18.217 PROBLEM SET 1 (due Friday, October 20, 2023)

Problem 1. Show that the linear span of Schubert polynomials $\mathfrak{S}_w$, $w \in S_n$ (as a linear subspace of the space of all polynomials in $x_1, \ldots, x_n$) coincides with the linear span of the set of monomials $x_1^{a_1} \ldots x_n^{a_n}$, for all $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $0 \le a_k \le n-k$, for $k = 1, \ldots, n$.

Problem 2. The *coinvariant algebra* C_n is defined as the quotient algebra $C_n := \mathbb{C}[x_1, \ldots, x_n]/I_n$, where $I_n = \langle e_1, \ldots, e_n \rangle$ is the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ generated by the *elementary symmetric polynomials*

$$e_k = e_k(x_1, \ldots, x_n) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

for $k = 1, \ldots, n$.

Prove that the coinvariant algebra C_n is $n!$ dimensional (as a linear space over $\mathbb{C}$), and the cosets of monomials $x_1^{a_1} \ldots x_n^{a_n}$, for all $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $0 \le a_k \le n-k$, for $k = 1, \ldots, n$, form a linear basis of the coinvariant algebra C_n.

Problem 3. The *Lehmer code* of a permutation $w = w_1, \ldots, w_n \in S_n$ is defined as $\mathrm{code}(w) = (c_1, \ldots, c_n)$, where $c_i = \#\{j \mid j > i,\ w_j < w_i\}$. A permutation is called *dominant* if its Lehmer code is weakly decreasing $c_1 \ge c_2 \ge \cdots \ge c_n$.

Prove that a permutation w dominant if and only if it is 132-avoiding. Find the number of dominant permutations in S_n

Problem 4. A permutation is called *strictly dominant* if its Lehmer code satisfies $c_1 > c_2 > \cdots > c_k = \cdots = c_n = 0$.

Prove that a permutation w strictly dominant if and only if it is both 132-avoiding and 231-avoiding. Find the number of strictly dominant permutations in S_n.

Problem 5. Prove that, for a dominant permutation $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w$ equals $\mathfrak{S}_w = x_1^{c_1} \cdots x_n^{c_n}$, where $(c_1, \ldots, c_n) = \mathrm{code}(w)$.

Problem 6. Let $\partial_i := (x_i - x_{i+1})^{-1}(1 - s_i)$ be the ith divided difference operator; and let $\partial_w := \partial_{i_1} \cdots \partial_{i_l}$ for a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$ of a permutation $w \in S_n$.

Prove that, for the longest permutation $w_\circ \in S_n$, the operator $\partial_{w_\circ}$ acts on polynomials $f = f(x_1, \ldots, x_n)$ as

$$\partial_{w_\circ} : f \mapsto \prod_{1 \le i < j \le n} (x_i - x_j)^{-1} \sum_{w \in S_n} (-1)^{\ell(w)} f(x_{w_1}, \ldots x_{w_n}).$$

Problem 7. The ith Demazure operator is given by $D_i : f \mapsto \partial_i(x_i\, f)$. For a permutation $w \in S_n$ and a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, define the operator D_w by $D_w := D_{i_1} \cdots D_{i_l}$.

Characterize all permutations $w \in S_n$ for which the operator D_w coincides with the operator $f \mapsto \partial_w(x_1^{c_1} \cdots x_n^{c_n}\, f)$, where $(c_1, \ldots, c_n) = \mathrm{code}(w)$.

Problem 8. Use the symmetry of the RSK correspondence ($A \mapsto (P, Q)$ iff $A^T \mapsto (Q, P)$) to deduce a closed-form product formula for the (infinite) sum of Schur polynomials $\sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} s_\lambda(x_1, \ldots, x_n)$ over all partitions λ with at most n parts.

Problem 9. Recall that f^λ denotes the number of standard Young tableaux of shape λ. A *skew Young diagram* λ/μ is the set-theoretic difference of two Young diagrams $\lambda \supset \mu$ (considered as collections of boxes on the plane). Let $f^{\lambda/\mu}$ be the number of standard Young tableaux of skew shape λ/μ.

Find a closed-form formula for the sum

$$\sum_{\lambda,\mu,\nu} f^\lambda\, f^{\lambda/\mu}\, f^{\nu/\mu}\, f^\nu$$

over triples of partitions λ, μ, ν such that $\lambda \supset \mu \subset \nu$, $|\lambda| = |\nu| = 2n$, and $|\mu| = n$.

For example, for $n = 1$, this sum equals 4.

Problem 10. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a Young diagram that fits inside the $n \times n$ square i.e., $\lambda_1 \le n$ and $\lambda'_1 \le n$. The number placements of n non-attacking rooks on the "chessboard" of shape λ equals $\prod_{k=0}^{n-1}(\lambda_{n-k} - k)$. Clearly, by symmetry, the number of such rook placements can also be calculated using parts of the conjugate partitions λ' as $\prod_{k=0}^{n-1}(\lambda'_{n-k} - k)$.

Prove that the multiset $\{\lambda_n, \lambda_{n-1}-1, \lambda_{n-2}-2, \dots, \lambda_1 - n+1\}$ coincides with the multiset $\{\lambda'_n, \lambda'_{n-1}-1, \lambda'_{n-2}-2, \dots, \lambda'_1 - n+1\}$, for any Young diagram λ that fits inside the $n \times n$ square. Can you describe a permutation of entries of the first multiset that gives entries of the second multiset?

Problem 11. Find a closed-form expression for the sum

$$\sum_{\lambda \subset n\times n} \prod_{k=0}^{n-1} (\lambda_{n-k} - k)$$

over all Young diagrams $\lambda = (\lambda_1, \dots, \lambda_n)$ that fit inside the $n \times n$ square.

Problem 12. Prove the following "tropical hooklength-type formula".

Fix a Young diagram λ with n boxes. The *content* of a box (i,j) in λ is defined as $j-i$. Let $\mathrm{SYT}(\lambda)$ be the set of standard Young tableaux of shape λ. For $T \in \mathrm{SYT}(\lambda)$ and $k \in [n]$, let $\mathrm{cont}(k,T)$ be the content $j-i$ of the box (i,j) with entry k in T. Let H_a be the hook at box a of λ. Let $x_s, x_{s+1}, \dots, x_t$ be some variables, where s and t are the minimal and the maximal contents among all boxes of λ.

Show that

$$\max_{T\in \mathrm{SYT}(\lambda)} \left(\sum_{i=1}^{n} \min_{k\in\{i,i+1,\dots,n\}} x_{\mathrm{cont}(k,T)} \right) = \sum_{a\in\lambda} \min_{(i,j)\in H_a} x_{j-i}.$$

For example, for $\lambda = (2,1)$, we have

$$\begin{aligned} &\max(x_1 + \min(x_1, x_{-1}) + \min(x_1, x_{-1}, x_0),\\ &\qquad x_{-1} + \min(x_{-1}, x_1) + \min(x_{-1}, x_1, x_0))\\ &= x_1 + x_{-1} + \min(x_1, x_0, x_{-1}). \end{aligned}$$

Problem 13. Let $h_1(x), \dots, h_{n-1}(x)$ be the elements of the nilCoxeter algebra given by $h_i(x) = 1 + x\, u_i$, where x is a commutative parameter. The $h_i(x)$ satisfy the Yang-Baxter relations:

(1) $h_i(x)\, h_i(y) = h_i(x+y)$,
(2) $h_i(x)\, h_j(y) = h_j(y)\, h_i(x)$ if $|i-j| \geq 2$,
(3) $h_i(x)\, h_{i+1}(x+y)\, h_i(y) = h_{i+1}(y)\, h_i(x+y)\, h_{i+1}(x)$.

Let

$$\mathfrak{S}(x_1,\dots,x_{n-1};y_1,\dots,y_{n-1}) := \prod_{i=1}^{n-1}\prod_{j=n-i}^{1} h_{i+j-1}(x_i - y_j)\,.$$

Use the Yang-Baxter relations to show that

$$\mathfrak{S}(x_1,\dots,x_{n-1};y_1,\dots,y_{n-1}) =$$
$$= \mathfrak{S}(0,\dots,0;y_1,\dots,y_{n-1})\ \mathfrak{S}(x_1,\dots,x_{n-1};0,\dots,0).$$

For example, for $n=3$, we have the identity

$$h_2(x_1-y_2)h_1(x_1-y_1)h_2(x_2-y_1) =$$
$$= h_2(-y_2)h_1(-y_1)h_2(-y_1)\ \ h_2(x_1)h_1(x_1)h_2(x_2).$$

Problem 14. Calculate the volume of the polytope $P \subset \mathbb{R}^{n^2}$ of $n\times n$ matrices $A=(a_{ij})$ with real entries $a_{ij}\in[0,1]$ that weakly increase in rows and columns. (In other words, P is the polytopes of $\mathbb{R}$-valued reverse plane partitions of shape $n\times n$ with entries bounded by 1.)

Problem 15. Let $\lambda=(\lambda_1,\dots,\lambda_n)$. Let $GT(\lambda)\subset\mathbb{R}^{\binom{n}{2}}$ be the convex polytope of all $\mathbb{R}$-valued Gelfand-Tsetlin patterns with top row $\lambda_1,\dots,\lambda_n$. Find an explicit expression for the volume of the polytope $GT(\lambda)$. For example, for $\lambda=(3,2,1)$, $GT(\lambda)$ is given by

$$\{(x,y,z)\in\mathbb{R}^3 \mid 3\ge x\ge 2\ge y\ge 1\,,\ x\ge z\ge y\}.$$

One can show that its volume is 1.

Problem 16. In class, we gave a recursive construction of the *generalized RSK correspondence* $\phi_\kappa : A \mapsto B$ from the set of non-negative matrices A of shape κ to the set of reverse plane partitions B of shape κ using toggle operations. We also formulated the *generalized Greene's theorem* that gives a non-recursive description of the map ϕ_κ in terms of maximums of certain sums over certain collections of non-crossing lattice paths. Prove the generalized Greene's theorem by showing that these expressions in terms lattice paths satisfy the toggle recurrence. Hint: It might be easier to prove a "detropicalized version" of this theorem.