18.217 (OPTIONAL) PROBLEM SET 3

due Wednesday, December 14, 2022

This is an optional problem set. Solve and submit as many problems as you like. This problem set will not affect your final letter grades for the course. But it might affect modifiers (+/−) after letter grades.

Problem 1. Let $SSYT(\lambda/\mu, \beta)$ be the set of semi-standard Young tableaux of shape λ/μ and weight β.

In class, we discussed how to construct a bijection between the sets $SSYT(\lambda/\mu, (k, \beta_1, \ldots, \beta_l)$ and $SSYT(\lambda/\mu, (\beta_1, \ldots, \beta_l, k)$ by performing k jeu-de-taquin evacuations.

(a) Check that this construction works correctly and indeed gives a bijection. In particular, you need to check that that the jeu-de-taquin evacuations paths are non-crossing.

(b) We can construct another bijection between between the sets $SSYT(\lambda/\mu, (k, \beta_1, \ldots, \beta_l)$ and $SSYT(\lambda/\mu, (\beta_1, \ldots, \beta_l, k)$ by performing l Bender-Knuth involutions $\sigma_1, \sigma_2, \ldots, \sigma_l$. Does the bijection based on jeu-de-taquin evacuation coincide with the bijection based on Bender-Knuth involutions?

Problem 2. A *Littlewood-Richadson tableau* T is a semi-standard Young tableau of a skew shape λ/μ whose semitic (or Hebrew) reading word (i.e., the word obtained reading the entries of T by rows right-to-left, top-to-bottom) is a Yamanouchi word.

(a) Prove that a semi-standard tableau T is a Littlewood-Richardson tableau if and only if its Chinese reading word (i.e., the word obtained reading the entries of T by columns top-to-bottom, right-to-left) is a Yamanouchi word.

(b) In this part, you will characterise all reading words that have the same property as semitic and Chinese reading words.

For a skew shape λ/μ with n boxes, let ϕ be one of $n!$ total ordering of boxes of λ/μ. For a semi-standard tableau T of shape λ/μ, define its *ϕ-reading word* as the word obtained by listing all entries of T in the order given by ϕ. Characterize all linear ordering ϕ of boxes of λ/μ such that, for any semi-standard tableau T of shape λ/μ, T is a Littlewood-Richardson tableau if an only if its ϕ-reading word is a Yamanouchi word.

Problem 3. In class, we discussed Zelevinsky's picture rule for the inner product $\langle s_{\lambda/\mu}, s_{\nu/\gamma} \rangle$ of two skew Schur functions.

(a) Assume that $\gamma = \emptyset$. Find a bijection between Zelevinsky's pictures and the set of Littlwood-Richardson tableaux for $c^{\lambda}_{\mu\nu}$.

(b) Assume that $\mu = \emptyset$ and the skew shape ν/γ is obtained by "gluing" two (non-skew) Young diagrams α and β. Find a bijection between Zelevinsky's pictures and the set of Littlewood-Richardson tableaux for $c^{\lambda}_{\alpha\beta}$.

Problem 4. Explicitly describe a bijection between Littlewood-Richardson tableaux and Berenstein-Zelevinsky triangles.

Problem 5. Find a triple of integer weights λ, μ, ν such that the Berenstein-Zelevinsky polytope $BZ(\lambda, \mu, \nu)$ (a.k.a. the honeycomb polytope) is not an integer lattice polytope.

Problem 6. For a triple of integer weight λ, μ, ν, prove the Berenstein-Zelevinsky polytope $BZ(\lambda, \mu, \nu)$ is either empty or has at least one integer vertex.

Problem 7. Show that Knutson-Tao-Woodward's puzzle rule for $c^{\lambda}_{\mu\nu}$ is equivalent to the classical Littlewood-Richarson rule. In other words, describe a bijection between puzzles and Littlewood-Richardson tableaux.