18.217 Combinatorial Theory

LECTURE 1 9/6/23

Schur Polynomials S_λ

Schubert Polynomials $\mathfrak{S}_w$

and their generalizations

Professor: Alex Postnikov

Website: math.mit.edu/~apost/courses/18.217/

Alternatively try math.mit.edu/18.217/

Schur polynomials $S_\lambda(x_1, \ldots, x_n)$

- rep theory: characters of irreps of GL_n / SL_n
- alg geometry: reps of cohomology classes of Schubert varieties in Grassmanian $Gr_{k,n}$

Generalizations: Demazure characters (aka key polynomials) rep theory characters →

Generalizations to flag varieties → Schubert polynomials $\mathfrak{S}_w$ homogeneous

K-theory ⇝ Grothendieck polynomials $\mathfrak{G}_w$ non-homogenious, smallest homogeneous part is $\mathfrak{S}_w$

Classical def of Schur polyn. $S_\lambda(x_1, \ldots, x_n)$

(fixed # of variables. Schur functions have ∞ many)

$\lambda = (\lambda_1, \ldots, \lambda_n)$ Partition with at most n parts

$\lambda_1 \geq \ldots \geq \lambda_n \geq 0$

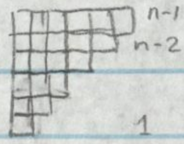
$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$

$$a_\alpha(x_1, \ldots, x_n) = \det \begin{bmatrix} x_1^{\alpha_1} & \cdots & x_n^{\alpha_n} \\ \vdots & \ddots & \vdots \\ x_1^{\alpha_n} & \cdots & x_n^{\alpha_n} \end{bmatrix} = \det(x_i^{\alpha_j})_{1 \leq i,j \leq n}$$

$$= \sum_{\substack{w = w_1, \ldots, w_n \in S^n \\ \text{a perm. of } 1, \ldots, n}} \operatorname{sign}(w)\, w(x^\alpha) \qquad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

$$w(x^\alpha) = x_{w_1}^{\alpha_1} \cdots x_{w_n}^{\alpha_n}$$

$$= \Big(\sum_{w \in S^n} \operatorname{sign}(w)\, w\Big) x^\alpha$$

$\delta = (n-1, n-2, \dots, 1, 0) =$ 

$a_\delta = \prod_{1 \le i < j \le n} (x_i - x_j)$

<u>Def</u>: $S_\lambda(x_1, \dots, x_n) = \dfrac{a_{\lambda+\delta}}{a_\delta}$

(type A Weyl's character formula)

<u>Properties</u>:

- This is a symmetric polynomial $\left(\dfrac{\text{anti-symetric}}{\text{anti-symetric}} = \text{symetric}\right)$
- non-obvious: It has positive integer coeff

Schubert Polynomials

$\mathfrak{S}_w(x_1, \dots, x_n)$ permutation $w \in S^n$

(Right) weak Bruhat order on S^n

E.x. $\boxed{n=3}$

$w_0 = 321$ ← longest permutation

$s_1 s_2 = 312$ $\quad$ $s_2 s_1 = 231$

$s_1 = 213$ $\quad$ $s_2 = 132$

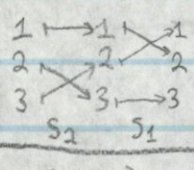
$id = 123$

Convention:

Two line notation $\begin{pmatrix} 1 & 2 & \dots & n \\ w_1 & \dots & \dots & w_n \end{pmatrix}$ = the map $[n] \to [n]$, $i \mapsto w_i$

One line notation $w = w_1 \cdots w_n$

Read like maps $s_1 s_2$ means (acts from right to left)



$= 231$

Covering relations in (right) week Bruhat order

$ws_i > w \qquad w = \dots w_i w_{i+1} \dots \qquad \text{if } w_i < w_{i+1}$

Divided difference operators

$\partial_i : \mathbb{C}[x_1, \dots, x_n] \longrightarrow \mathbb{C}[x_1, \dots, x_n] \qquad i = 1, \dots, n-1$

$$f(x_1, \dots, x_n) \longmapsto \frac{f(x_1, \dots, x_n) - f(x_1, \dots x_{i+1}, x_i, \dots, x_n)}{x_i - x_{i+1}} = \frac{f - s_i(f)}{x_i - x_{i+1}}$$

$$\partial_i = \frac{1}{x_i - x_{i+1}} (\underbrace{1}_{\text{id operator}} - s_i)$$ Order matters here! Act by $(1 - s_i)$ first.

Classical def of Schubert Polynomials

Def: ① $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^{1} = x^{\delta}$

② $\mathfrak{S}_w = \partial_i (\mathfrak{S}_{ws_i})$ if $w \lessdot ws_i$ in weak Bruhat order

(use divided difference to go down)

Lemma: This is well-defined (going down different ways still gives same polynomial)

$$\begin{tikzcd}[column sep=tiny]
& \mathfrak{S}_{321} = x_1^2 x_2 \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & \\
\mathfrak{S}_{231} = \frac{x_1^2 x_2 - x_2^2 x_1}{x_1 - x_2} = x_1 x_2 \arrow[d] & & \mathfrak{S}_{231} = x_1^2 \arrow[d] \\
\mathfrak{S}_{213} = x_1 \arrow[dr, "\partial_1"'] & & \mathfrak{S}_{132} = x_1 \arrow[dl, "\partial_2"] \\
& \mathfrak{S}_{123} = 1 &
\end{tikzcd}$$

Claim Schubert polynomials always have non-negative int. coeff

(Easier to see from other definitions)

Schur vs Schubert

$s_\lambda = \prod_{i<j} \frac{1}{x_i - x_j} \left(\sum_w \text{sign}(w) w \right) (x^{\lambda + \delta})$	$\mathfrak{S}_w = \frac{1}{x_i - x_{i+1}} (1 - s_i)(x^{\delta})$

18.217 LECTURE 2 (Missed for math retreat)

Summary of Material covered

<u>Grassmanian Polynomials</u> Fix n, k $0 \le k \le n$

$w = w_1 < w_2 < \dots < w_k$, $w_{k+1} < w_{k+2} \dots < w_n \in S_n$

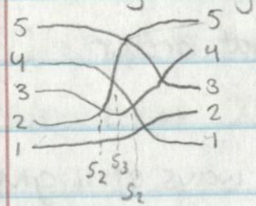
Identified with partitions via $w(\lambda) = \lambda_k + 1, \lambda_{k-1} + 2, \dots, \lambda_1 + k$
where λ in $k \times (n-k)$ rectangle

<u>Thrm</u>: For a Grassmanian perm $w(\lambda)$, $\lambda \subset k \times (n-k)$

$$\mathfrak{S}_{w(\lambda)}(x_1, \dots, x_k) = S_\lambda(x_1, \dots, x_k)$$

Wiring diagrams of permutations

E.X.



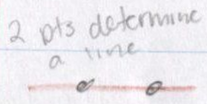
<u>Def</u>: Length $\ell(w)$ of $w \in S_n$ is $\min(\ell \text{ s.t. } w = s_{k_1} \cdots s_{k_\ell})$
$= \mathrm{inv}(n) = \#\{i < j \mid w_i > w_j\}$

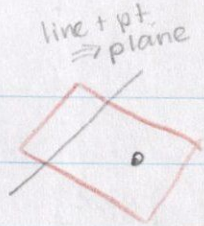
<u>Def</u>: Reduced decomposition for w is any $w = s_{k_1} \cdots s_{k_\ell}$ s.t. $\ell = \ell(w)$

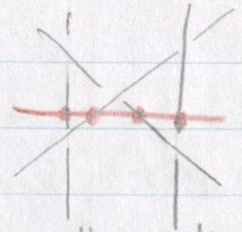
Covers properties of divided difference operators
(Many of which are sufficiently reviewed in the following lectures, so hopefully without the last bit of lecture you will still be able to follow the rest of the notes ...)

18.217 LECTURE 3   9/11

How many lines in $\mathbb{C}^3$ intersect 4 given generic lines?







$\mathbb{C}^3$

$\# = ?$

→ Schubert calculus is tool to solve these kinds of geometric problems

Use $H^*(Fl_n, \mathbb{C})$ ← flag manifold

Thrm (Borel): $H^*(Fl_n, \mathbb{C}) \simeq \mathbb{C}[x_1, \dots, x_n]/I_n$ ← coinveriant algebra of S_n

(← can by any field of char 0)

$I_n := \langle e_i(x_1, \dots, x_n) \;\; i = 1, \dots, n \rangle$ (All sym polys w/ no const. term)

elementary sym. poly $e_i := \sum_{j_1 < j_2 < \dots < j_i} x_{j_1} \cdots x_{j_i}$

Ex. $n=3$ $\mathbb{C}[x_1, x_2, x_3] / \langle x_1 + x_2 + x_3,\; x_1x_2 + x_2x_3 + x_1x_3,\; x_1x_2x_3 \rangle$

A linear basis:

Degree	elts	dimension
deg 3	$x_1^2 x_2$	1
deg 2	$x_1x_2,\ x_1^2$ ← pick any 2 of (x_1^2, x_2^2, x_1x_2)	2
deg 1	$x_1,\ x_2$ ← pick any 2 of (x_1, x_2, x_3)	2
deg 0	1	1

} 6

Lemma: All monomials $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ s.t.
$0 \le a_1 \le n-1,\; 0 \le a_2 \le n-2,\; \dots,\; 0 \le a_{n-1} \le 1,\; a_n = 0$
form a linear basis of the coinvariant alg $\mathbb{C}[x_1, \dots, x_n]/I_n$

Bernstein-Gelfand-Gelfand

Defined divided difference operators $\partial_i \cdot f \mapsto \frac{1}{x_i - x_{i+1}}(1 - s_i) f$

$w = s_{i_1} \cdots s_{i_\ell}$ reduced decomp

$\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$

Schubert basis is given by $\partial_{w^{-1}w_0}(f)$ ← Almost any poly of degree $\binom{n}{2}$ (Max degree)

Lascoux-Schützenberger

Schubert polynomials

$\mathfrak{S}_w = \partial_{w^{-1}w_0}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 x_n^0)$ ← x^δ

Properties

General! (1) $\{\mathfrak{S}_w, w \in S_n\}$ mod I_n is the linear basis of the coinvariant algebra given by Schub class [BGG]

Works only b/c we choose x^δ as top poly

(2) (nonnegativity) $\mathfrak{S}_w$ have nonneg. int. coeffs

(3) Stability $S_n \hookrightarrow S_{n+1}$

$w = w_1 \cdots w_n \mapsto \tilde{w} = w_1 \cdots w_n\, n+1$

then $\mathfrak{S}_w(x_1, \dots, x_n) = \mathfrak{S}_{\tilde{w}}(x_1, \dots, x_{n+1})$.

Side Remark: For type B Schub polys can get any 2 of above properties ~~but not~~ all 3

(A) $S_\lambda(x_1, \dots, x_k) = \mathfrak{S}_{w(\lambda)} = \partial_{w(\lambda)^{-1} w_0}(x^\delta)$ — Grassmanian perm

$\lambda = (\lambda_1, \dots, \lambda_k)$ $0 \le k \le n$

(B) $S_\mu(x_1, \dots, x_n) = \partial_{w_0}(x^{\lambda+\delta})$ $\quad \mu = \mu_1, \dots, \mu_n$

Claim: (A) $\Leftrightarrow$ (B)

Key: Realize $x^{\lambda+\delta} = \mathfrak{S}_u$ for some u in S_m $\quad m > n$

Q: Which $\mathfrak{S}_w$ are monomials, and which monomials are Schub polys?

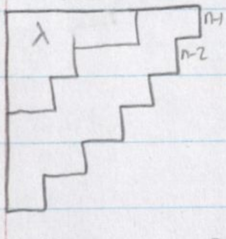
$$\begin{tikzcd}
 & \mathfrak{S}_{321} = x_1^2 x_2 \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & \\
\mathfrak{S}_{231} = x_1 x_2 \arrow[d, "\partial_2"'] & & \mathfrak{S}_{312} = x_1^2 \arrow[d, "\partial_1"] \\
\mathfrak{S}_{213} = x_1 \arrow[dr, "\partial_1"'] & & \mathfrak{S}_{132} = x_1 + x_2 \arrow[dl, "\partial_2"] \\
 & \mathfrak{S}_{123} = 1 &
\end{tikzcd}$$

Thrm: Any monomial $x^\lambda = x^{\lambda_1} \cdots x^{\lambda_n}$ s.t.

(1) x^λ divides x^δ (i.e. $0 \le \lambda_1 \le n-1,\ 0 \le \lambda_2 \le n-2,\ \dots,\ 0 \le \lambda_{n-1} \le 1,\ \lambda_n = 0$)

(2) $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n$ is a certain Schub poly $\mathfrak{S}_w$ for some $w \in S_n$

$\#$ Such monomials $= C_n = \frac{1}{n+1}\binom{2n}{n}$



$=$ # of Young diagrams that fit in staircase $\overset{\text{bij}}{\longleftrightarrow}$ Dyck paths

Q: When is $\partial_i(x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_n})$ a single monomial?

$$\partial_i (x_i^a x_{i+1}^b) =$$
$$= \begin{cases} 0 & \text{if } a=b \\ x_i^{a-1} x_{i+1}^{b} + x_i^{a-2} x_{i+1}^{b+1} \cdots x_i^{b} x_{i+1}^{a-1} & \text{if } a>b \\ -(x_i^{b-1} \cdot x_{i+1}^{a} + \ldots + x_i^{a} x_{i+1}^{b-1}) & \text{if } a<b \end{cases}$$

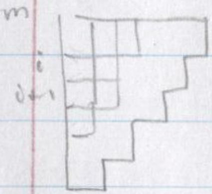
$\Rightarrow$ get single monomial iff $a = b+1$

$$\partial_i (x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}) = x_1^{\alpha_1} \cdots x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} \quad \text{if } \alpha_i = \alpha_{i+1}+1$$

$$\begin{tikzcd} & \mathfrak{S}_{321} = x_1^2 x_1 \arrow[dl] \arrow[dr] & \\ \mathfrak{S} \arrow[d] & & \mathfrak{S}_{312} \\ \mathfrak{S} \arrow[dr] & & \\ & \mathfrak{S} & \end{tikzcd}$$

Proof: Induct on $\binom{n}{2} - |\lambda|$

of Thrm



x^λ if If $\lambda \neq \delta$

find i s.t. $\lambda_i = \lambda_{i+1}$

$\mu = (\lambda_1, \lambda_{i+1}, \cdots \lambda_n)$ $\qquad (\lambda_{i-1} > \lambda_i$ or $\lambda = i)$

$\partial_i : x^\mu \to x^\lambda$

1

18.211 LECTURE 4 9/13

Last time: Any monomial $x^\lambda = x_1^\lambda \cdots x_n^\lambda$ s.t.

(1) $0 \leq \lambda_i \leq n-i$ for $i = 1, \ldots, n$

(2) $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda$

ie.

is a certain Schubert polynomial $\mathfrak{S}_w$, $w \in S^n$

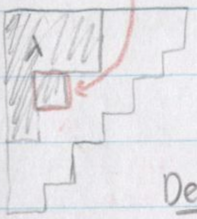
Q: $\lambda \overset{?}{\longleftrightarrow} w$

"special" divided difference:

$$\partial_i \cdot x^\lambda \longmapsto x_1^{\lambda_1} \cdots x_i^{\lambda_i - 1} \cdots x_n^{\lambda_n} \quad \text{if } \lambda_i = \lambda_{i+1} + 1$$

$$= x_1^{\lambda_1} \cdots x_i^{\lambda_{i+1}} x_{i+1}^{\lambda_{i+1}} \cdots x_n^{\lambda_n}$$

can remove



We can remove boxes is #(boxes in i^{th} row) = #(boxes in $i+1^{st}$ row) + 1

Def: Lehmer code of perm $w \in S_n$

$\text{code}(w) := (c_1, c_2, \ldots, c_n)$ s.t. $c_i = \#\{j > i \mid w_j < w_i\}$

E.x. $w = 15243$

$\text{code}(w) = (0, 3, 0, 1, 0)$

Lemma: The map $w \mapsto \text{code}(w)$ is a bij. between S_n & the set

$\{(c_1, \ldots, c_n) \mid 0 \leq c_i \leq n-i\}$

(To prove, just need a decoding alg. — Start at beginning and keep going)

Lemma: TFAE for $w \in S_n$

(A) $\text{Code}(w) = (c_1, \ldots, c_n)$ satisfies $c_1 \geq c_2 \geq \ldots \geq c_n$

(B) w is 132-avoiding ($\nexists\, i<j<k$ s.t. $w_i < w_k < w_j$)

E.x. $w = 21453$

does form inversion

$\text{code}(w) = (1, \overset{1}{0}, \underset{3}{1}, 1, \underset{2}{0})$

doesn't form inversion

Def: 132-avoiding perms are called dominant perms

Thrm: For a dominant perm $w \in S_n$, $\mathfrak{S}_w = x^{\text{code}(w)}$

This tells us everything since we know their are catalan many dominant perms & catalan many possible partitions

Recall: $\partial_i : f \longmapsto \frac{1}{x_i - x_{i+1}}(1 - s_i)(f)$

Satisfy (1)' $\partial_i^2 = 0$

(2) $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

(3) $\partial_i \partial_j = \partial_j \partial_i$ for $|i-j| \geq 2$

Nill coxeter relations

Def: The Demazure operators D_i (aka isoberic div. diff. operators)

$$D_i : f \mapsto \frac{f - \frac{x_{i+1}}{x_i} s_i(f)}{1 - \frac{x_{i+1}}{x_i}}$$

generalizable beyond type A

$f = f(x_1, \dots, x_n)$

$i = 1, \dots, n-1$

$$= \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}$$

$$= \partial_i (x_i f)$$

Lemma: D_i's satisfy

(1)'' $D_i^2 = D_i$

(2) $D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$

(3) $D_i D_j = D_j D_i$ for $|i-j| \geq 2$

zero Hecke relations

$\forall w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ (reduced)

$D_w = D_{i_1} D_{i_2} \cdots D_{i_\ell}$

Def: Key polynomials (aka Demazure characters)

$ch_{\lambda, w}(x_1, \dots, x_n) := D_w(x^\lambda)$

$\lambda = (\lambda_1, \dots, \lambda_n)$ partition, $w \in S^n$

Thrm: $w = w_0 \in S_n$

$ch_{\lambda, w_0} = S_\lambda(x_1, \dots, x_n)$ (Schur polynomial)

Ex. $n = 3$, $\lambda = (4, 2, 0)$

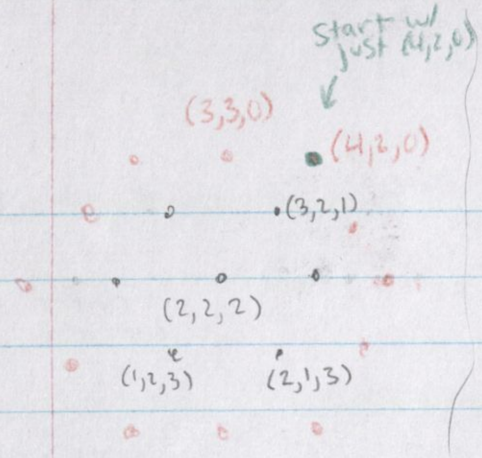
$S_{420}(x_1, x_2, x_3) = D_1 D_2 D_1 (x^\lambda)$

$\partial_i (x_i^a x_{i+1}^b) = x_i^{a-1} x_{i+1}^b + x_i^{a-2} x_{i+1}^{b+1} + \dots + x_i^b x_{i+1}^{a-1}$

$D_i (x_i^a x_{i+1}^b) = x_i^a x_{i+1}^b + x_i^{a-1} x_{i+1}^{b+1} + \dots + x_i^b x_{i+1}^a$

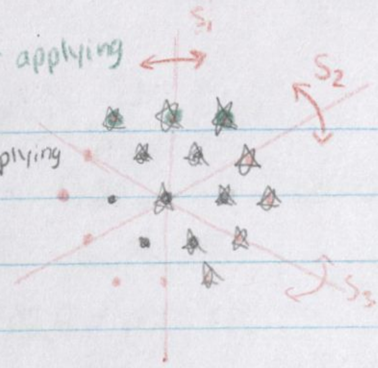
$x_1^a x_2^b x_3^c \rightsquigarrow (a, b, c) \in \mathbb{Z}^3$

$a + b + c = 4 + 2 + 0 = 6$

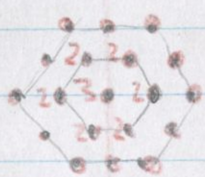


● After applying D_1

☆ After applying D_2 next



After applying the 2nd D_1



Prove: $S_\lambda = \partial_{w_0}(x^{\lambda+\delta})$

$\stackrel{?}{=} D_{w_0}(x^\lambda)$

Claim: $\forall f,\ \ \partial_{w_0}(x^\delta f) = D_{w_0}(f)$

E.x. $\partial_1 \partial_2 \partial_1 (x_1^2 x_2 f) \stackrel{?}{=} \partial_1(x_1(\partial_2(x_2(\partial_1(x_1 f)))))$