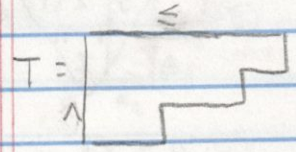
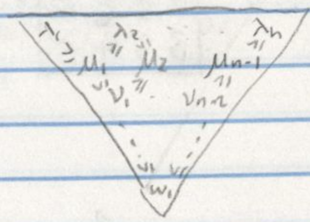
LECTURE 10 9/29

SSYT $\xleftrightarrow{\text{bij}}$ Gelfand Tsetlin Patterns



$\longrightarrow$ $\tilde{T}$ =



shape $= (\lambda_1, \ldots, \lambda_n)$
entries $\in \{1, \ldots, n\}$

<u>Rule</u>: i^{th} row of $\tilde{T}$ from the bottom
= shape formed by entries $\leq i$ in T

<u>Ex.</u> T =

1	1	1	2	2	5	5
2	2	3	3	3		
3	4	4	5			
5	5					

$\rightsquigarrow$

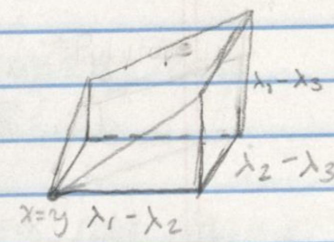
$$\begin{array}{ccccccccc} 7 & & 5 & & 4 & & 2 & & 0 \\ & 5 & & 5 & & 3 & & 0 & \\ & & 5 & & 5 & & 1 & & \\ & & & 4 & & 2 & & & \\ & & & & 3 & & & & \end{array}$$

Can think of GT patterns using polytopes

<u>Ex.</u> $n=3$

$$\begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & x & & y & \\ & & z & & \end{array}$$

$$\begin{cases} \lambda_2 \leq x \leq \lambda_1 \\ \lambda_3 \leq y \leq \lambda_2 \\ y \leq z \leq x \end{cases}$$

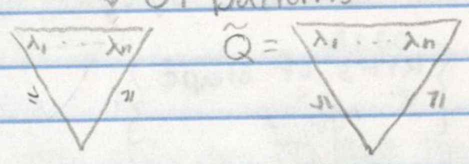


Integer pts of polytope correspond to SSYT

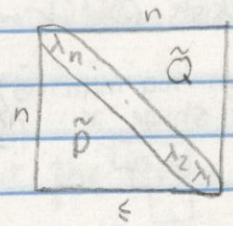
<u>Volume of polytope in $\mathbb{R}^3$</u>
$= (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)/2$

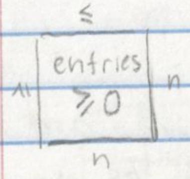
<u>RSK</u> (P,Q) SSYT's of shape $\lambda = (\lambda_1, \ldots, \lambda_n)$
$\updownarrow$ GT-patterns

$\tilde{P}$ =



$\tilde{Q}$ = $\xrightarrow{\text{RSK}}$





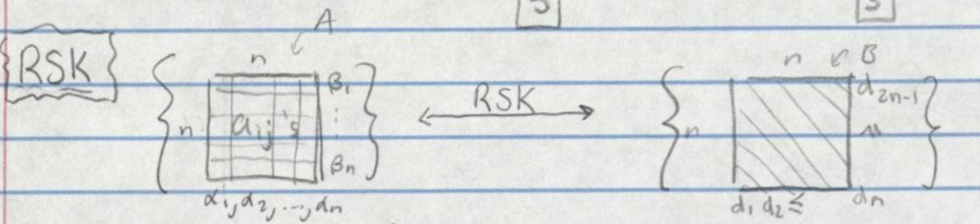
Reverse plane partitions
(RPP's)

Ex. From RPP get back P & Q

An RPP $\begin{array}{|cccc|} 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ 2 & 2 & 3 & 5 \\ 3 & 3 & 4 & 5 \end{array} \longrightarrow \tilde{P} = \begin{array}{cccc} 5 & 3 & 1 & 0 \\ & 4 & 2 & 1 \\ & 3 & 2 & \\ & & 3 & \end{array}, \quad \tilde{Q} = \begin{array}{cccc} 5 & 3 & 1 & 0 \\ & 5 & 2 & 1 \\ & 4 & 1 & \\ & & 2 & \end{array}$

$P = \begin{array}{ccccc} 1 & 1 & 1 & 3 & 4 \\ 2 & 2 & 4 & & \\ 3 & & & & \end{array} \qquad Q = \begin{array}{ccccc} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & & \\ 3 & & & & \end{array}$

RSK



non-neg int $n \times n$ matrices

Colum sums $\alpha_1, \ldots, \alpha_n \longleftrightarrow$ diag sums of B

Row sums $\beta_1, \ldots, \beta_n$ $\qquad d_1, d_2, \ldots, d_{2n-1}$

$d_1 = \alpha_1$

$d_2 = \alpha_1 + \alpha_2$

$\vdots$

$d_n = \alpha_1 + \ldots + \alpha_n = \beta_1 + \ldots + \beta_n$

$d_{n+1} = \beta_1 + \beta_2 + \ldots + \beta_{n-1}$

$\vdots$

$d_{2n-1} = \beta_1$

* Transpose of matrix A gives transpose of matrix B

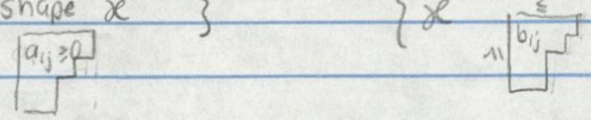
$\Rightarrow$ switch P & Q.

A generalization of RSK

$\varkappa$ be any Young diagram

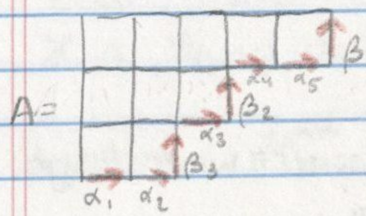
Thrm: There exists (and we will give explicitly) a certain bijection $\varphi_\varkappa$

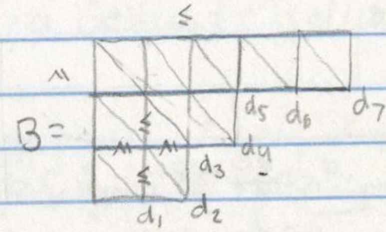
$\left\{ \begin{array}{l} \text{non-neg matrices} \\ \text{of shape } \varkappa \end{array} \right\} \xrightarrow{\varphi_\varkappa} \left\{ \begin{array}{l} \text{RPP's of shape} \\ \varkappa \end{array} \right\}$ satisfying



- $\varphi_\varkappa$ a piecewise linear map
- column sums $\alpha_1, \ldots, \alpha_n$ & row sums $\beta_1, \ldots, \beta_n$ of A are related to diagonal sums $d_1, d_2, \ldots, d_{m+n-1}$ of B as shown on top of next page
- Symmetry. If $\varphi_\varkappa : A \mapsto B$

  Then $\varphi_{\varkappa'} : A^T \mapsto B^T$

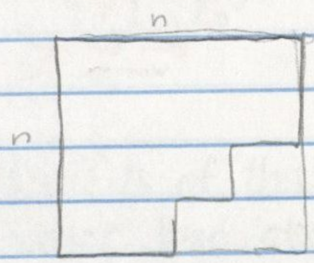




$$\begin{cases} d_1 = \alpha_1 \\ d_2 = \alpha_1 + \alpha_2 \\ d_3 = \alpha_1 + \alpha_2 - \beta_3 \quad \cdot \quad \text{etc} \\ d_4 = \alpha_1 + \alpha_2 - \beta_3 + \alpha_3 \end{cases}$$

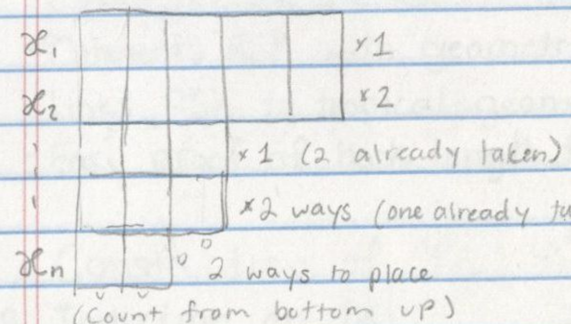
<u>Special case</u>: Rook placements & oscillating Tableaux

$$\alpha_1 = \dots = \alpha_n = \beta_1 = \dots = \beta_n = 1$$



permutation matrix whose nonzero entries fit in Young Diagram of shape λ.

Aka ways to place n non-attacking rooks in shape λ. (Rook placements)



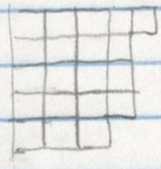
$2 \cdot 2 \cdot 1 \cdot 2 \cdot 1 = 8$ rook placements

→ UUDUUDDUU

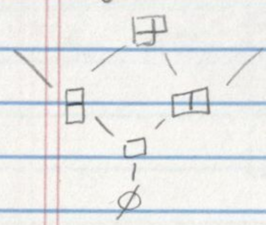
In general # rook placements $= \lambda_n(\lambda_{n-1}-1)(\lambda_{n-2}-2)\cdots(\lambda_1-n+1)$ $\boxed{\text{Using row sums}}$

ALSO $= \lambda'_n(\lambda'_{n-1}-1) \cdots (\lambda'_1-n+1)$ $\boxed{\text{column sums}}$

Claim: Not only are products the same, but the terms in the products are permutations of each other

E.x.  $3 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 1 \cdot 3 \cdot 3 \cdot 2 \cdot 1$

Young's Lattice



start at empty shape Count # ways to go up or down at each step

UUDUDDUUDD

$1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1$

$$(2n-1)!! = (2n-1)(2n-3)\cdots 3 \cdot 1$$

LECTURE 11 10/2

Last Time: RSK (via Gelfand–Tsetlin patterns)

$\varkappa$ a Young diagram

bijection $\varphi_{\varkappa}^{RSK} : \{ a_{ij} \geq 0 \} \xrightarrow{\sim} \{ b_{ij} \}$

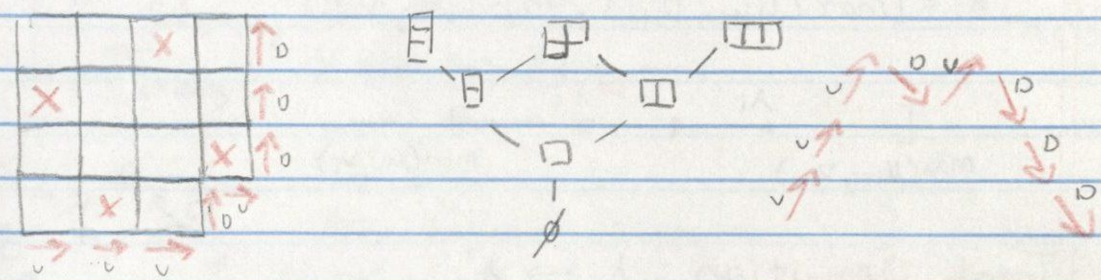
non-neg int. matrices of shape $\varkappa$ → RPP's of shape $\varkappa$

row & column sums ⟷ diagonal sums

Classical RSK: The case where $\varkappa = n \times n$

Special Case

$\{\text{Rook placements of shape } \varkappa\} \xrightarrow{\sim} \{\text{oscillating Tableaux}\}$



Benifits of this construction

- Easier than classical construction of RSK
- Easier to see symmetry & other nice properties
- Connects RSK with geometry of convex polytopes
- Links RSK to tropical geometry & cluster algebras
- Easy proof of hook length formula

Construction of $\varphi_{\varkappa} = \varphi_{\varkappa}^{RSK}$ bijection

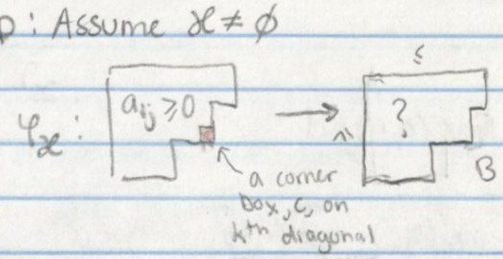
By Induction on $|\varkappa|$

Base: $\varkappa = \emptyset$

$\varphi_{\emptyset} : \{\emptyset\} \longrightarrow \{\emptyset\}$, $\emptyset \mapsto \emptyset$

set of single element, whose single elt is empty matrix

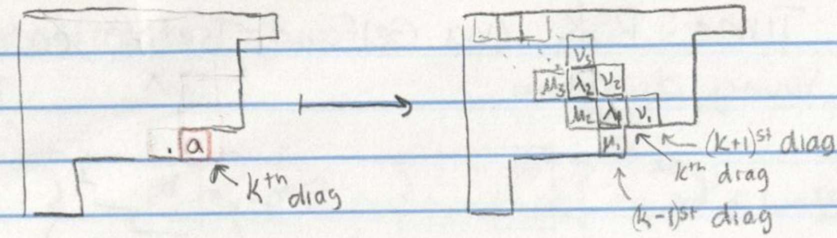
Ind Step: Assume $\varkappa \neq \emptyset$



$\tilde{\varkappa} = \varkappa \setminus \{c\}$

$\tilde{A} = A$ without entry in box c

$\varphi_{\tilde{x}}$:

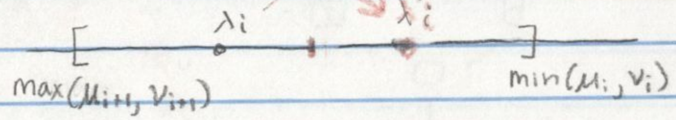


$(k-1)^{st}$ diag $\quad \mu_1 \quad\quad \mu_2 \geq \quad \geq \mu_3 \; \ldots$

k^{th} diag $\quad\quad \lambda_1 \quad\quad \lambda_2 \; \ldots$

$(k+1)^{st}$ diag $\quad \nu_1 \quad\quad \nu_2 \quad\quad \nu_3 \; \ldots$

$$\lambda_i \in [\max(\mu_{i+1}, \nu_{i+1}), \min(\mu_i, \nu_i)]$$



Toggle Operation : $\lambda_i \to \lambda_i^*$

$$\boxed{\lambda_i^* = \min(\mu_i, \nu_i) + \max(\mu_{i+1}, \nu_{i+1}) - \lambda_i}$$

i.e reflect λ over center of interval

$\lambda_0^* = \max(\mu_1, \nu_1) + a \longleftarrow$ entry of matrix A in corner box c.

Step: B is obtained from $\tilde{B}$ by replacing the diag $\lambda_1, \lambda_2, \ldots$ with $\lambda_0^*, \lambda_1^*, \lambda_2^*, \ldots$

Ex. $A = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & \\ \cline{1-2} f & & \\ \cline{1-1} \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline ? & ? & ? \\ \hline ? & ? & \\ \cline{1-2} ? & & \\ \cline{1-1} \end{array}$

$\boxed{a} \longmapsto \boxed{a}$ $\quad a = \max(0,0) + a$

$\boxed{a \mid b} \longmapsto \boxed{a \mid a+b}$ $\quad a+b = \max(a,0) + b$

$\begin{array}{|c|c|} \hline a & b \\ \hline d & \\ \cline{1-1} \end{array} \longmapsto \begin{array}{|c|c|} \hline a & a+b \\ \hline a+d & \\ \cline{1-1} \end{array}$ $\quad a+d = \max(a,0) + d$

$\begin{array}{|c|c|} \hline a & b \\ \hline d & e \\ \hline \end{array} \longmapsto \begin{array}{|c|c|} \hline \max(b,d) & a+b \\ \hline a+d & \\ \hline \end{array}$ $\quad \max(b,d) = \max(0,0) + \min(a+b, a+d) - a$

$\max(b,d) + a + e = \max(a+b, a+d) + e$

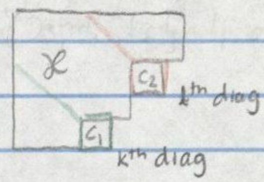
$$\begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & \\ \hline f & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline \min(b,d) & a+b & a+b+c \\ \hline a+d & \max(b,d)+a+e & \\ \hline a+d+f & & \\ \hline \end{array}$$

(both the last 2 insertions)

Whole construction is symmetric $\Rightarrow$ get symmetry

<u>Lemma</u>: The map $\varphi_{\mathcal{X}}$ is independent of order we add boxes.

<u>Proof</u>:



$\tilde{\mathcal{X}} = \mathcal{X} - c_1$

$\tilde{\tilde{\mathcal{X}}} = \mathcal{X} - c_2$

$\tilde{\tilde{\tilde{\mathcal{X}}}} = \mathcal{X} - \{c_1, c_2\}$

$$\begin{tikzcd} & \mathcal{X} & \\ \tilde{\mathcal{X}} \arrow[ur, "c_1"] & & \tilde{\tilde{\mathcal{X}}} \arrow[ul, "c_2"'] \\ & \tilde{\tilde{\tilde{\mathcal{X}}}} \arrow[ul, "c_2"] \arrow[ur, "c_1"'] & \\ \emptyset \arrow[ur, squiggly] & & \end{tikzcd}$$

NTS diagram commutes

We always have $|k-\ell| \geq 2$

(otherwise boxes would be adjacent, so the same one would have had to be added first)

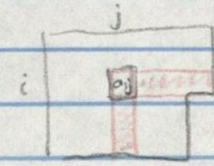
Then toggles don't affect each other because each toggle only depends on neighboring diagonals

<u>Relation to Hook Length Formula</u>

<u>Lemma</u>: $\varphi_{\mathcal{X}} : A \longmapsto B$ (RPP)

$\sum b_{ij} = \sum h_{ij} a_{ij}$

h_{ij} = hook length of box $(i,j) \in \mathcal{X}$



LECTURE 12 10/4

Hook Length formula

λ Young Diagram

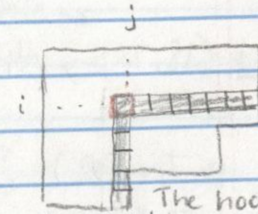
$f^\lambda :=$ SSYT of shape λ

Thrm: Frame-Robinson-Thrall

$$\boxed{f^\lambda = \frac{N!}{\prod_{(i,j)\in\lambda} h_{ij}}}$$

$N = |\lambda|$

h_{ij} are the hook lengths



The hook at (i,j)

h_{ij} = # boxes in hook

Ex.

1	3	5..4
2	4..5	

2 options

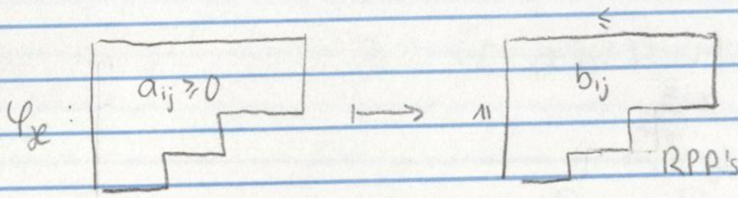
1	2	3
3		

3 options

And indeed $f^\lambda = \frac{5!}{1\cdot 1\cdot 2\cdot 3\cdot 4} = 5$

- The original proof did not explain the role of the hook lengths
- More probabalistic "probabilistic hook walk" Proof by Greene-Nijenhuis-Wilf
- We will do a proof using RSK

RSK



From last Time

Ex.

a	b	c
d	e	

$\longrightarrow$

min(b,d)	a+b	a+b+c
a+d	max(b,d) +a+e	

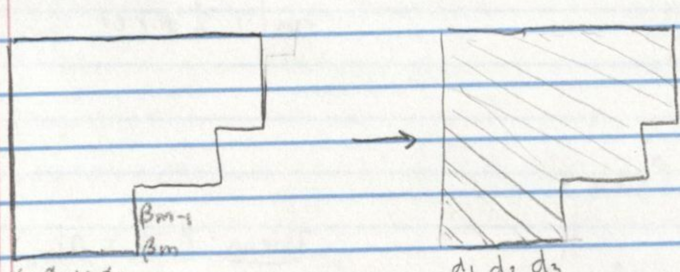
Claim: This is a bijective piece-wise linear map which is volume preserving.

Proof Sketch: Why does it preserve volume?

Each toggle has 2 areas of linearity. On each area of linearity it has det = 1.

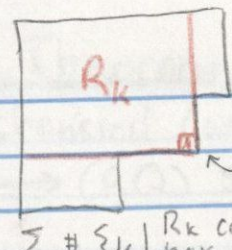
$$\sum b_{ij} = b+d + a+b + a+b+c + a+d + a+e = 4a+3b+c+2d+e$$

Lemma: $\sum_{(i,j)\in\chi} b_{ij} = \sum_{(i,j)\in\chi} h_{ij}a_{ij}$ ↖weighted by hook lengths



$$\begin{cases} d_1 = \alpha_1 \\ d_2 = \alpha_1 + \alpha_2 \\ \vdots \\ d_s = \alpha_1 + \cdots + \alpha_s \\ d_{s+1} = \alpha_1 + \cdots + \alpha_s - \beta_m \\ d_{s+2} = \alpha_1 + \cdots + \alpha_s - \beta_m - \beta_{m-1} \end{cases}$$

$$d_k = \sum_{(i,j)\in R_k} a_{ij}$$

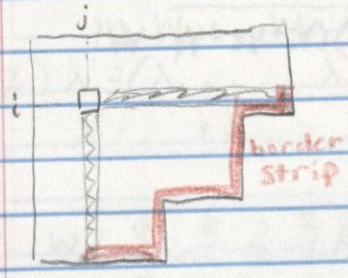


$$\sum b_{ij} = \sum d_i = \sum_{(i,j)\in\varkappa} \#\left\{k \,\middle|\, \begin{array}{l} R_k \text{ contains the} \\ \text{box } (i,j) \end{array}\right\} a_{ij}$$

Q: What is # of such rectangles R_k?

A: = # of boxes on border strip (each box gives such an R_k)

= hook length of (i,j)



Proof (of hook length formula):

$\varphi_\varkappa$ is a volume preserving map between the following 2 polytopes:

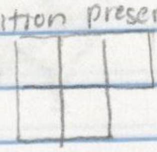
$$P_\varkappa = \left\{ \begin{array}{l} A=(a_{ij}) \\ \text{of shape } \varkappa \end{array} \,\middle|\, \begin{array}{l} a_{ij}\geq 0 \\ \sum h_{ij}a_{ij}\leq 1 \end{array} \right\} \subset \mathbb{R}^N$$

$$Q_\varkappa = \left\{ \begin{array}{l} B=(b_{ij}) \\ \text{of shape } \varkappa \end{array} \,\middle|\, \begin{array}{l} \mathbb{R} \text{ valued RPP's} \\ \sum b_{ij}\leq 1 \end{array} \right\} \subset \mathbb{R}^N$$

Claim: $P_\varkappa$ & $Q_\varkappa$ has the same volume (b/c $\varphi_\varkappa$ is volume preserving as we argued before)

Note: Bdry condition preserved under $\varphi_\varkappa$ because $\sum h_{ij}a_{ij} = \sum b_{ij}$.

Ex. $\varkappa =$



$$P_\varkappa = \{(a_{11},a_{12},a_{13},a_{21},a_{22})\in\mathbb{R}^5,\ a_{ij}\geq 0,\ 5a_{11}+3a_{12}+a_{13}+2a_{21}+a_{22}\leq 1\}$$

$$Q_\varkappa = \left\{ \begin{array}{ccccc} 0 & & & & \\ \rotatebox{90}{$\leq$} & & & & \\ b_{11} & \leq & b_{12} & \leq & b_{13} \\ \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \\ b_{21} & \leq & b_{22} & & \end{array},\ \sum b_{ij}\leq 1 \right\}$$

$$\mathrm{Vol}\, Q_\varkappa = f^\varkappa \cdot \mathrm{Vol}\{(x_1,\dots,x_N)\in\mathbb{R}^N \mid 0\leq x_1\leq \dots \leq x_N,\ x_1+\dots+x_N\leq 1\}$$

Rescale along each x_i to pull this out

$$\mathrm{Vol}\, P_\varkappa = \frac{1}{\prod h_{ij}} \mathrm{Vol}\left((\tilde{a}_{ij}) \,\middle|\, \begin{array}{l} \tilde{a}_{ij}\geq 0 \\ \sum \tilde{a}_{ij}\leq 1 \end{array}\right\} \qquad \tilde{a}_{ij} = h_{ij}a_{ij}$$

for ordering the x_i

$$= \frac{N!}{\prod h_{ij}} \mathrm{Vol}\left((x_1,\dots,x_n)\in\mathbb{R}^N \,\middle|\, \begin{array}{l} 0\leq x_1\leq\dots\leq x_n \\ \sum x_i\leq 1 \end{array}\right\}$$

$$\mathrm{Vol}\, Q_\varkappa = \mathrm{Vol}\, P_\varkappa \Rightarrow f^\varkappa = \frac{N!}{\prod h_{ij}}$$

Multivariable Hook Length Formula (Weighted Diagonals)

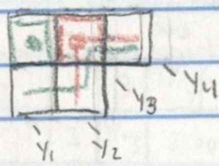
1	2	3
4	5	

y_1 y_2 y_3 y_4 y_2

$$\rightsquigarrow \frac{1}{y_2(y_2+y_4)(y_2+y_4+y_1)(y_2+y_4+y_1+y_3)} \qquad (y_2+y_4+y_3+y_2)$$

Thrm: $\sum_{T\in SYT(\lambda)} wt(T) = \prod_{(i,j)\in\lambda} \frac{1}{h^y_{ij}}$

where $h^y_{ij} = \sum_{\substack{k \text{ are indices} \\ \text{of boxes in the} \\ \text{border strip for} \\ \text{box } (i,j)}} y_k$

Ex. 

$$\frac{1}{y_2(y_1+y_2)\,y_4(y_2+y_3+y_4)(y_1+y_2+y_3+y_4)}$$

$w_1 \cdots w_n \longmapsto (P,Q)$ SYT of shape λ

$\lambda_1 =$ max size of incr. subseq. in $w_1,\dots,w_K$

Thrm: (Greene's Thrm): $\forall k$

$\lambda_1 + \cdots + \lambda_k = \max(j \mid$ w has a subseq of size j that can be covered by k incr. subseq$)$

$w = 3\ 1\ 5\ 2\ 7\ 6\ 4$

$\lambda_1 = 3$

$\lambda_1 + \lambda_2 = 6$

LECTURE 13 10/6

Greene's Theorem

Robinson-Schensted Correspondence

perm $w \in S_n \longmapsto (P, Q)$ SYT of the same shape $\lambda = (\lambda_1, \dots, \lambda_\ell)$
called the (Schensted) shape of the permutation

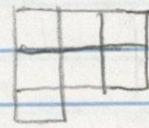
Thrm (Greene's Thrm) For $r = 1, 2, \dots$

1.) $\lambda_1 + \dots + \lambda_r$ is max size of the union of r disjoint incr. subseq. in $w_1, \dots, w_n$

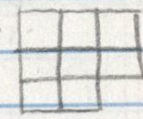
2.) $\lambda' = (\lambda'_1, \dots, \lambda'_s)$ the conjuge partition
$\lambda'_1 + \dots + \lambda'_r$ = max size of union of r disjoint decreasing subseq.

Ex. $w = 3\,1\,5\,2\,7\,6\,4$

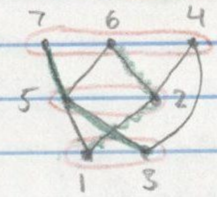
$\lambda_1 = 3$, $\lambda_1 + \lambda_2 = 6$, $\lambda_1 + \lambda_2 + \lambda_3 = 7$

$\lambda = (3,3,1) =$ 

$\lambda'_1 = 3$, $\lambda'_1 + \lambda'_2 = 5$, $\lambda'_1 + \lambda'_2 + \lambda'_3 = 7$

$\lambda' = (3,2,2) =$ 

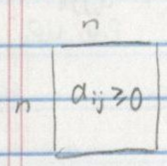
P_w poset on $[n]$: $w_i < w_j$ if $i < j$ (i.e. if these elts are not transposed)



In general, naive way to calculate λ (i.e. find longest incr. subseq, remove it, find new longest, remove it, etc) does NOT work

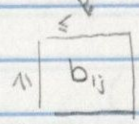
Exercise: Find an example where it fails.

Generalization to RSK



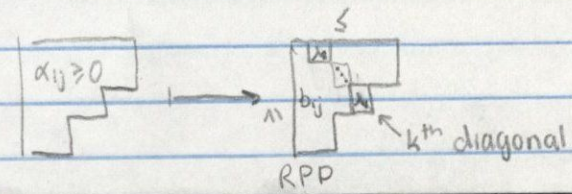
$\xmapsto{RSK} (P, Q)$ SSYT's of same shape λ

$\}$ GT-patterns



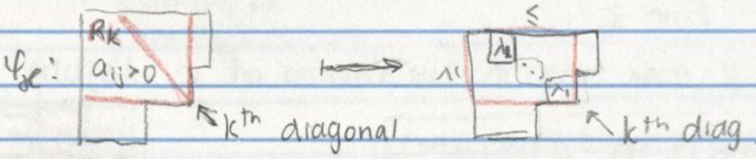
RPP
reverse plane partition

Diagonal is $\lambda_n, \dots, \lambda_2, \lambda_1$

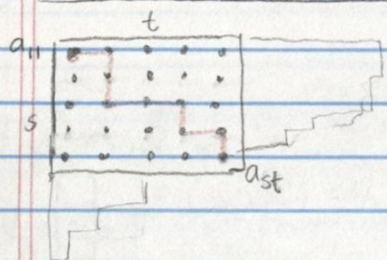
φ_k : 

Greene's theorem should be a non-recursive definition for this correspondence.

Strategy: First fill in boxes in whole rectangle. That already gives λ.

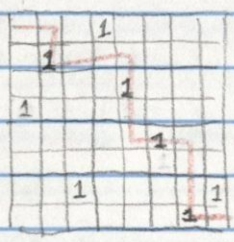


Generalized Greene's Theorem

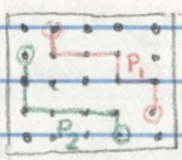


$$\lambda_{11} = \max_{\substack{p \text{ lattice path} \\ \text{from } a_{11} \text{ to } a_{st}}} \left(\sum_{(i,j)\in p} a_{ij} \right)$$

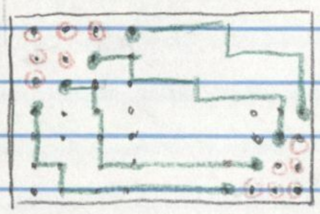
Ex.



$\lambda_1 = 4$

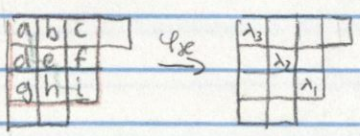


$$\lambda_1 + \lambda_2 = \max_{P_1, P_2} \left(a_{11} + a_{st} + \sum_{(i,j)\in P_1 \cup P_2} a_{ij} \right)$$



$$\lambda_1 + \dots + \lambda_k = \max_{\substack{k\text{-tuple of} \\ \text{non-crossing} \\ \text{lattice path}}} \left(\text{circled } a_{ij}\text{'s} + \sum_{\substack{(i,j)\in \\ P_1 \cup \dots \cup P_k}} a_{ij} \right)$$

Ex.



$\lambda_1 = \max(a+b+c+f+i,\ a+b+e+f+i,\ \dots)$

$\lambda_1 + \lambda_2 = a+i+\max(b+c+f+d+e+h,\ b+c+f+d+g+h,\ \dots)$

$\lambda_1 + \lambda_2 + \lambda_3 = a+b+c \dots + i$

Can use tropical geometry for this

(subtraction-free) Rational Calculus	Tropical Calculus ← "calucluls"
$A \cdot B$	$a+b$
A/B	$a-b$
$A+B$	$\max(a,b)$
const.	0
$A+B(C+D/E)$ →	$\max(a, b+\max(c, d-e))$

Lemma: Given any expression (w/out minus signs), the tropicalization is well defined.
i.e. whatever way you write expression, after tropicalizing, you still get equivalent expressions.

Check on: $A = c_1 t^a + c_2 t^{a-1} + \ldots$ $\qquad t^a \rightsquigarrow at$, constants go away
$c_1 > 0,\ c_2, c_3, \ldots \geq 0$

$$\frac{A^2-B^2}{A-B} = A+B \xrightarrow{\text{tropicalize}} \max(A,B)$$
$$= \frac{2A^2+3AB+B^2}{2A+B} \rightsquigarrow \max(2a, a+b, 2b) - \max(a,b)$$

Inverse operation not well defined in general.

We want to reverse-tropicalize our expression for $\lambda_1 + \ldots + \lambda_k$