

18.217 PROBLEM SET 2 (due Wednesday, December 7, 2022)

Solve 5 (or more) problems.

**Problem 1.** Prove the Saturated Newton Polytope (SNP) property for the product

$$f(x_1, \dots, x_n) := \prod_{1 \leq i \leq j \leq n} (x_i + x_{i+1} + \dots + x_j).$$

In other words, show that, for any integer lattice point  $(a_1, \dots, a_n) \in \mathbb{Z}^n$  that belongs to the Newton polytope of  $f(x_1, \dots, x_n)$ , the monomial  $x_1^{a_1} \cdots x_n^{a_n}$  appears in the expansion of  $f(x_1, \dots, x_n)$  with a non-zero coefficient.

**Problem 2.** Describe all vertices of the Newton polytope of the polynomial  $f(x_1, \dots, x_n)$  from the previous problem. What is the number of vertices of this Newton polytope?

**Problem 3.** In class, we constructed the differential poset  $\mathbb{F}$  called the *Fibonacci lattice*. Prove that  $\mathbb{F}$  is indeed a lattice.

**Problem 4.** Let  $U$  and  $D$  be the up and down operators that act on the space  $\mathbb{C}[\mathbb{Y}]$  of formal linear combinations of Young diagrams by adding/removing boxes. (These are exactly the operators  $H_1$  and  $H_1^*$  from Problem 7 in Problem Set 1.)

Give explicit expressions for the coefficient of  $\emptyset$  in  $F(\emptyset)$ , for the following operators  $F$ :

- (a)  $F = (U + D)^n$
- (b)  $F = ((D + 1)(U + 1))^n$
- (c)  $F = (U + D + UD)^n$

Your answers might involve the Bell numbers

$$B_m := \#\{\text{set partitions of } [m]\}.$$

**Problem 5.** Prove the following  $q$ -hook length formula for the specialization of Schur function  $s_\lambda$  (in infinitely many variables):

$$s_\lambda(1, q, q^2, q^3, q^4, \dots) = \frac{q^{n(\lambda)}}{\prod_{a \in \lambda} (1 - q^{h(a)})},$$

where  $h(a)$  is the hook length of a box in  $\lambda$  and  $n(\lambda) := \sum_{i \geq 1} (i-1)\lambda_i$ .

**Problem 6.** A Young diagram  $\lambda$  is called an  $r$ -core if it is impossible to remove an  $r$ -ribbon from  $\lambda$ . Prove that  $\lambda$  is an  $r$ -core if and only if the value of  $\prod_{a \in \lambda} (1 - q^{h(a)})$  at the  $r$ -th primitive root of unity  $q = e^{2\pi i/r}$  is non-zero.

(First, you may want to show that if  $h$  is a hook length of  $\lambda$  than any divisor of  $h$  is a hook length of  $\lambda$ .)

**Problem 7.** Let  $A$  be a generic upper-triangular  $n \times n$  matrix. Prove bijectively that the number of non-zero minors of  $A$  (including the empty minor) equals the  $(n+1)$ st Catalan number  $C_{n+1}$ .

For example, for  $n = 2$ , matrix  $A = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$  has  $C_3 = 5$  non-zero minors:  $A_{\emptyset, \emptyset} = 1$ ,  $A_{\{1\}, \{1\}} = x$ ,  $A_{\{1\}, \{2\}} = y$ ,  $A_{\{2\}, \{2\}} = z$ , and  $A_{\{1,2\}, \{1,2\}} = xz$ .

**Problem 8.** Let  $E_i(x)$ ,  $i \in [n-1]$ ,  $x \in \mathbb{R}$ , be the  $n \times n$  matrix  $E_i(x) := I + x M_{i, i+1}$ , where  $I$  is the identity matrix and  $M_{i, i+1}$  the matrix that has only one non-zero entry, equal to 1, in the position  $(i, i+1)$ .

Show that, for any triple  $(x, y, z)$  of positive real numbers, there is a unique triple  $(x', y', z')$  of positive real numbers such that

$$E_i(x) E_{i+1}(y) E_i(z) = E_{i+1}(x') E_i(y') E_{i+1}(z').$$

**Problem 9.** Find an explicit formula for the determinant of the  $n \times n$  matrix

$$A_n = (a_{ij}) = \begin{pmatrix} 2 & 6 & 20 & 70 & \cdots \\ 1 & 2 & 6 & 20 & \cdots \\ 0 & 1 & 2 & 6 & \cdots \\ 0 & 0 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with entries

$$a_{ij} = \begin{cases} \binom{2(j-i+1)}{j-i+1} & \text{for } j \geq i-1, \\ 0 & \text{for } j < i-1. \end{cases}$$

For example,  $\det(2) = 2$ ,  $\det \begin{pmatrix} 2 & 6 \\ 1 & 2 \end{pmatrix} = -2$ ,  $\det \begin{pmatrix} 2 & 6 & 20 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix} = 4$ .

**Problem 10.** Find an explicit formula for the determinant of the  $n \times n$  matrix

$$B_n = (b_{ij}) = \begin{pmatrix} 1 & 2 & 5 & 14 & \cdots \\ 1 & 1 & 2 & 5 & \cdots \\ 0 & 1 & 1 & 2 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with entries

$$b_{ij} = \begin{cases} C_{j-i+1} & \text{for } j \geq i-1, \\ 0 & \text{for } j < i-1, \end{cases}$$

where  $C_m$  is the Catalan number.

For example,  $\det(1) = 1$ ,  $\det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = -1$ ,  $\det \begin{pmatrix} 1 & 2 & 5 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} = 2$ .

**Problem 11.** Let  $T$  be any filling of boxes of a Young diagram by  $1, 2, \dots, n$ . Let  $a_T, b_T \in \mathbb{C}[S_n]$  be the symmetrizer and the antisymmetrizer that appear in Young symmetrizer.

Show that there exist two invertible elements  $f_T, g_T$  in the group algebra  $\mathbb{C}[S_n]$  such that

$$a_T b_T = f_T b_T a_T g_T.$$

Give an explicit construction for  $f_T$  and  $g_T$ .

**Problem 12.** Let  $X_i := (1, i) + (2, i) + \cdots + (i-1, i) \in \mathbb{C}[S_n]$ ,  $i \in [n]$  be the Jucys-Murphy elements.

For a partition  $\lambda$  of  $n$ , let  $c_\lambda \in \mathbb{C}[S_n]$  be the sum of all permutations  $w \in S_n$  of cyclic type  $\lambda$ .

(a) Find an expression for  $c_{(n)}$  as a polynomial in the Jucys-Murphy elements  $X_1, \dots, X_n$ .

(b) For any partition  $\lambda$ , show that  $c_\lambda$  can be expressed as a polynomial in  $X_1, \dots, X_n$ .

**Problem 13.** An *allowed transposition* in a standard Young tableau  $T$  is the switching the entries  $i$  and  $i+1$  in  $T$  if they are located in non-adjacent boxes. (So the result of an allowed transposition is another standard Young tableau.)

Prove that any two standard Young tableaux of the same shape can be obtained from each other by a sequence of allowed transpositions.

**Problem 14.** Let  $V_\lambda$  be the irreducible representation of  $S_n$  defined in terms of Young's orthogonal form. Show that  $V_{(n-1,1)}$  is isomorphic to the representation of  $S_n$  on the  $(n-1)$ -dimensional space  $\{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 + \dots + x_n = 0\}$  where the symmetric group  $S_n$  acts by permutations of the coordinates  $x_i$ .

**Problem 15.** Let  $\chi_\lambda := \chi_{V_\lambda}$  be the character of the irreducible representation  $V_\lambda$  of  $S_n$ . Let  $w_\circ$  be the longest permutation in  $S_n$ . Explicitly describe the set of partitions  $\lambda$  of  $n$  such that character value  $\chi_\lambda(w_\circ)$  is non-zero. Give a rule for the sign of this character value.

**Problem 16.** Let

$$\langle i_1, i_2, \dots, i_k \rangle := \frac{1}{x_{i_1} - x_{i_2}} \cdot \frac{1}{x_{i_2} - x_{i_3}} \cdot \frac{1}{x_{i_3} - x_{i_4}} \cdots \frac{1}{x_{i_{k-1}} - x_{i_k}}.$$

Let  $\text{Shuffle}(A, B)$  be the set of all sequences obtained by shuffling the sequences  $A$  and  $B$ , i.e., by permuting the entries of  $A \cup B$  in any way so that the order of entries in  $A$  is preserved and the order of entries in  $B$  is preserved. For example,  $\text{Shuffle}((1, 2), (3, 4)) =$

$$\{(1, 2, 3, 4), (1, 3, 2, 4), (1, 3, 4, 2), (3, 1, 2, 4), (3, 1, 4, 2), (3, 4, 1, 2)\}.$$

Prove the following identities:

(a) Let  $A$  and  $B$  be two non-empty disjoint sequences of integers in  $[n]$  without repeated entries. Then

$$\sum_{C \in \text{Shuffle}(A, B)} \langle C \rangle = 0.$$

(b) Let  $A$  and  $B$  be two non-empty disjoint sequences of integers in  $\{2, 3, \dots, n\}$  without repeated entries. Then

$$\sum_{C \in \text{Shuffle}(A, B)} \langle 1, C, 1 \rangle = 0.$$

**Problem 17.** Let  $T$  be a directed tree on the set of vertices  $[n]$ . (We view  $T$  as the Hasse diagram of a certain poset on  $[n]$ .) A *linear extension* of  $T$  is a permutation  $w = w_1, \dots, w_n$  of  $[n]$  such that, for any directed edge  $i \rightarrow j$  in  $T$ , we have  $w^{-1}(i) < w^{-1}(j)$ . Then

$$\sum_{w \text{ is a linear extension of } T} \langle w_1, \dots, w_n \rangle = \prod_{i \rightarrow j \text{ is an edge in } T} \frac{1}{x_i - x_j}.$$

(Here we use the notation from the previous problem.)

**Problem 18.** (a) In class, we mentioned the set of monomial elements associated with increasing forests on the set of vertices  $[n]$ . Prove that this set of elements forms a linear basis of the Orlik-Solomon algebra  $OS_n$ .

- (b) Calculate the Hilbert series of the Orlik-Solomon algebra  $OS_n$ .
- (c) Calculate the Hilbert series of the Orlik-Terao algebra  $OT_n$ .

**Problem 19.** In class, we discussed the sets of non-crossing alternating trees and non-nesting alternating trees.

Prove bijectively that the number of non-crossing alternating trees on  $[n]$  equals the number of non-nesting alternating trees on  $[n]$  equals the Catalan number  $C_{n-1}$ .

**Problem 20.** Let  $P_n \subset \mathbb{R}^n$  be the  $(n-1)$ -dimensional polytope defined as the convex hull of all points  $\mathbf{e}_i - \mathbf{e}_j$ , for  $i \neq j \in [n]$ . (Here  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are the standard coordinate vectors in  $\mathbb{R}^n$ .) Let  $\text{Vol } P_n$  be the  $(n-1)$ -dimensional volume of the image of  $P_n$  in  $\mathbb{R}^{n-1}$  under the projection  $p : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$ .

Construct a triangulation of the polytope  $P_n$  and calculate  $\text{Vol } P_n$ .