18.217 Problem Set 2 (due Monday, December 4, 2023)

Problem 1. For a sequence $I = (i_1, i_2, \dots)$ of integers such that $0 \leq i_k \leq k$, for any k, and $i_k = 0$ for all sufficiently large k, define the polynomial

$$e_I := e_{i_1}(x_1)\, e_{i_2}(x_1, x_2)\, e_{i_3}(x_1, x_2, x_3) \cdots .$$

(Here $e_i(x_1, \dots, x_k)$ is the i-th elementary symmetric polynomial in k variables.)

Prove that the polynomials e_I form a linear basis of the ring of polynomials $\mathbb{C}[x_1, x_2, \dots]$ in infinitely many variables $x_1, x_2, \dots$.

Problem 2. Give a combinatorial proof of Monk's formula

$$(x_1 + \cdots + x_k)\ \mathfrak{S}_w = \sum_{i \leq k < j,\ \ell(w\, t_{ij}) = \ell(w)+1} \mathfrak{S}_{w\, t_{ij}}$$

based on the pipe dream formula for Schubert polynomials $\mathfrak{S}_w$. (Here we assume that $w \in S_\infty$.) In other words, construct a bijection between combinatorial objects that represent terms in both sides of the above formula.

Problem 3. Prove the following formula of Macdonald:

$$\sum_{s_{i_1} s_{i_2} \cdots s_{i_N} = w_\circ} i_1 \cdot i_2 \cdots i_N = \binom{n}{2}!\,,$$

where the sum is over all reduced decompositions $s_{i_1} s_{i_2} \cdots s_{i_N} = w_\circ$ of the longest permutation $w_\circ \in S_n$ and $N = \binom{n}{2}$.

Problem 4. Recall that the Schubert polynomials $\mathfrak{S}_w$, $w \in S_\infty$, form a linear basis of the polynomial ring $\mathbb{C}[x_1, x_2, x_3, \dots]$ in infinitely many variables $x_1, x_2, x_3, \dots$. For positive integers $i < j$, the *Bruhat operator* T_{ij} acts on the basis of Schubert polynomials by

$$T_{ij} : \mathfrak{S}_w \mapsto \begin{cases} \mathfrak{S}_{w\, t_{ij}} & \text{if } \ell(w\, t_{ij}) = \ell(w) + 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Also assume that $T_{ji} = -T_{ij}$ and $T_{ii} = 0$.

Prove the following formula for the product of a Schubert polynomial $\mathfrak{S}_w$ with a square-free quadratic monomial $x_{i_1}x_{i_2}$, $i_1 \neq i_2$,

$$x_{i_1}x_{i_2}\,\mathfrak{S}_w = \sum_{(j_1,j_2)} (T_{i_1\,j_1}T_{i_2\,j_2} + T_{i_2\,j_1}T_{i_1\,j_2})(\mathfrak{S}_w),$$

where the sum is over all pairs (j_1, j_2) such that $j_1, j_2 \notin \{i_1, i_2\}$ and $j_1 \leq j_2$. (You may use Monk's formula.)

Problem 5. Prove that the product of any Schubert polynomial $\mathfrak{S}_w$ with any quadratic monomial $x_{i_1}x_{i_2}$ (not necessarily square-free) is a linear combination $\sum_u c_u\,\mathfrak{S}_u$ of Schubert polynomials $\mathfrak{S}_u$ with coefficients in $c_u \in \{1, -1, 0\}$.

Problem 6. Prove that a product of any Schubert polynomial $\mathfrak{S}_w$ with any square-free monomial $x_{i_1}x_{i_2}\dots x_{i_k}$ $(i_1 < i_2 < \cdots < i_k)$ is a linear combination $\sum_u c_u\,\mathfrak{S}_u$ of Schubert polynomials $\mathfrak{S}_u$ with coefficients in $c_u \in \{1, -1, 0\}$. Give an explicit rule for coefficients (and their signs) in this expansion.

Problem 7. Recall that the Fomin-Kirillov algebra FK_n is the associative algebra[1] with generators t_{ij}, for $i, j \in [n]$, $i \neq j$, satisfying the relations: (a) $t_{ij} = -t_{ji}$; (b) $t_{ij}^2 = 0$; (c) $t_{ij}t_{kl} = t_{kl}t_{ij}$, when all four indices i, j, k, l are distinct; and (d) $t_{ij}t_{jk} = t_{ik}t_{ij} + t_{jk}t_{ik}$.

Prove the following identity in the Fomin-Kirillov algebra. For any distinct $a, b_1, \dots, b_m \in [n]$, we have

$$\sum_{(c_1,\dots,c_m)} t_{ac_1}t_{ac_2}\cdots t_{ac_m}t_{ac_1} = 0,$$

where the sum is over all cyclic shifts $(c_1, \dots, c_m)$ of $(b_1, \dots, b_m)$.

For example, for $m = 3$, we have

$$t_{ab_1}t_{ab_2}t_{ab_3}t_{ab_1} + t_{ab_2}t_{ab_3}t_{ab_1}t_{ab_2} + t_{ab_3}t_{ab_1}t_{ab_2}t_{ab_3} = 0.$$

Problem 8. Recall that a *balanced tableau* T of the staircase shape $\delta = (n-1, n-2, \dots, 1)$ is a way to fill the boxes of the shape δ by numbers $1, 2, \dots, \binom{n}{2}$ (without repetitions) such that, for any hook H in T, the median entry among all entries of H is located at the upper-left corner of the hook H.

[1] For $n \geq 3$, FK_n is not a commutative algebra.

Define a *proper reflection ordering* as a linear ordering "$\prec$" of all pairs (i, j), for $1 \le i < j \le n$, such that, for any three indices $i_1, i_2, i_3 \in [n]$, $i_1 < i_2 < i_3$, we have either $(i_1, i_2) \prec (i_1, i_3) \prec (i_2, i_3)$ or $(i_2, i_3) \prec (i_1, i_3) \prec (i_1, i_2)$.

Construct a bijection between balanced tableaux and proper reflection orderings (and prove that it is indeed a bijection).

Problem 9. For $k < n$, let $\binom{[n]}{k+1}$ denote the set of $(k+1)$-element subsets of $[n]$. The *higher Bruhat order* $B(n, k)$ is defined as the partial order by containment on all subsets $\mathcal{I} \subseteq \binom{[n]}{k+1}$ that satisfy the following conditions:

- There is no $(k+2)$-element subset $J \subset [n]$ and $j_1 < j_2 < j_3$ in J such that $J \setminus j_1 \in \mathcal{I}$, $J \setminus j_2 \notin \mathcal{I}$, and $J \setminus j_3 \in \mathcal{I}$.
- There is no $(k+2)$-element subset $J \subset [n]$ and $j_1 < j_2 < j_3$ in J such that $J \setminus j_1 \notin \mathcal{I}$, $J \setminus j_2 \in \mathcal{I}$, and $J \setminus j_3 \notin \mathcal{I}$.

(a) Prove that $B(n, 1)$ is isomorphic to the weak Bruhat order on the symmetric group S_n.

(b) For a proper reflection ordering "$\prec$" (as in the previous problem), let $\mathcal{I} \subseteq \binom{[n]}{3}$ be the set of triples (i_1, i_2, i_3) (with $i_1, i_2, i_3 \in [n]$, $i_1 < i_2 < i_3$) such that $(i_2, i_3) \prec (i_1, i_3) \prec (i_1, i_2)$. Prove that $\mathcal{I}$ is an element of the higher Bruhat order $B(n, 2)$ and that any element of $B(n, 2)$ comes from some proper reflection ordering.

Problem 10. Calculate cardinality of the higher Bruhat order $B(n, k)$ for $k \ge n - 3$.

Problem 11. For $1 \le k \le n$, define three numbers A_{kn}, B_{kn}, and C_{kn}:

- A_{kn} is the number of acyclic orientations of the complete bipartite graph $K_{k,n-k}$. (Recall, that an *acyclic orientation* of a graph is a way to direct all its edges, so that no directed cycles are created.)
- B_{kn} is the cardinality of the interval $[id, w_{kn}]$ in the strong Bruhat order on the symmetric group S_n, where $w_{kn} \in S_n$ is the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & n \\ n-k+1 & n-k+2 & \cdots & n & 1 & 2 & \cdots & n-k \end{pmatrix}$$

- C_{kn} is the number of Le-diagrams of the rectangular shape $k \times (n-k)$. Recall that a Le-diagram D is a way to place dots in some boxes the $k \times (n-k)$ rectangle, such that, if D contains dots in boxes (i_1, j_2) and (i_2, j_1), for some indices $i_1 < i_2$ and $j_1 < j_2$, then D should contain a dot in the box (i_2, j_2).

Prove that $A_{kn} = B_{kn} = C_{kn}$.

Problem 12. For three permutations $u, v, w \in S_\infty$, let c_{uv}^w be the generalized Littlewood-Richardson coefficients defined by the expansion of the product of two Schubert polynomials in the basis of Schubert polynomials

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_w c_{uv}^w \, \mathfrak{S}_w \, .$$

In class, we defined the *dual Schubert polynomials* $\mathfrak{D}_w(x_1, \dots, x_n)$, for $w \in S_n$. Prove that

$$\mathfrak{D}_w(x_1+y_1, x_2+y_2, \dots, x_n+y_n) =$$
$$= \sum_{u,v \in S_n} c_{uv}^w \, \mathfrak{D}_u(x_1, x_2, \dots, x_n) \cdot \mathfrak{D}_v(y_1, y_2, \dots, y_n).$$

Problem 13. The *permanent* of a square $N \times N$-matrix $A = (a_{i,j})$ is defined as $\mathrm{per}(A) := \sum_{w \in S_N} \prod_{i=1}^N a_{i,w(i)}$. (The same expression as $\det(A)$ by without signs.)

Let B be the $\binom{n}{2} \times n$ matrix whose rows are labelled by pairs (i, j) (with $1 \le i < j \le n$) and the row labelled (i, j) is given by the n-vector $(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$ with 1 in i-th position and -1 in j-th position.

Prove that $\mathrm{per}(B \cdot B^T)$ equals $1!\,2!\,\cdots\,n!$.

For example, for $n = 3$, we have

$$\mathrm{per}\left(\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}\right) = 1!\,2!\,3!.$$

Problem 14. Recall that Zelevinsky's picture rule is a combinatorial rule for the inner product of two skew Schur functions $\langle s_{\lambda/\mu}, s_{\nu/\gamma} \rangle$. A *picture* is defined as a bijective map ϕ between boxes of λ/μ and ν/γ such that (a) if we list the boxes of λ/μ by rows right-to-left top-to-bottom, then the images of these boxes under the map ϕ form a standard Young tableau of shape ν/γ; (b) the same condition for the inverse map ϕ^{-1}. According to the picture rule the inner product $\langle s_{\lambda/\mu}, s_{\nu/\gamma} \rangle$ equals the number of pictures $\phi : \lambda/\mu \to \nu/\gamma$.

Prove that, in case $\gamma = \emptyset$, Zelevinsky's picture rule is equivalent to the classical Littlewood-Richardson rule. In other words, construct a bijection between pictures and Littlewood-Richardson tableaux.

Problem 15. Define two partial orders "$\swarrow$" and "$\nwarrow$" of the set of boxes of a skew shape. For boxes $a = (i, j)$ and $b = (i', j')$, we have

- $a \swarrow b$ if $i \geq i'$, $j \leq j'$, and $a \neq b$. (Box a is located to the South-West of box b.)
- $a \nwarrow b$ if $i \leq i'$, $j \leq j'$, and $a \neq b$ (Box a is located to the North-West of box b.)

Prove that Zelevinsky's pictures $\phi : \lambda/\mu \to \nu/\gamma$ (defined as in the previous problem) can be described by the following conditions:

(a) There are no boxes a, b in λ/μ such that $a \swarrow b$ and $\phi(a) \nwarrow \phi(b)$.

(b) There are no boxes c, d in λ/μ such that $c \nwarrow d$ and $\phi(c) \swarrow \phi(d)$.

Problem 16. In class, we gave Stembridge's "concise proof". Actually, it proves the following extension of the Littlewood-Richardson rule.

Theorem. Let $\lambda, \mu, \nu, \gamma$ be partitions with at most n parts. The coefficient of s_γ in the Schur expansion of the product $s_\lambda \cdot s_{\mu/\nu}$ equals the number of semi-standard Young tableaux T of shape μ/ν and weight $\gamma - \lambda$ such that, for any j, $\lambda + \mathrm{weight}(T_{\geq j})$ is a partition. (Here $T_{\geq j}$ denotes the subtableau of T formed by all boxes of T in columns $j, j+1, j+2, \ldots$.)

Notice that the above theorem gives a rule for the Hall inner product $\langle s_{\gamma/\lambda}, s_{\mu/\nu} \rangle$.

Prove bijectively that this rule is equivalent to Zelevinsky's picture rule. In other words, construct a bijection between semi-standard tableaux T as above and Zelevinsky's pictures $\phi : \gamma/\lambda \to \mu/\nu$.

Problem 17. For three partitions λ, μ, ν with n parts, let $BZ(\lambda, \mu, \nu)$ be the polytope of $\mathbb{R}$-valued BZ-triangles with boundary conditions given by λ, μ, ν. (Equivalently, it is the polytopes of honeycombs with boundary rays given by parts of λ, μ, ν.)

Find a triple of integer partitions λ, μ, ν such that the polytope $BZ(\lambda, \mu, \nu)$ has a non-integer vertex.

Problem 18. Fix two positive integers $k < n$. Also fix k complex numbers $c_{n-k+1}, c_{n-k+2}, \ldots, c_n \in \mathbb{C}$.

Consider the factor algebra $A := \mathbb{C}[x_1, \ldots, x_k]^{S_k}/I$ of the algebra $\mathbb{C}[x_1, \ldots, x_k]^{S_k}$ of symmetric polynomials in k variables modulo the ideal

$$I := \langle h_i(x_1, \ldots, x_k) - c_i \mid \text{ for } n-k+1 \leq i \leq n \rangle \,,$$

where the $h_i(x_1, \ldots, x_k)$ are the complete homogeneous symmetric polynomials. Prove that A is a finite-dimensional (as a vector space over $\mathbb{C}$) and calculate its dimension $\dim_{\mathbb{C}} A$.

Problem 19. In class, we defined the cylindric Schur function $s_{\lambda/d/\mu}$ as the sum $s_{\lambda/d/\mu} := \sum_T x^{\mathrm{weight}(T)}$ over semi-standard Young tableaux T of cylindric shape $\lambda/d/\mu$. Show that $s_{\lambda/d/\mu}$ is a symmetric function.