LECTURE 15 10/13

(Chevelley)-Monk's formula

[Monk] 1959 Monk's formula

[Bernstein-Gelfand-Gelfand] 1973, divided diff.

[Lasceux-Schützenburger] 1982 Schub. Polys

Several ways to formulate Monk's formula

Thrm: $x_i \mathfrak{S}_w \equiv \sum \pm \mathfrak{S}_{wt_{ij}}$ (mod I_n) $\boxed{t_{ij} = \text{transpose } i \,\&\, j}$

$I_n = \langle e_1, \dots, e_n \rangle$ ideal of sym. polys in $\mathbb{C}[x_1, \dots, x_n]$

Thrm': $\mathfrak{S}_{s_i} \mathfrak{S}_w \equiv \sum \mathfrak{S}_{wt_{ij}}$ (mod I_n)

More generally

$\ell(x) = \ell_1 x_1 + \dots + \ell_n x_n$ any linear form

Thrm'': $w \in S_n$,

$$\ell(x) \cdot \mathfrak{S}_w \equiv \sum_{\substack{i<j \\ \ell(wt_{ij}) = \ell(w)+1}} (\ell_i - \ell_j)\, \mathfrak{S}_{wt_{ij}} \pmod{I_n}$$

The coinvarient algebra

$C_n = \mathbb{C}[x_1, \dots, x_n] / I_n$

- $\dim C_n = n!$
- one linear basis of C_n is (the cosets of) $x_1^{a_1} \cdots x_n^{a_n}$ s.t. $0 \le a_i \le n-i \ \forall i$ } on pset
- Second lin. basis σ_w, cosets of $\mathfrak{S}_w$ (mod I_n)
- Third lin. basis

$$e_{i_1}(x_1)\, e_{i_2}(x_1, x_2)\, e_{i_3}(x_1, x_2, x_3) \cdots e_{i_{n-1}}(x_1, \dots, x_{n-1}) \qquad 0 \le i_k \le k$$

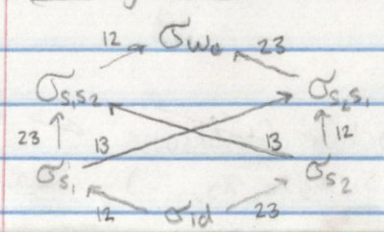
NOTE: Coeffs to express 2^{nd} basis in terms of first are same as coeff's to express 3^{rd} basis in terms of 2^{nd}

Operators on C_n

$$\underset{i<j}{T_{ij}} : \sigma_w \longmapsto \begin{cases} \sigma_{wt_{ij}} & \text{if } \ell(wt_{ij}) = \ell(w)+1 \\ 0 & \text{otherwise} \end{cases}$$

$T_{ji} = -T_{ij}$, $T_{ii} = 0$.

Strong Bruhat order on S_n



consider as directed graph (rather than partial order)

$X_i : f \pmod{I_n} \mapsto x_i f \pmod{I_n}$

~~Lemma~~: In terms of T_{ij} operators, can formulate Monk's formula as follows

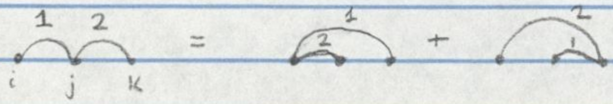
<u>Monk's Formula</u>: $X_i = \sum_{j=1}^{n} T_{ij}$

<u>Lemma</u> (Fomin, Kivillov) The operators T_{ij} satisfy the relations

(1) $T_{ij}^2 = 0$

(2) $T_{ij} T_{k\ell} = T_{k\ell} T_{ij}$ if $\#\{i,j,k,\ell\} = 4$ (i.e none are equal)

(3) $T_{ij} \cdot T_{jk} = T_{ik} \cdot T_{ij} + T_{jk} \, T_{ij}$



(4) $T_{jk} T_{ij} = T_{ij} \cdot T_{ik} + T_{ik} \cdot T_{jk}$

If you think about T_{ij}'s as formal generators of an algebra, Monk's formula as axiom, then you can use these to construct all of Schubert calculus axiomatically

<u>Lemma</u>: Fomin-Kivillov relations $\Rightarrow$ X_i's are commutative

[Subalg. generated by X_i is isomorphic to coinvarient algebra

Sum of all X_i's cancel (will have sum of all T_{ij}'s and T_{ji}'s, which cancel)

<u>Stability property of $\mathfrak{S}_w$</u>

$w = w_1 \cdots w_n \in S_n$ $\qquad\qquad S_n \hookrightarrow S_{n+1}$

$\tilde{w} = w_1 \cdots w_n \; n{+}1 \in S_{n+1}$ $\qquad\qquad w \mapsto \tilde{w}$

$\mathfrak{S}_w = \mathfrak{S}_{\tilde{w}}$ as we can see from pipe dream formula

$S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots$

$S_\infty = \varinjlim S_n$

infinite permutations $w = \begin{pmatrix} 1 & 2 & 3 & \cdots \\ w_1 & w_2 & w_3 & \cdots \end{pmatrix}$

s.t. $w_i = i$ for all $i > N$ ← some sufficiently large N

<u>$\mathfrak{S}_w$ for $w \in S_\infty$</u>

<u>Lemma</u>: $\mathfrak{S}_w$, $w \in S_\infty$ forms a lin. basis of $\mathbb{C}[x_1, \ldots, x_n]$

<u>Proof</u>: PSET problem 1: $\langle \mathfrak{S}_w \mid w \in S_n \rangle = \langle x_1^{a_1} \cdots x_n^{a_n} \mid 0 < a_i \leq n-i \rangle$

Take limit as $n \to \infty$

Compare with <u>Schur polynomials</u> vs. <u>Schur functions</u> ← NOT polynomials include infinitely many terms

$s_\lambda(x_1, \ldots, x_n)$ $\qquad$ $s_\lambda(x_1, x_2, x_3, \ldots)$ $\qquad$ Ex. $e_1 = x_1 + x_2 + \cdots$

Thrm: $\forall\, w \in S_\infty$ and any linear form

$$l(x) = l_1 x_1 + l_2(x_2) + \cdots \qquad (l_i = 0 \text{ for all sufficiently large } i)$$

$$l(x)\cdot \mathfrak{S}_w = \sum_{\substack{i<j \\ l(wt_{ij}) = l(w)+1}} (l_i - l_j)\, \mathfrak{S}_{wt_{ij}}$$

Lemma: If $w \in S_\infty$ but $w \notin S_n$ (i.e. permutes more than first n indices)

then $\mathfrak{S}_w \equiv 0 \pmod{\langle I_n, x_{n+1}, x_{n+2}, \ldots \rangle}$

Now have many equivalent formulations of Monk's formula, but how do we prove Monk's formula?

Key Ingredient:

Leibniz rule for divided diff

$$\partial_i(x_j f) = \underbrace{\partial_i(x_j)}_{= \begin{cases} 1 & \text{if } j = i \\ -1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}} \cdot f + x_{s_i(j)} \cdot \partial_i(f)$$

Using this rule repeatedly gives Monk's Formula

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Last Time: $\begin{pmatrix}\text{Monk's}\\\text{Formula}\end{pmatrix} \overset{?}{\Longleftrightarrow} \begin{pmatrix}\text{Divided}\\\text{differences}\end{pmatrix}$

~strong Bruhat order on S_n — ~weak Bruhat order on S_n

(strong) Bruhat: $u \lessdot w$ if $w = u t_{ij}$, $\ell(w) = \ell(u)+1$

$s_i = t_{i,i+1}$

(right) weak Bruhat: $u \lessdot w$ if $w = u s_i$ $\ell(w) = \ell(u)+1$

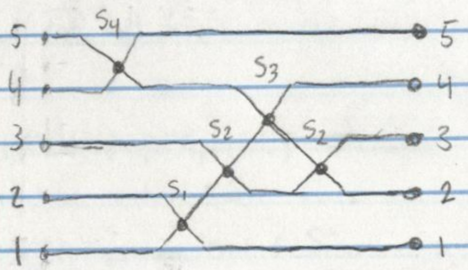
Note: For strong Bruhat order, right & left are the same
$w = u t_{ij} = t_{i'j'} u$ $i' = u(i)$, $j' = u(j)$ $\ell(w) = \ell(u)+1$ either way

Lemma: Fix a reduced decomp $w = s_{i_1} \cdots s_{i_\ell}$.
All permutations $u \in S_n$ covered by w in strong order $u \lessdot w$ have the following form

$u = s_{i_1} s_{i_2} \cdots \hat{s}_{i_k} \cdots s_{i_\ell}$ s.t. this is a reduced decomp of u

skip s_{i_k}

Ex. $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix}$



inversion set $Inv(w) =$
$\{(4,5), (1,2), (1,3), (1,5), (3,5)\}$

$w = s_2 s_3 s_2 s_1 s_4$

$u = s_2 s_3 \hat{s}_2 s_1 s_4$ below w

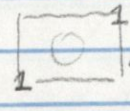
$u = s_2 \hat{s}_3 s_2 s_1 s_4$ is NOT b/c no longer reduced in this form

$\underbrace{s_2 \hat{s}_3 s_2}_{} \; = s_1 s_4$

In terms of perm matrices

$$w = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

inversion good if 1's form rectangle



everything inside is 0

inversion bad if there are 1's contained in the rectangle.

anything below w in Bruhat order

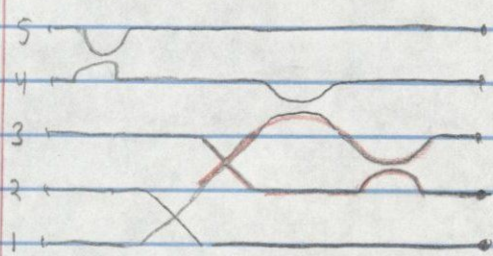
Cor: All Bruhat intervals $u \in [id, w]$ (in the strong order)
Fix a reduced decomp $w = s_{i_1} \cdots s_{i_\ell}$
Take subwords $J = \{j_1 < \ldots < j_2\} \subset [\ell]$
s.t. $u = s_{i_{j_1}} \cdots s_{i_{j_2}}$ is a reduced decomp.
This gives all $u \in [id, w]$

Note: In pipe dream, this corresponds to replacing some crossings by non-crossing in a way that doesn't make two wires cross more than once

Note: Word doesn't need to be reduced to be subward (if we reduce, then it is also a subward that is reduced now), but we don't need to consider these.

Def: March-Bietsch: All subwords whose wiring diagrams satisfy
- no double crossings 
- no 

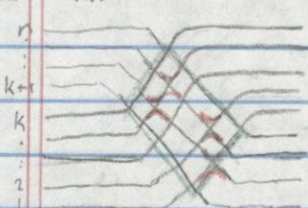
Ex.



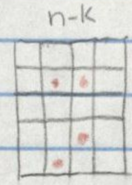
NOT allowed
But if we switched the order of the ✕ and ⌣ then it is allowed

Lemma: Such subwords of reduced decomp. are in bijection with elts of $[id, w]$

Ex. $w_{kn} = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ n-k+1 & \cdots & n & 1 & \cdots & n-k \end{pmatrix}$



$\curvearrowright 45°$

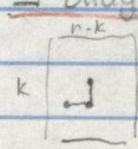


⅃-diagram

˪ letter "el" ⅃ letter "le"

← THIS IS WRONG HE FIXES DEF & Example next lecture

Thrm: Elts of Bruhat interval $[id, w_{kn}]$ are in bijection with ⅃-diagrams of shape $k \times (n-k)$



not allowed

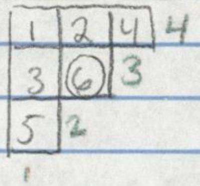
Reduced decomp $w = s_{i_1} \cdots s_{i_\ell}$ of $w \in S_n$
are in bijection with saturated chains from id to w in the weak Bruhat order
$id \lessdot s_{i_1} \lessdot s_{i_1} s_{i_2} \lessdot \cdots \lessdot s_{i_1} s_{i_2} \cdots s_{i_\ell} = w$

Thrm: (Stanley) For the longest perm $w_0 \in S_n$, # reduced decomp of w_0
= # SYT of the staircase shape $(n-1, n-2, \ldots, 1)$

Edelman-Greene gave a bijective proof

<u>Edelman-Greene's correspondence</u>

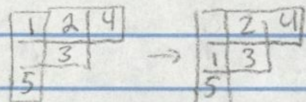
E.x. $n=4$



1.) • Find maximal entry

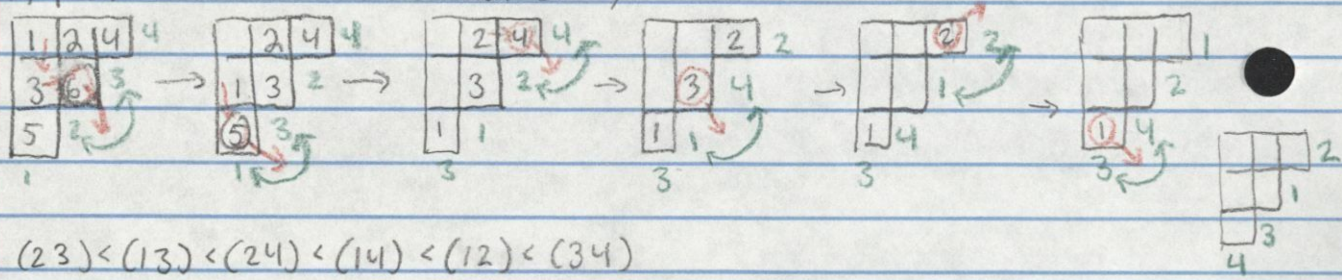
• Pull out of tableaux

• Jeu de taquin

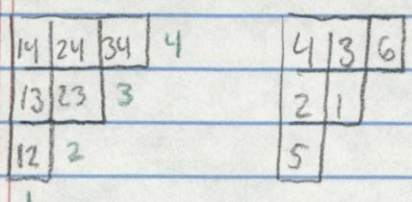


Keep sliding entries into empty box while making sure it remains a SYT at every step

2.) Repeat with new maximal entry



$(23) < (13) < (24) < (14) < (12) < (34)$



a balanced tableau

$\forall$ triple $i<j<k$

$(ij) < (ik) < (jk)$ or

$(jk) < (ik) < (ij)$

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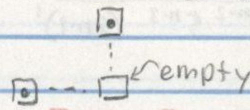
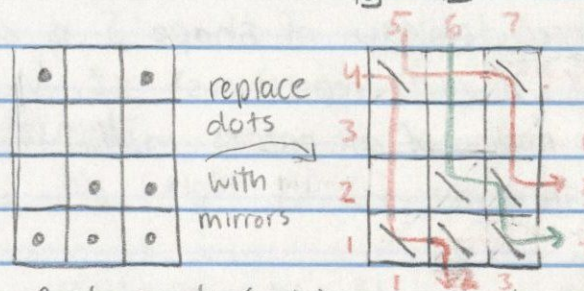
<u>Last Time</u>: Combinatorics of reduced decompositions
strong & weak Bruhat orders on S_n.

The correct definition of Ⅎ-diagrams
$k \times (n-k)$ rectangle with dots in some boxes
(equiv. $k \times (n-k)$ matrix w/ entries $\in \{0,1\}$)
that avoids the pattern



<u>Ex.</u> $k=4$, $n=7$



$$\longleftrightarrow u = \begin{pmatrix} 1&2&3&4&5&6&7 \\ 1&3&6&2&5&4&7 \end{pmatrix}$$

<u>Thm</u>: Ⅎ-diagrams of shape $k \times (n-k)$ are in bijection w/ permutations $u \in [id, w_{kn}]$ (interval in the strong order)

$$w_{kn} = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ n-k+1 & \cdots & n & 1 & \cdots & n-k \end{pmatrix}$$ (bijection illustrated above)

0-1 matrices that avoid the patterns

$$\begin{matrix} \boxed{1} & \cdots & \boxed{1} \\ \vdots & & \vdots \\ \boxed{1} & \cdots & \boxed{0} \end{matrix} \quad \& \quad \begin{matrix} \boxed{0} & \cdots & \boxed{1} \\ \vdots & & \vdots \\ \boxed{1} & \cdots & \boxed{0} \end{matrix}$$

# of these are =

0-1 matrices that avoid the patterns

$$\begin{matrix} \boxed{0} & \cdots & \boxed{1} \\ \vdots & & \vdots \\ \boxed{1} & \cdots & \boxed{0} \end{matrix} \quad \& \quad \begin{matrix} \boxed{1} & \cdots & \boxed{0} \\ \vdots & & \vdots \\ \boxed{0} & \cdots & \boxed{1} \end{matrix}$$

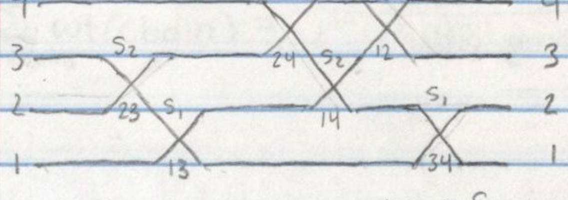
<u>Lemma</u>: For the longest perm. $w_0 \in S_n$

$$\left\{\begin{matrix}\text{reduced} \\ \text{decomp of } w_0\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{max chains in the} \\ \text{weak Bruhat order}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{Proper inversion} \\ \text{orderings of } Inv(w_0)\end{matrix}\right\}$$

$$Inv(w_0) = \{(i,j) \mid 1 \le i < j \le n, \ w_i > w_j\}$$

<u>Ex.</u> $n=4$



<u>reduced decomp</u>: $w_0 = s_1 s_3 s_2 s_3 s_1 s_2$

<u>Max chain</u>: $id <\cdot s_1 <\cdot s_1 s_2 <\cdot \ldots$

<u>Inversion ordering</u> $(23) < (13) < (24) < (14) < (12) < (34)$

$t_{a_1b_1}$ $t_{a_2b_2}$ $t_{a_3b_3}$

$$w_0 = s_{i_1} \cdots s_{i_{\ell-1}} s_{i_\ell}$$

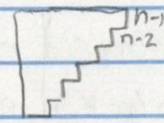
$$w_0 = (s_{i_\ell})(s_{i_\ell} s_{i_{\ell-1}} s_{i_\ell})(s_{i_\ell} s_{i_{\ell-1}} s_{i_{\ell-2}} s_{i_{\ell-1}} s_{i_\ell}) \cdots (s_{i_\ell} s_{i_{\ell-1}} \cdots s_0 s_1 s_2 \cdots s_{i_{\ell-1}} s_{i_\ell})$$

$\rightsquigarrow (a_1 b_1) < (a_2 b_2) < (a_3 b_3) < \cdots$

<u>Thm</u>: (Stanley. Bijective proof Edelman-Greene)

\# reduced decomp of $w_0 \in S_n$ =

= \# SYT's of shape $\delta = (n-1, n-2, \ldots, 1)$



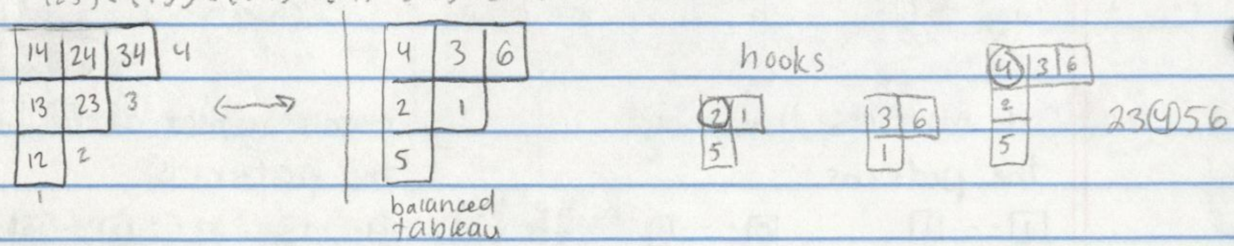
$\left(= \dfrac{\binom{n}{2}!}{1^{n-1} \cdot 3^{n-2} \cdot 5^{n-3} \cdots (2n-1)^1} \right.$ via hook length formula $\left. \right)$.

<u>Def</u> [E-G] A <u>balanced tableau</u> of shape δ is a filling of boxes of δ by $1, 2, \ldots, \binom{n}{2}$ (w/o repititions) s.t. $\forall$ hooks H_a, the entry at box a is the median of all entries in H_a

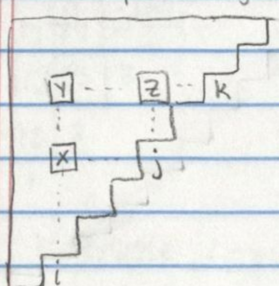
(In SYT it is the minimum)

<u>Thrm</u> [E-G] The Edelman-Greene's correspondences (that we constructed last class) gives a bijection between SYT's of shape δ & balanced tableaux of shape δ

$(23) < (13) < (24) < (14) < (12) < (13)$



$\forall$ triple $i<j<k$



$x<y<z$

or $x>y>z$

<u>Weighted chain enumeration</u>

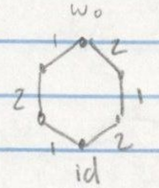
<u>Macdonald's formula</u> weak order $w \lessdot ws_i$ has weight i

<u>Stembridge's formula</u> strong order

Macdonald's formula

edge $w <\cdot\, ws_i$ has weight i

$n=3$



$1\cdot2\cdot1 + 2\cdot1\cdot2 = 3!$

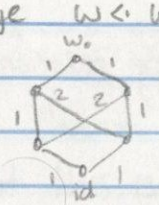
Macdonald:

$$\sum_{\substack{w_0 = s_{i_1}\cdots s_{i_\ell} \\ \text{(reduced)}}} i_1 \cdot i_2 \cdot \cdots \cdot i_\ell = \binom{n}{2}!$$

Stembridge's formula

edge $w <\cdot\, wt_{ij}$ has weight $j-i$

$n=3$



$1\cdot1\cdot1 + 1\cdot1\cdot1 + 1\cdot2\cdot1 + 1\cdot2\cdot1 = 3!$

Stembridge:

$$\sum_{\substack{\text{max chains in} \\ \text{the strong order} \\ \text{on } S_n}} wt(\text{chain}) = \binom{n}{2}!$$

$wt(\text{chain}) =$ product of edge weights

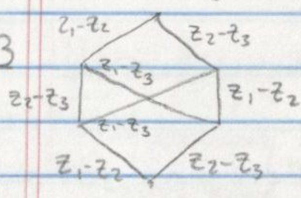
- Is there bijection showing equivalence of these formulas?
- Yes, same idea as divided diff. vs. Monks formula

Variable $z_1, z_2, \ldots, z_n$

Covering rel $w <\cdot\, wt_{ij}$ has $z_i - z_j$

is the strong order

$n=3$



Thrm (Stembridge)

$$\sum_{\text{max chains}} wt(\text{chain}) = \frac{\binom{n}{2}!\prod_{i<j}(z_i - z_j)}{1^{n-1}\cdot 2^{n-2}\, 3^{n-3} \cdots (n-1)^1}$$