18.217 PROBLEM SET 1 (due Friday, October 21, 2022)

Solve 5 (or more) problems.

Problem 1. Prove that, for two partitions λ and μ of n, the Kostka number $K_{\lambda\mu}$ (the number of semi-standard tableaux of shape λ and weight μ) is non-zero if and only if $\lambda \geq \mu$ in the dominance order, that is, if and only if

$$\begin{aligned} &\lambda_1 \geq \mu_1, \\ &\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2, \\ &\lambda_1 + \lambda_2 + \lambda_3 \geq \mu_1 + \mu_2 + \mu_3, \\ &\text{etc.} \end{aligned}$$

and $|\lambda| = |\mu|$.

Problem 2. Use the symmetry $w \mapsto (P, Q)$, $w^{-1} \mapsto (Q, P)$ of the Robinson-Schensted correspondence to obtain an explicit expression for the total number of standard Young tableaux of an arbitrary shape with n boxes. Your answer might involve a single sum over an integer.

Problem 3. Let $A \overset{RSK}{\longmapsto} (P, Q)$ be the Robinson-Schensted-Knuth correspondence constructed using the Schensted insertion algorithm, where A is a nonnegative integer $n \times n$ matrix and P, Q are SYTs of the same shape λ with n boxes. Let R be the reverse plane partition of the square shape $n \times n$ obtained by gluing the Gelfand-Tsetlin patterns of P and Q along their top row λ.

Let $\phi_n^{RSK} : A \mapsto R$ be the resulting map from $n \times n$ matrices A to reverse plane partitions R of shape $n \times n$.

In class, we constructed the map $\phi_n^{\text{toggle}} : A \mapsto R$ from nonnegative integer $n \times n$ matrices A to $n \times n$ reverse plane partitions R using toggle operations.

Check that the map ϕ_n^{RSK} coincides with the map ϕ_n^{toggle}. Basically, you need to carefully check that each Schensted insertion step from the classical construction of RSK is obtained by applying a certain sequence toggle operations.

Problem 4. A *linear extension* of poset P on n elements is a bijective map $f : P \to [n]$ such that $f(x) < f(y)$ whenever $x < y$ in P. Let $\text{ext}(P)$ denote the number of linear extensions of P. Notice standard Young tableaux of shape λ can be identified with linear extensions of a certain poset associated with λ. In this problem we will attempt to invent a hook length formula for an arbitrary finite poset P.

For an element x of a finite poset P, define the *naive hook length* as $h(x) := \#\{y \in P \mid y \geq x\}$. (Note that the usual hook lengths in a Young diagram are usually less than the naive hook lengths.) We say that a poset P on n elements is *naive* if the "naive hook length formula"

$$\text{ext}(P) = \frac{n!}{\prod_{x \in P} h(x)}$$

holds.

(A) Show that, for an arbitrary poset P on n elements, we have

$$\text{ext}(P) \geq \frac{n!}{\prod_{x \in P} h(x)}.$$

(B) Characterize the class of naive posets P.

Problem 5. Prove the "broken leg" hook length formula for the number of standard Young tableaux of shifted shape.

Problem 6. Define the weight $wt(T)$ of a standard Young tableau T of the square shape $\lambda = n \times n$ as

$$wt(T) := \left(\prod_{i=1}^{n} i^{d_{i+1} - d_i} \right)^{-1},$$

where $d_1 < \cdots < d_n$ are the entries of T on the main diagonal, and $d_{n+1} = n^2 + 1$. Prove that

$$\sum_{T \in \text{SYT}(n \times n)} wt(T) = 1.$$

For example, among $9!/(1 \cdot 2^2 \cdot 3^3 \cdot 4^2 \cdot 5) = 42$ standard Young tableaux of the square shape 3×3, there are

- 12 tableaux with diagonal vector $(1, 4, 9)$,
- 18 tableaux with diagonal vector $(1, 5, 9)$,

- 12 tableaux with diagonal vector $(1,6,9)$.

We obtain

$$\frac{12}{1^3\,2^5\,3^1}+\frac{18}{1^4\,2^4\,3^1}+\frac{12}{1^5\,2^3\,3^1}=1.$$

Hint: Possible approaches to Problems 5 and 6 might be based on methods similar to the polytopal proof of the usual hook length formula we gave in class.

Problem 7. Let $\mathbb{C}[\mathbb{Y}]$ be the linear space of formal linear combinations of Young diagrams. For $k \geq 0$, define the four operators H_k, E_k, H_k^*, E_k^* that act on the space $\mathbb{C}[Y]$ as

$$H_k : \lambda \longmapsto \sum_{\mu\supset\lambda\,:\,\mu/\lambda \text{ is a horizontal } k\text{-strip}} \mu$$

$$E_k : \lambda \longmapsto \sum_{\mu\supset\lambda\,:\,\mu/\lambda \text{ is a vertical } k\text{-strip}} \mu$$

$$H_k^* : \lambda \longmapsto \sum_{\mu\subset\lambda\,:\,\lambda/\mu \text{ is a horizontal } k\text{-strip}} \mu$$

$$E_k^* : \lambda \longmapsto \sum_{\mu\subset\,:\,\lambda/\mu \text{ is a vertical } k\text{-strip}} \mu$$

In particular, $H_0 = E_0 = H_0^* = E_0^* = \mathrm{Id}$. We assume that $H_k = E_k = H_k^* = E_k^* = 0$ for $k < 0$.

If we idenify the space $\mathbb{C}[\mathbb{Y}]$ with by space Λ of symmetric functions by $\lambda \mapsto s_\lambda$, then the operator H_k corresponds to the operatior of multiplication by the complete homogeneous symmetric function h_k and the operator E_k corresponds to the operator of multiplication by the elementary symmetric function e_k.

Prove combinatorially the following relations for these operators, for any $k, l \in \mathbb{Z}$. Here $[A,B] := AB - BA$ is the commutator of A and B.

(A) The operators H_k, H_l, E_k, E_l commute with each other. Similarly, the operators $H_k^*, H_l^*, E_k^*, E_l^*$ commute with each other.

(B) We have $[H_k^*, H_l] = H_{k-1}^*\, H_{l-1}$. Similarly, $[E_k^*, E_l] = E_{k-1}^*\, E_{l-1}$.

(C) We have $[H_k^*, E_l] = E_{l-1}\, H_{k-1}^*$. Similarly, $[E_k^*, H_l] = H_{l-1}\, E_{k-1}^*$.

Notice that the relations in parts (B) and (C) can be written as

$$H_k^*\, H_l = \sum_{0\le i\le \min(k,l)} H_{l-i}\, H_{k-i}^*$$

$$E_k^*\, E_l = \sum_{0\le i\le \min(k,l)} E_{l-i}\, E_{k-i}^*$$

$$H_k^*\, E_l = \sum_{i\in\{0,1\}} E_{l-i}\, H_{k-i}^*$$

$$E_k^*\, H_l = \sum_{i\in\{0,1\}} H_{l-i}\, E_{k-i}^*$$

Problem 8. Let H_1 and H_1^* be the operators from Problem 7. Show that, for a nonnegative integer vector $(a_1,\dots,a_n)$ such that $a_1+\cdots+a_n=n$, the coefficient of $\emptyset$ in $H_1^*(H_1)^{a_n}\cdots H_1^*(H_1)^{a_2}H_1^*(H_1)^{a_1}(\emptyset)$ equals

$$a_1(a_1+a_2-1)(a_1+a_2+a_3-2)\cdots(a_1+a_2+\cdots+a_n-(n-1)),$$

if all factors in this product are positive; otherwise, the coefficient is zero.

Problem 9. Define the *shadow* $Sh(w)$ of a permutation $w\in S_n$ as $Sh(w) := \{(i,j)\in[n]\times[n] \mid j\ge w(k) \text{ for some } k\le i\}$. Let $C(w)$ be the number of Young diagrams μ that fit inside the complement $([n]\times[n])\setminus Sh(w)$ of the shadow of w. For example, for the identity permutation 1, we have $C(1)=1$; and for the longest permutation $w_\circ\in S_n$, we have $C(w_\circ)=\frac{1}{n+1}\binom{2n}{n}$ (the Catalan number).

Prove the identity

$$\sum_{w\in S_n} C(w) = (2n-1)!!$$

Recall that $(2n-1)!! := 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)$.

Problem 10. For a partition $\lambda\vdash 2n$, define a *standard domino tableau* T of shape λ as a reverse plane partition of shape λ such that, for any $i=1,\dots,n$, T contains exactly 2 entries i located in 2 boxes adjacent to each other. (In other words, the 2 boxes containing i form either a horizonal or a vertical domino.) Let $f_\lambda^{\mathrm{domino}}$ be the number of standard domino tableaux of shape n. For example, $f^{\text{domino}}_{(3,3)} = 3$ counts the following domino tableaux:

$$\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2&3\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 1&3&3\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 1&2&3\\\hline\end{array}$$

In this problem, you will prove the following identity:

$$\sum_{\lambda\vdash 2n}\left(f^{\text{domino}}_{\lambda}\right)^2 = 2^n\, n! \tag{1}$$

For example, for $n = 3$, we have $f^{\text{domino}}_{(6)} = 1$, $f^{\text{domino}}_{(5,1)} = 1$, $f^{\text{domino}}_{(4,2)} = 3$, $f^{\text{domino}}_{(4,1,1)} = 2$, $f^{\text{domino}}_{(3,3)} = 3$, $f^{\text{domino}}_{(3,2,1)} = 0$, $f^{\text{domino}}_{(3,1,1,1)} = 2$, $f^{\text{domino}}_{(2,2,2)} = 3$, $f^{\text{domino}}_{(2,2,1,1)} = 3$, $f^{\text{domino}}_{(2,1,1,1,1)} = 1$, $f^{\text{domino}}_{(1,1,1,1,1,1)} = 1$. We get

$$1^2 + 1^2 + 3^2 + 2^2 + 3^2 + 0^2 + 2^2 + 3^2 + 3^2 + 1^2 + 1^2 = 2^3\, 3!$$

(A) Show that, for any two standard domino tableaux T_1 and T_2 of the same shape, the parities of the numbers of vertical dominos in T_1 and T_2 are the same. For example, each of the three domino tableaux of shape $(3, 3)$ shown above has either 1 or 3 vertical dominos.

(B) Show that the left hand side of the identity (1) equals the coefficient of $\emptyset$ in $(H_2^* - E_2^*)^n(H_2 - E_2)^n(\emptyset)$. Here H_2, E_2, H_2^*, E_2^* are the operators from Problem 7.

(C) Use the relations from Problem 7 to show that the coefficient of $\emptyset$ in $(H_2^* - E_2^*)^n(H_2 - E_2)^n(\emptyset)$ equals the alternating sum $\sum_G(-1)^{v(G)}$ over colored graphs G such that

- G is a graph on vertices $1, 2, \ldots, 2n$ (possibly with multiple edges).
- For each edge (i, j) of G we have $i \in [n]$ and $j \in \{n+1, \ldots, 2n\}$.
- Each vertex of G has degree 2.
- Vertices of G are colored in two colors: Hazel and Violet.
- G has at most 1 edge between a pair of vertices of different colors.

Here $v(G)$ is the number of violet vertices of G.

(D) Construct a sign reversing involution of the set of colored graphs G from part (C) to show that the alternating sum $\sum_G(-1)^{v(G)}$ equals $2^n\, n!$.

Problem 11. Give a bijective proof of the identity (1) from the previous problem. (In other words, you need to construct an analog of the Robinson-Schensted correspondence for domino tableaux.)

Problem 12. Give a combinatorial characterization of partitions $\lambda \vdash 2n$ such that $f_\lambda^{\mathrm{domino}} \neq 0$.

For example, $f_\lambda^{\mathrm{domino}} \neq 0$ for all partitions $\lambda \vdash 6$, except $\lambda = (3,2,1)$, see example in Problem 10.

Problem 13. In class, we defined the divided difference operator $\partial_w := \partial_{i_1} \cdots \partial_{i_l}$, for a reduced decomposition $w = s_{i_1} \cdots s_{i_l} \in S_n$.

Show that, for the longest permutation $w_\circ = \begin{pmatrix} 1 & 2 & \cdots & n \\ n-1 & n-2 & \cdots & 1 \end{pmatrix}$ in S_n, the divided difference operator $\partial_{w_\circ}$ is given by

$$\partial_{w_\circ} : f(x_1, \dots, x_n) \longmapsto \frac{\displaystyle\sum_{w \in S_n} (-1)^{\ell(w)}\, f(x_{w(1)}, x_{w(2)}, \dots, x_{w(n)})}{\displaystyle\prod_{1 \le i < j \le n} (x_i - x_j)}$$

Problem 14. In class, we defined the Demazure operator (a.k.a. the isobaric divided difference operator) $D_w := D_{i_1} \cdots D_{i_l}$, for a reduced decomposition $w = s_{i_1} \cdots s_{i_l} \in S_n$. We noted that the Schur polynomial can be expressed as $s_\lambda(x_1, \dots, x_n) = D_{w_\circ}(x^\lambda)$, where $w_\circ$ is the longest permutation in S_n, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, is a partition, and $x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$.

Find an explicit expression for $D_{w_\circ}(x^\beta)$, for an arbitrary nonnegative integer vector $\beta = (\beta_1, \dots, \beta_n)$.

Problem 15. In class, we showed that, for the longest permutation $w_\circ \in S_n$, the Demazure and the divided difference operators are related to each other as $D_{w_\circ} = \partial_{w_\circ} X^{(n-1,n-2,\dots,1,0)}$, where $X^{(\alpha_1,\dots,\alpha_n)}$ is the operator of multiplication by $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We also noticed that a similar relation holds for all permutations in S_3 except one.

Describe the class of permutations $w \in S_n$ such that $D_w = \partial_w X^{\mathrm{code}(w)}$, where $\mathrm{code}(w) = (c_1, \dots, c_n)$, $c_i = \#\{j > i \mid w_i > w_j\}$, is the Lehmer code of permutation w.

Problem 16. In class, we mentioned two multiplicative formulas for the specialization of Schur function $s_\lambda(1, \dots, 1)$ (with n 1's): Weyl's dimension formula

$$s_\lambda(1, \dots, 1) = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i},$$

and Stanley's hook-content formula

$$s_\lambda(1,\dots,1) = \prod_{a \text{ is a box of } \lambda} \frac{n+c(a)}{h(a)},$$

where $h(a)$ is the hook length and $c(a) = j - i$ is the content of box $a = (i,j)$ of Young diagram λ.

Show combinatorially that the two expressions in the right hand sides of these two formulas are equal to each other.

Problem 17. Use l'Hopital's rule to derive Weyl's dimension formula for $s_\lambda(1,\dots,1) = \prod_{1\le i<j\le n} \frac{(\lambda_i-\lambda_j+j-i)}{(j-i)}$ from the classical definition of Schur functions $s_\lambda = a_{\lambda+\delta}/a_\delta$.

Problem 18. Prove the identity

$$\sum_{\lambda\subseteq m\times n} s_\lambda(1^m)\, s_{\lambda'}(1^n) = 2^{m\cdot n},$$

where the sum is over Young diagrams λ that fit into the $m\times n$ rectangle, and $s_\lambda(1^m)$ denotes the specialization $s_\lambda(1,\dots,1)$ (with m 1's).

Problem 19. Prove bijectively that the number of Gelfand-Tsetlin patterns with top row $n-1, n-2, \dots, 1, 0$ equals $2^{\binom{n}{2}}$.

Problem 20. For a partition $\lambda = (\lambda_1,\dots,\lambda_n)$ and a nonnegative integer vector $\beta = (\beta_1,\dots,\beta_n)$, define the Gelfand-Tsetlin polytope $GT(\lambda,\beta) \in \mathbb{R}^{\binom{n}{2}}$ as the polytope of $\mathbb{R}$-valued Gelfand-Tsetlin patterns with the top row $\lambda_1,\dots,\lambda_n$ and whose row sums are $\beta_1, \beta_1+\beta_2, \beta_1+\beta_2+\beta_3, \dots$ (listed from the bottom). In class, we explained that integer lattice points of $GT(\lambda,\beta)$ correspond to semi-standard Young tableaux of shape λ and weight β.

Is it always true that $GT(\lambda,\beta)$ is an integer lattice polytope (i.e., all its vertices are integer lattice points)? Prove this claim or present a counterexample.