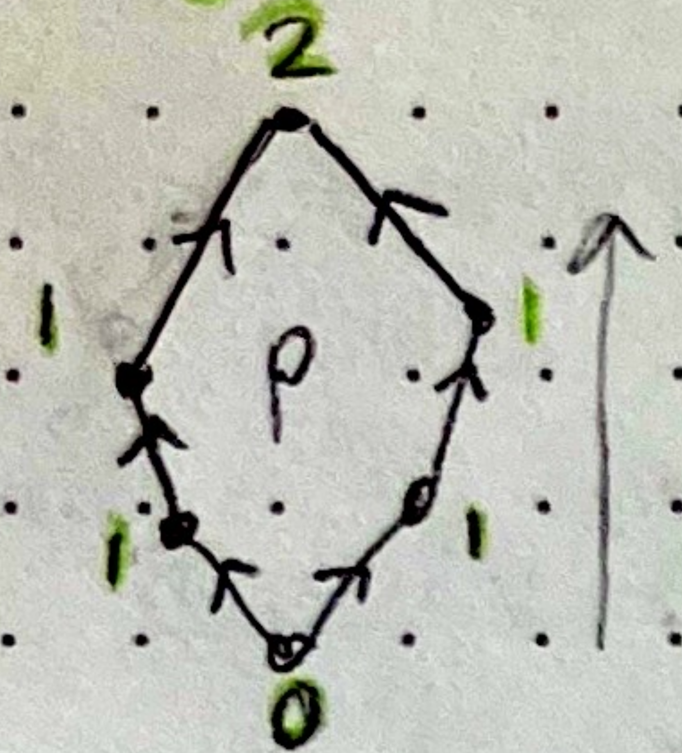
LECTURE 6 MON 9/16

Recall

<u>Thrm</u>: P simple d-dim polytope.
Its h-vector given by

in-degree $h_k(P) = \#$ vertices of P with in-degree k.

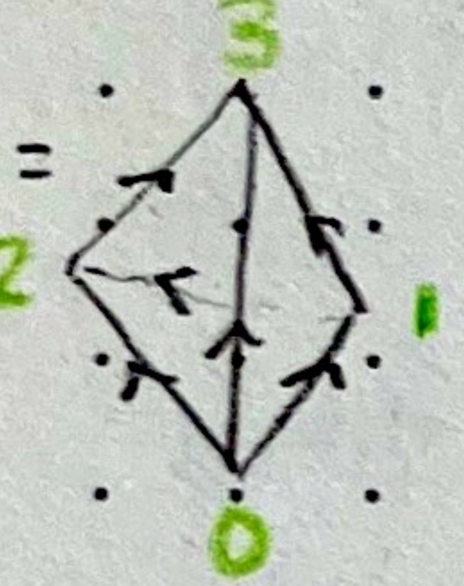


Orient the 1-skeleton on P by a generic linear function

Let P_n be a family of simple polytopes, $\dim P_n = n-1$.

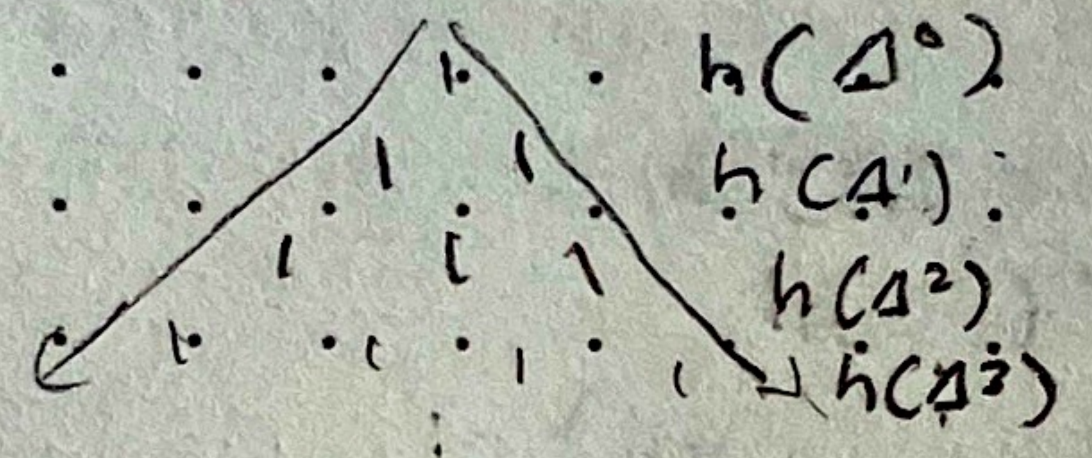
<u>Ex. 1</u>: $P_n = \Delta^{n-1}$ (the $(n-1)$-dim simplex)

$\Delta^3 =$



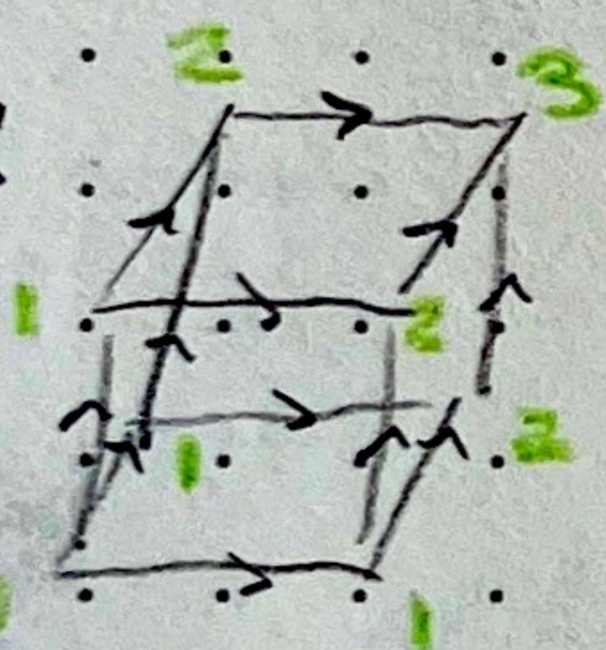
h-vector $= (1,1,1,1)$

In general, $h_k(\Delta^{n-1}) = 1 \ \forall k$



<u>Ex. 2</u>:

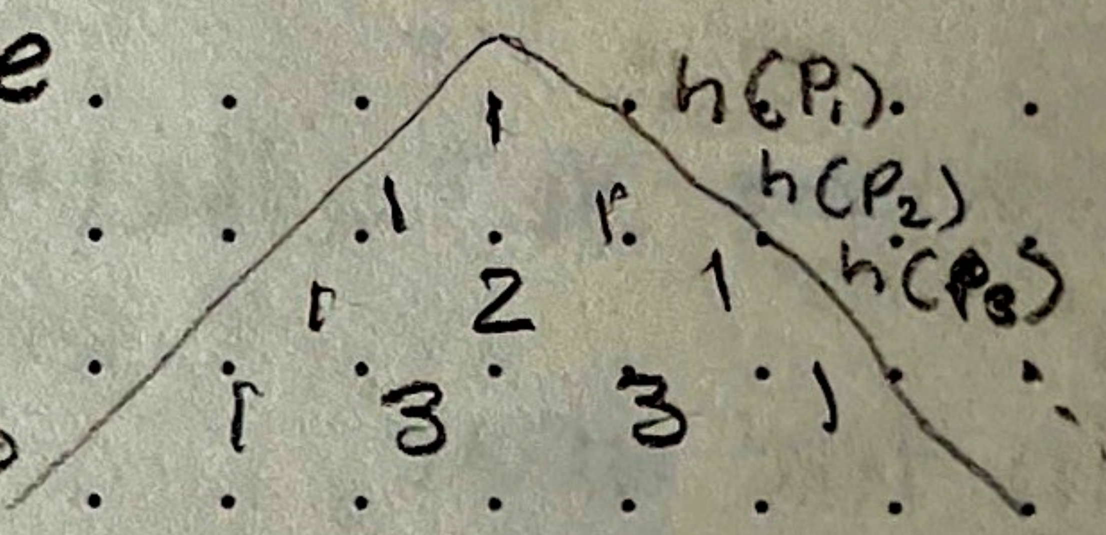
$P_4 =$



$P_n = (n-1)$-dim cube

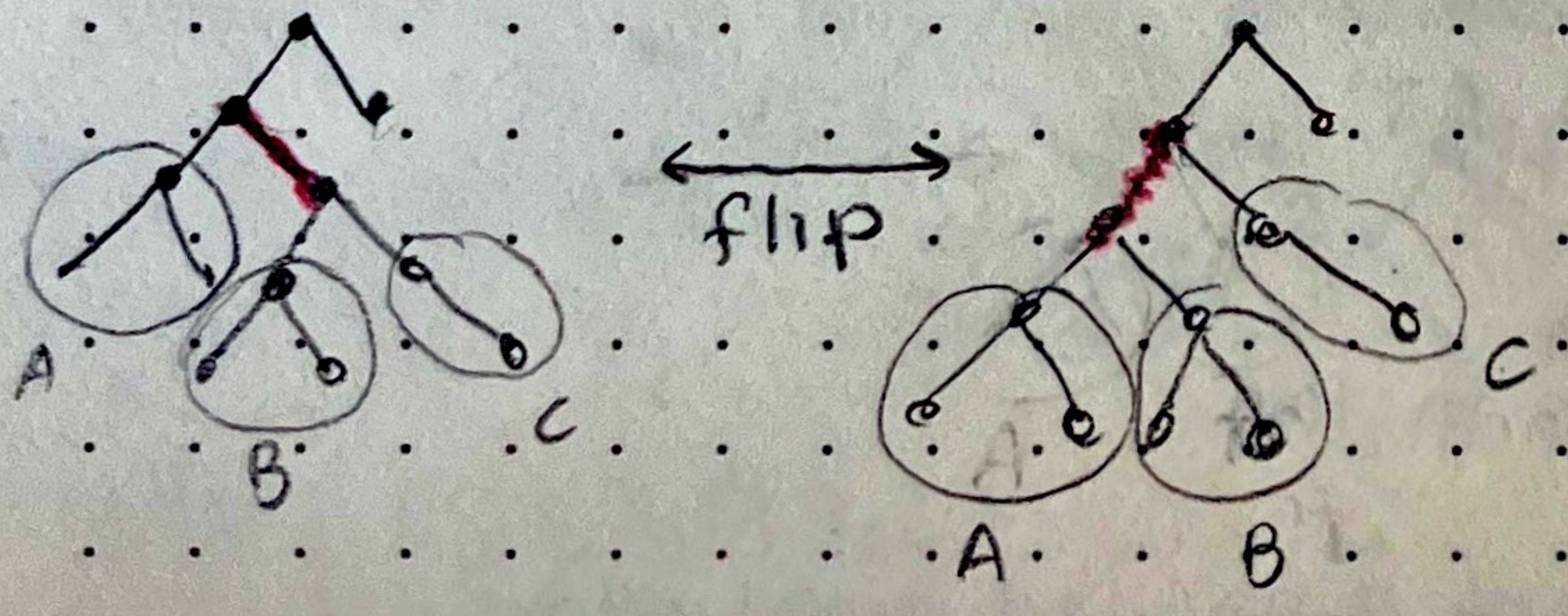
$h_k(P_n) = \binom{n-1}{k}$

Pascal's triangle →



<u>Ex. 3</u>: $P_n = A_n$ (associohedron)

vertices of $A_n \longleftrightarrow$ binary trees
edges of $A_n \longleftrightarrow$ flips of binary trees



(Edge corresponds to function maximized at 2 vertices, so scrunch those vertices together and then expand back in either way by red edge)

If we direct the 1-skeleton of A_n by linear function $(a_1 < a_2 < \dots < a_n)$

$h_k(A_n) := \#$ binary trees on n nodes w/ k right edges

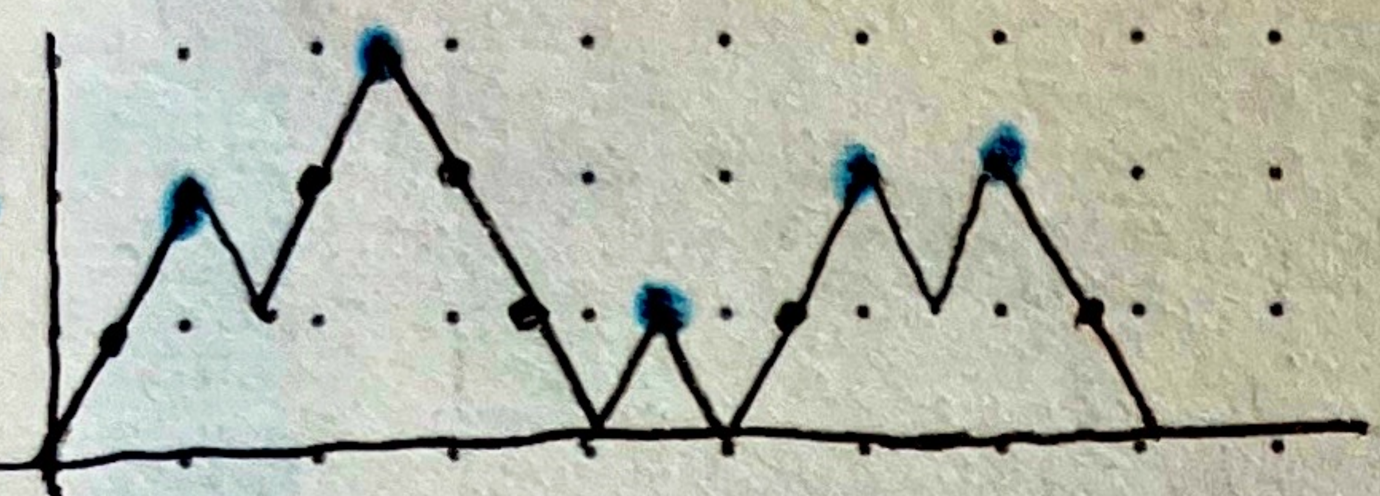
Thm: $h_k(A_n)$ is the Narayana number
$$N(n,k+1) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

Narayan Triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 3 & & 1 & & \\ & 1 & & 6 & & 6 & & 1 & \\ 1 & & 10 & & 20 & & 10 & & 1 \end{array}$$

Thm: $N(n,k)$ = # Dyck paths with $2n$ steps and k peaks.

$n=8$
$k=5$



Exercise: Find a bijection

$$\{\text{Dyck Paths}\} \longleftrightarrow \{\text{bin. trees}\}$$
s.t. # peaks -1 = # right edges

Ex. 4: P_n = $(n-1)$-dim. permutohedron

vertices $\longleftrightarrow$ permutations $\quad (\ldots i, \ldots, i+1, \ldots)$
edges $\longleftrightarrow$ adjacent transpositions (of values, not positions)

$(w^{-1}(1), \ldots, w^{-1}(n)) = (w_1, \ldots, w_n)$. $\quad (\ldots, w_i, w_{i+1}, \ldots)$

Then in inverse perm edge corresponds to swapping adjacent positions.

$h_k((n-1)\text{-dim permutohedra})$
:= The Eularian number $A(n,k)$
:= # perms of n letters w/ k descents (when $w_i > w_{i+1}$)
i.e. 1247 35 6
a descent (in spot 4)

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 4 & & 1 & & \\ & 1 & & 11 & & 11 & & 1 & \\ 1 & & 26 & & 66 & & 26 & & 1 \end{array}$$

Research Question: More triangles of these kinds?
And how do they relate?

The normal fan of a polytope

Def: A (polyhedral) cone is a polyhedron with a single minimal face F_0 (every other face contains F_0).

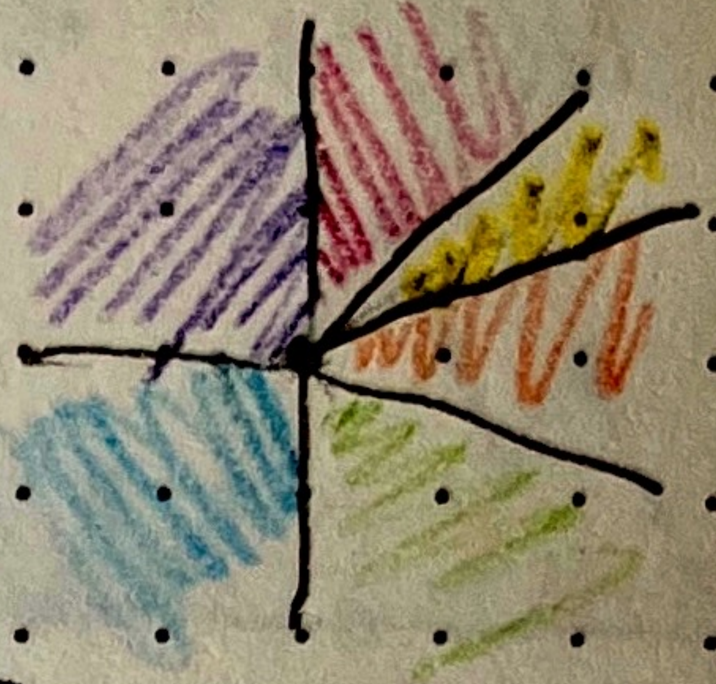
(Typically F_0 is a vertex).

Def: A fan is a collection of cones $\mathcal{F} = \{C_1, C_2, \dots, C_m\}$ in $\mathbb{R}^n$ s.t.

(1). All C_i's have the same min face

(2). Every face of C_i belongs to $\mathcal{F}$

(3). $C_i \cap C_j$ is the common face of C_i & C_j



A fan is complete if $\bigcup C_i = \mathbb{R}^n$

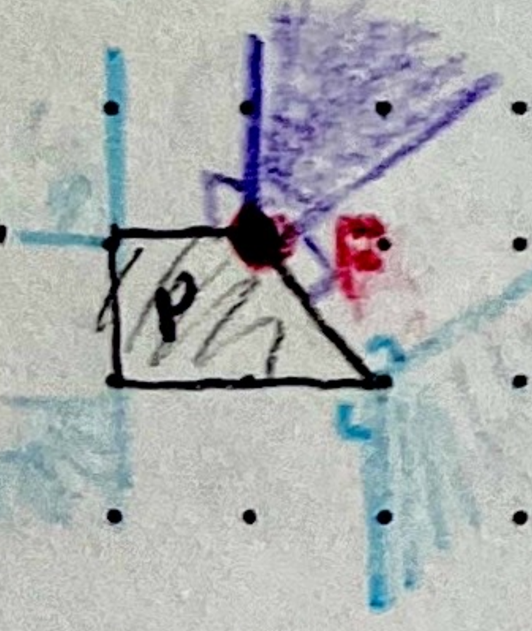
Polytope P in $\mathbb{R}^n \rightsquigarrow$ normal fan N_P in $(\mathbb{R}^n)^* \simeq \mathbb{R}^n$

For any face F of P, $F = F_{\vec{a}, P}$.

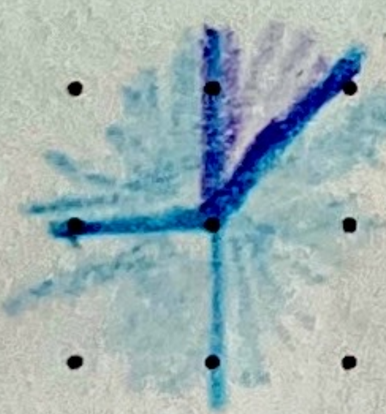
$$C_F := \{\vec{a} \in (\mathbb{R}^n)^* \mid F_{\vec{a},P} = F\}$$

$$= \{\vec{a} \in (\mathbb{R}^n)^* \mid F_{\vec{a},P} \supseteq F\}$$

Ex.

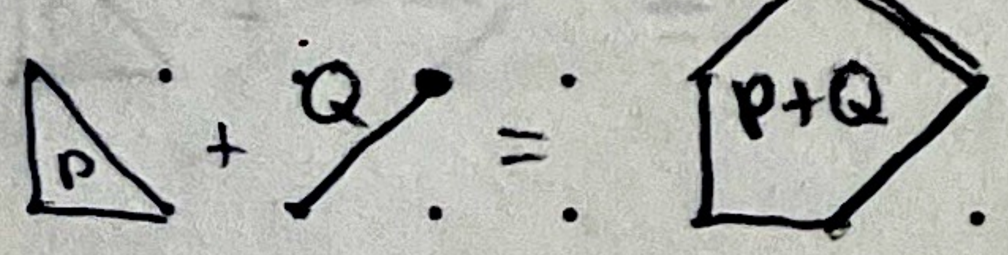


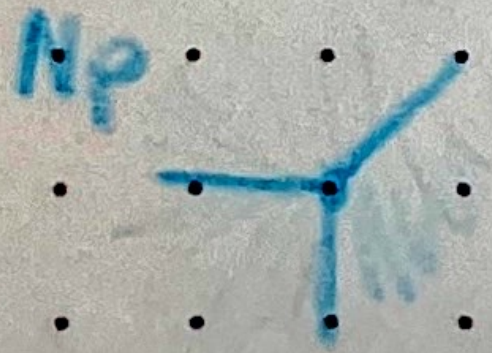
Any lin. function $\vec{a}$ s.t. $\vec{a} \cdot x$ maximized at F (for $x \in P$) is all the vectors in purple

$N_P =$  (stick the faces C_F for each F together at a common minimal vertex)

By definition, the face poset of P is dual to the face poset of N_P

Ex.





N_{P+Q}

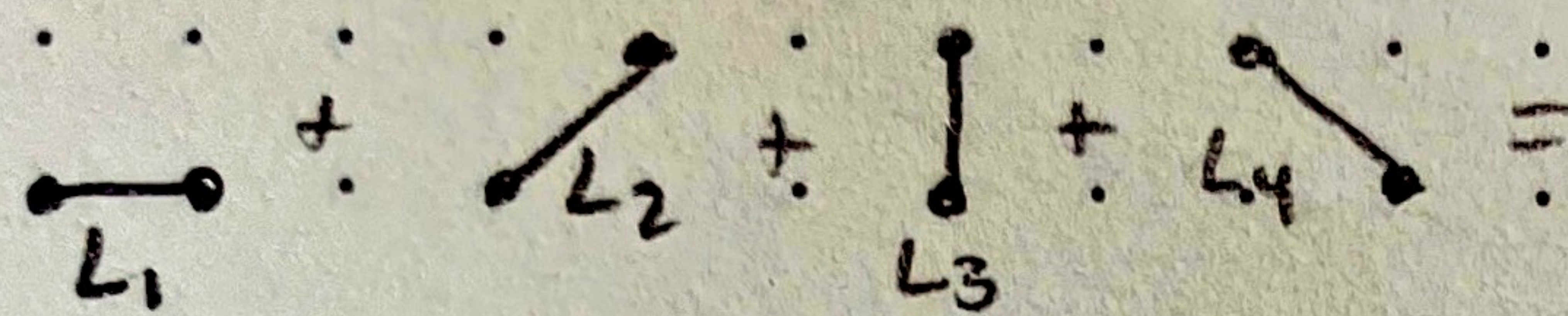
Lemma: N_{P+Q} = the common refinement of N_P & N_Q

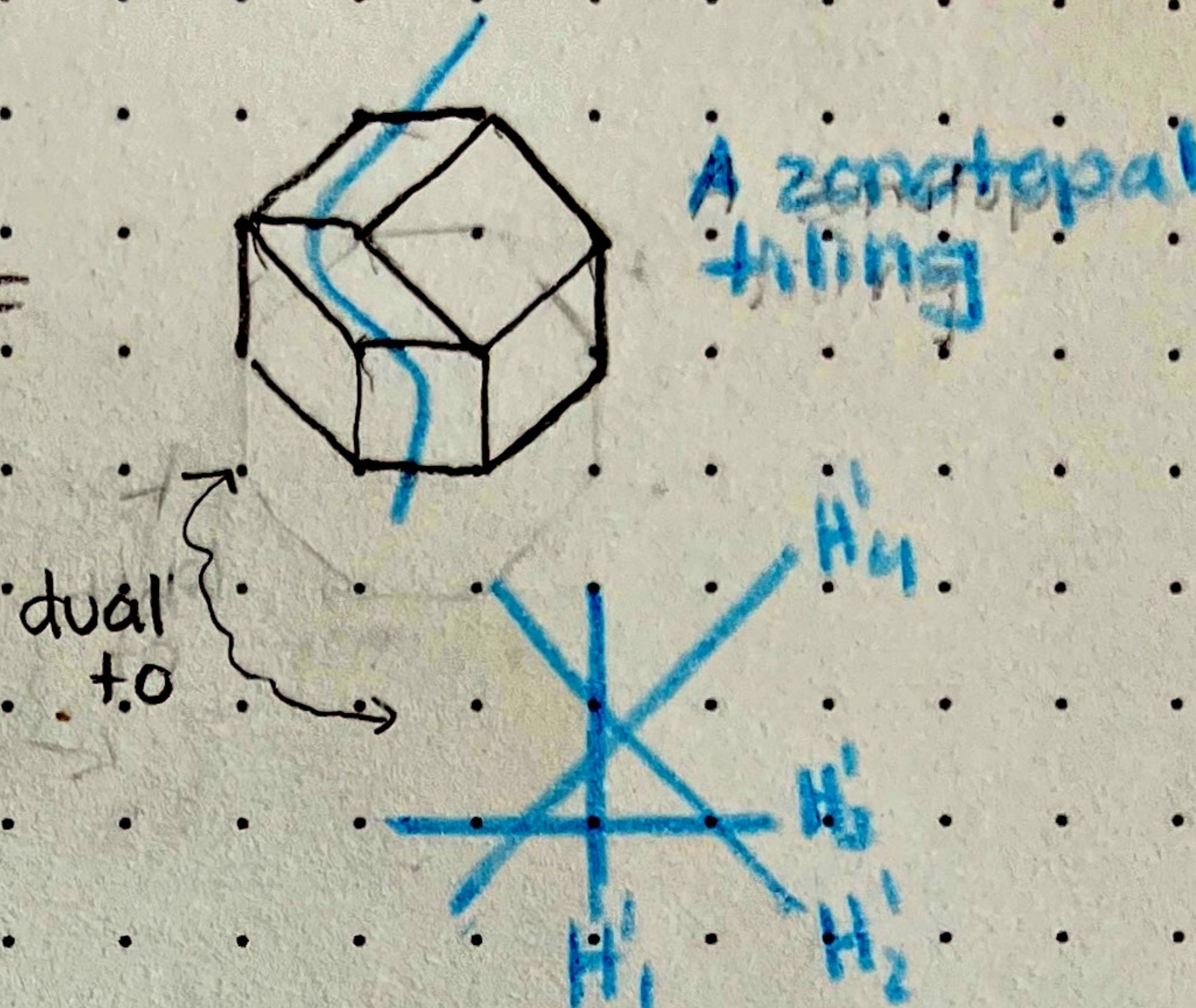
In particular,

$N_{\text{"zonotope"}}$ = (the fan associated with) hyperplane arrangement $\mathcal{H} = \{H_1, \dots, H_k\}$ where $H_i = L_i^{\perp}$

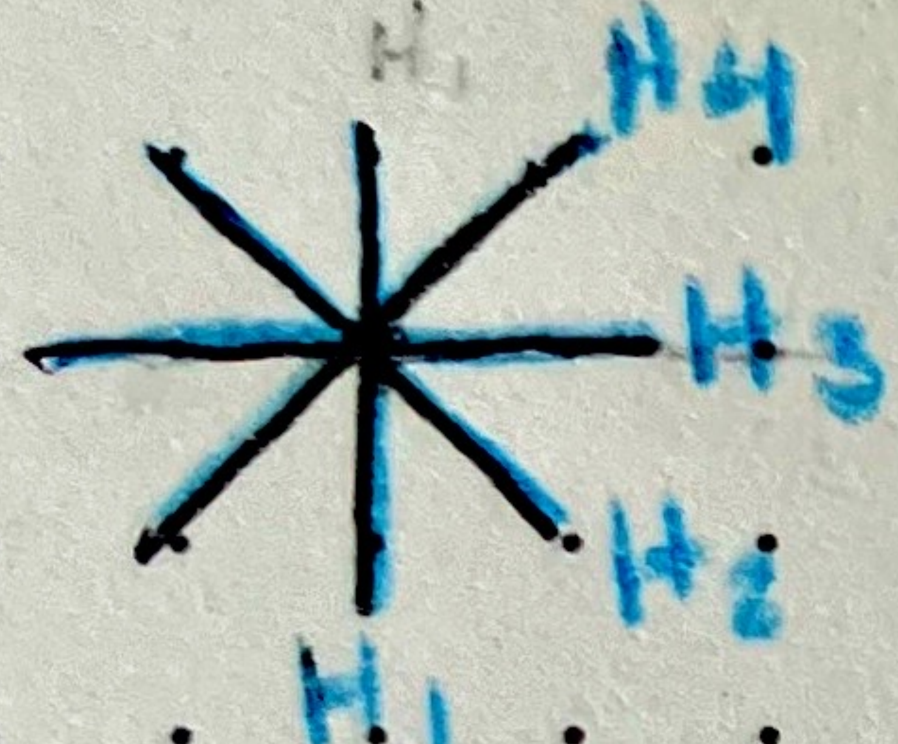
$Z = L_1 + \dots + L_m$

Ex. $Z =$





$N_Z =$



Duals of the zonotopal tiling are parallel translations of the hyperplanes in the normal fan.