ex. 1</u> $\vec V_1 = (1)$, $\vec V_2 = (2)$



NOT a tiling



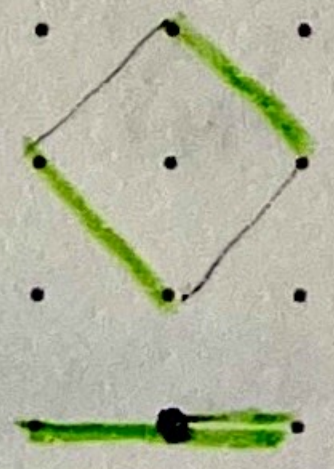
$\rightarrow$ 2 fine zonotopal tilings.

Ex. 2 

$d=1$ $\vec v_1 = \vec v_2 = (1)$

$Z_{\emptyset,\{1\}}$ $Z_{\{1\},\{2\}}$ $\Big\}$ 2 tilings

$Z_{\emptyset,\{2\}}$ $Z_{\{2\},\{1\}}$



$Z_{\emptyset,\{1\}}$ $Z_{\{1\},\{1\}}$

NOT a tiling

Fails rule (2)