18.217 LECTURE 27

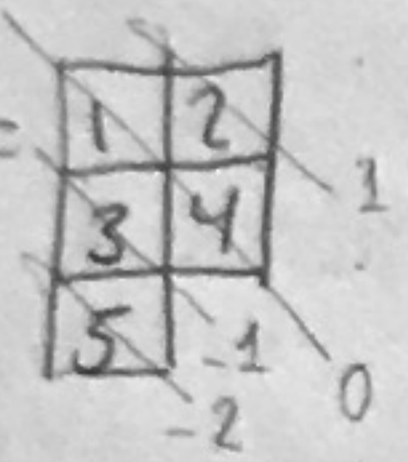
Last Time: <u>Young's Orthogonal Form</u>:

Irreps V_λ of S_n

Basis of V_λ: $\{v_T \mid T \in SYT(\lambda)\}$

$T \rightsquigarrow$ content vector $c(T) = (c_1, \dots, c_n)$ $\qquad c_i$ = content of box i in T

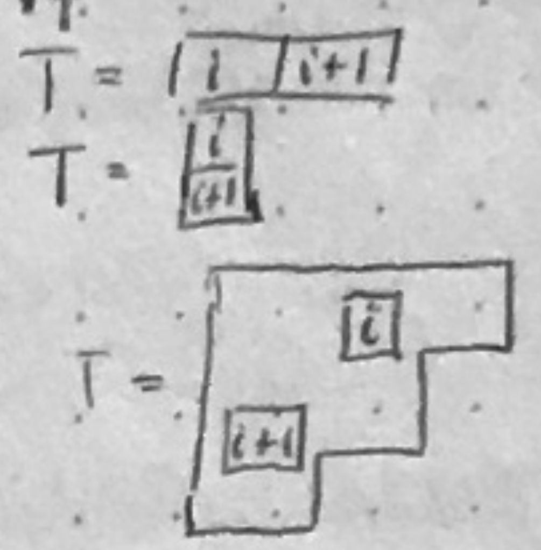
↖ diagonals on which entries fall

Ex. $T =$ 

$c(T) = (0, 1, -1, -2, 0)$

Action of generators $s_1, \dots, s_{n-1}$ on basis elts v_T

$$R_{s_i} \cdot v_T \longmapsto \begin{cases} v_T & \text{if } c_{i+1} - c_i = 1 \\ -v_T & \text{if } c_{i+1} - c_i = -1 \\ \frac{1}{c_{i+1} - c_i} v_T + \sqrt{1 - \frac{1}{(c_{i+1} - c_i)^2}}\, v_{\tilde{T}} & \text{if } c_{i+1} - c_i \neq \pm 1 \end{cases}$$



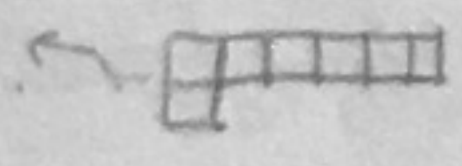
Ex. $n = 3$, $\lambda = \ydiagram{2,1}$ $\qquad v_1 = v_{\young(12,3)}$ $\qquad v_2 = v_{\young(13,2)}$

$R_{s_1}: \begin{array}{l} v_1 \longmapsto v_1 \\ v_2 \longmapsto -v_2 \end{array}$ $\qquad R_{s_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$R_{s_2}: \begin{array}{l} v_1 \longmapsto -\frac{1}{2} v_1 + \frac{\sqrt{3}}{2} v_2 \\ v_2 \longmapsto \frac{1}{2} v_2 + \frac{\sqrt{3}}{2} v_2 \end{array}$ $\qquad R_{s_2} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Action of S^3 on $\{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}$ by perm. of coord

<u>Exercise</u> show $V_{(n-1,1)} \cong$ Action of S_n by perm of coord on $\{(x_1, \dots, x_n) \in \mathbb{C}^n, x_1 + \dots + x_n = 0\}$



<u>Def</u>: <u>Character of Reps of S_n</u>

$R: S_n \longrightarrow GL_n(V)$ some rep of S_n

Its <u>character</u> $\chi_V: S_n \longrightarrow \mathbb{C}$

$\qquad\qquad\qquad \overset{\in}{w} \longmapsto tr(R(w))$

$$\boxed{\begin{array}{l} tr(A \cdot B) = tr(B \cdot A) \\ tr(A) = tr(CAC^{-1}) \end{array}}$$

<u>General Facts of characters</u>

1. χ_V does not depend on choice of basis in V.
2. χ_V is constant on conj. classes of S_n
3. $V_1 \cong V_2$ iff $\chi_{V_1} = \chi_{V_2}$
4. $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$
5. $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$

Character table for S_3:

conj. classes	id	(1,2)	(1,2,3)
$V_{\ydiagram{3}}$	1	1	1
$V_{\ydiagram{1,1,1}}$	1	-1	1
$V_{\ydiagram{2,1}}$	2	0	-1

$Tr(R_{s_1} R_{s_2}) = -1$

Given in previous ex.

__Def.__ $\chi_\lambda(\mu) = \chi_{V_\lambda}(w)$ w is of cyclic type μ

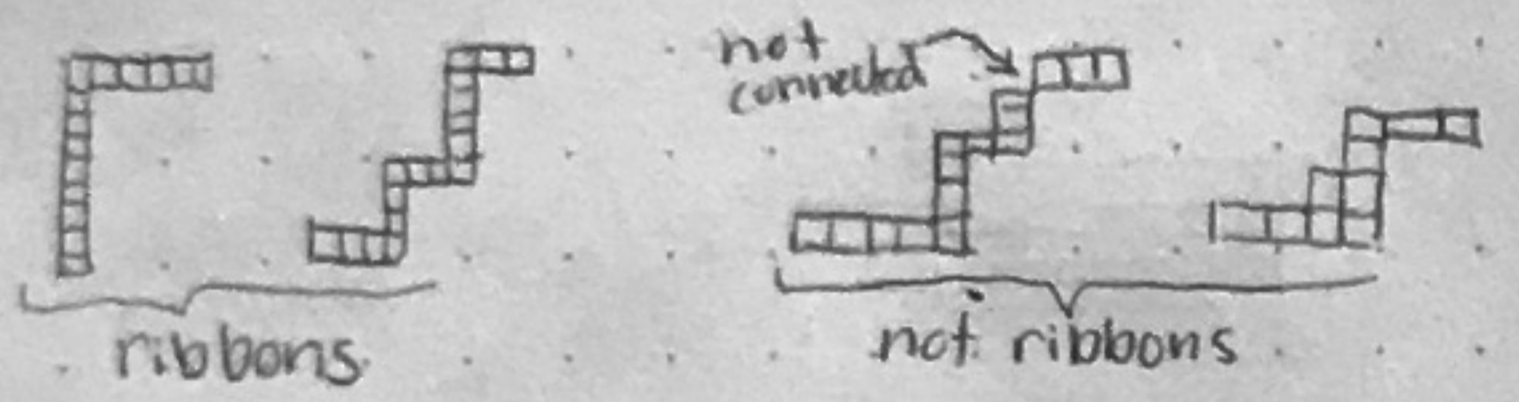
__Thm:__ (Murnaghan-Nakayama Rule)

$$\chi_\lambda(\mu) = \sum_{\substack{T \text{ ribbon} \\ \text{Tableaux} \\ \text{of shape } \lambda \\ \text{and type } \mu}} (-1)^{ht(T)}$$

__Def:__ A __ribbon__ is a skew shape φ/ψ s.t. it is connected and contains no 2×2 box

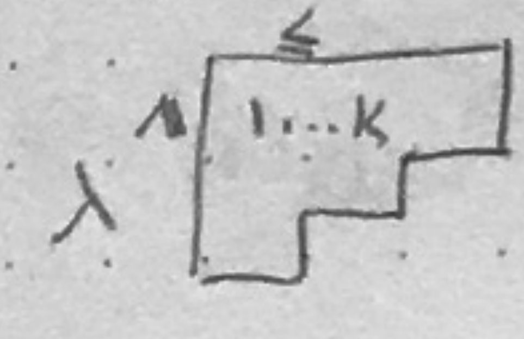
A __k-ribbon__ has k boxes

__E.x.__



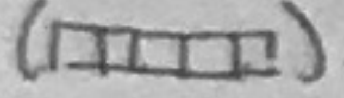
__Def:__ A __ribbon tableau__ of shape λ and type $\mu = (\mu_1, \ldots, \mu_k)$ is a reverse plane partition of shape λ s.t. all boxes with entries i form a μ_i-ribbon

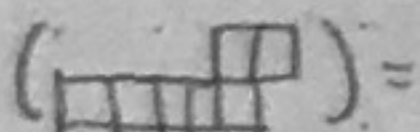
__E.x.__ Domino tableaux are ribbon tableaux of type $(2, 2, \ldots, 2)$



height

$ht(\text{ribbon}) := \# \text{rows} - 1$ $ht(T) = \sum_i ht(i^{\text{th}} \text{ ribbon})$

__E.x.__ ht() = 0

ht() = 1

__E.x.__ $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 1 & 2 & 2 & 4 \\ \hline 3 & 3 & 4 & 4 \\ \hline 3 & 4 & 4 \\ \cline{1-3} 3 & 5 & 5 \\ \cline{1-3} \end{array}$

shape $\lambda = (4, 4, 4, 3, 3)$

type $\mu = (3, 3, 4, 6, 2)$ ← NOTE: Not a partition in This case (and doesn't need to be for Thm)

$ht(T) = 1 + 1 + 2 + 3 + 0 = 7$

__E.x.__ $\chi_{(2,1)}((3,0)) = -1$ $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}$

$\chi_{(2,2)}((2,2)) = 1 + (-1)^2 = 2$ $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$ $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array}$

Thm (Murnagham-Nakayama Rule on skew shapes)

λ/μ skew shape $|\lambda/\mu| = n$

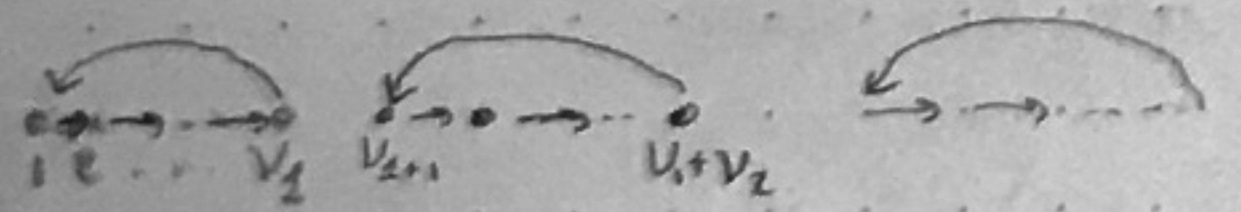
$V_{\lambda/\mu}$ rep of S_n by Young's Orth. Form

action $\{ v_T \mid T \in SYT(\lambda/\mu) \}$

$\chi_{\lambda/\mu} := \chi_{V_{\lambda/\mu}}$

λ/μ skew shape, $v = (v_1, \ldots, v_k)$ a composition of n ← will be easier to prove than on just regular tableause

__Proof:__ $R_{s_i} : v_T \longmapsto \frac{1}{c_{i+1} - c_i} v_T + \sqrt{\cdots} \, v_{\tilde{T}}$

W = 

$$= \underbrace{s_1 s_2 \cdots s_{v_1-1}}_{v_1-1} \underbrace{s_{v_1+1} \cdots s_{v_1+v_2-1}}_{v_2-1} \underbrace{\qquad\qquad}_{v_3-1}$$

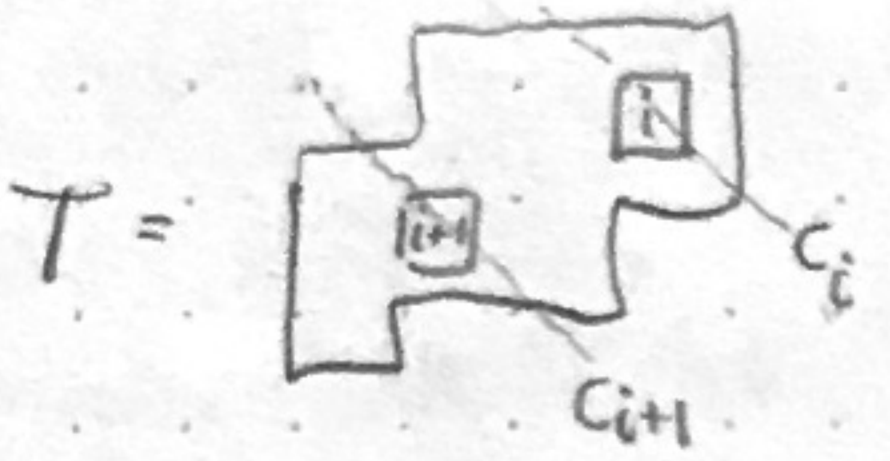
$tr(R_w) = \sum_T \text{coeff of } v_T \text{ in } R_w(v_T)$

MORE ON THIS PROOF NEXT TIME

18.217 LECTURE 28

Last time: λ/μ skew shape w/ n boxes
$V_{\lambda/\mu}$ rep of S_n given by Young's Orth. form
on the space / $\mathbb{C}$ w/ basis $\{v_T \mid T \in SYT(\lambda/\mu)\}$

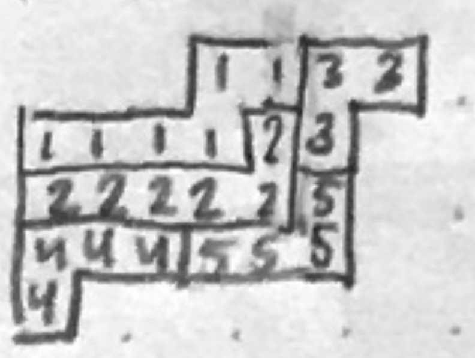
$$R_{s_i} : v_T \longmapsto \frac{1}{c_{i+1}-c_i} v_T + \sqrt{1-\frac{1}{(c_{i+1}-c_i)^2}}\, v_{\tilde{T}}$$

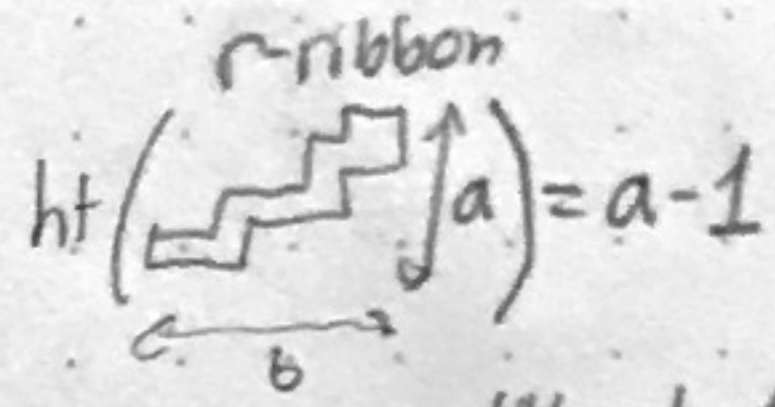


c_i is the content (diagonal) of box $\boxed{i}$ in T

$\nu = (\nu_1, \dots, \nu_k)$ composition of n
$\chi_{\lambda/\mu}(\nu)$ = the value of character of $V_{\lambda/\mu}$ on a perm $w \in S_n$ of cyclic type μ

MN-rule: $\chi_{\lambda/\mu}(\nu) = \sum_{\substack{\text{RT ribbon Tableau} \\ \text{of shape } \lambda/\mu \\ \text{and type } \nu}} (-1)^{ht(T)}$





width = b-1
$(a-1)+(b-1) = (r-1)$

Lemma: $V_{(\lambda/\mu)'} = V_{\lambda/\mu} \otimes V_{\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}\}n}$ ← the sign rep. of S_n

$$\chi_{(\lambda/\mu)'}(w) = \chi_{\lambda/\mu}(w) \cdot \mathrm{sign}(w)$$

$$\mathrm{sign}\left(\substack{\text{perm of} \\ \text{cyclic type } \nu}\right) = (-1)^{\Sigma(\nu_i - 1)}$$

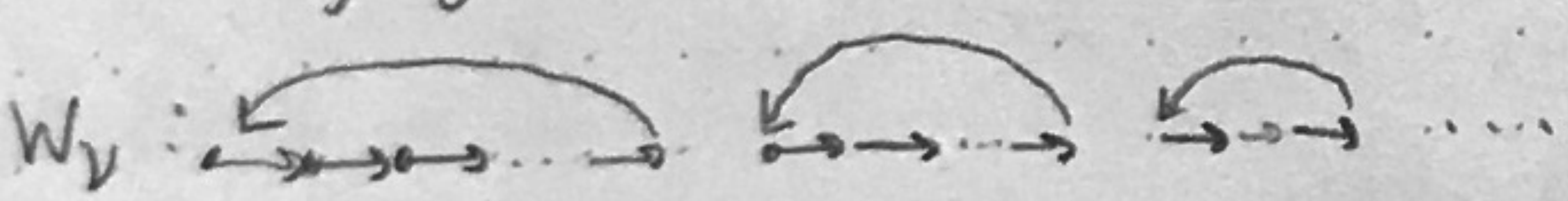
Cor If λ/μ is self-conj. shape $(\lambda/\mu)' = (\lambda, \mu)$, then for any perm w with $\mathrm{sign}(w) = -1$, $\chi_{\lambda/\mu}(w) = 0$.

E.x.

id	s_1	$s_1 s_2$
2	0	-1

← $\chi_{⊞}(w)$

Proof of MN rule $\nu = (\nu_1, \dots, \nu_k) \rightsquigarrow$



$$= (s_1 s_2 \cdots s_{\nu_1 - 1})(s_{\nu_1+1} s_{\nu_1+2} \cdots s_{\nu_1+\nu_2-1}) \cdots$$
$$= s_1 s_2 \cdots \widehat{s_{\nu_1}} \cdots \widehat{s_{\nu_1+\nu_2}} \cdots \widehat{s_{\nu_1+\nu_2+\dots+\nu_k}}$$

$$\chi_{\lambda/\mu}(\nu) = \chi_{\lambda/\mu}(w_\mu^{-1}) \overset{\text{def}}{=} tr\left(\widehat{R_{s_{\nu_1+\dots+\nu_k}}} \cdots \widehat{R_{s_{\nu_1+\nu_2}}} \cdots \widehat{R_{s_{\nu_1}}} \cdots R_{s_1} R_{s_2}\right)$$
$$= \sum_{T \in SYT(\lambda/\mu)} \text{coeff of } v_T \text{ in } (\quad) v_T$$

Observation: Only diagonal terms $\frac{1}{c_{i+1}-c_i} v_T$ (and not $\sqrt{\ }$ term) make contribution to $tr(\dots)$

Prop: $\chi_{\lambda/\mu}(\nu) = \sum_{T \in SYT(\lambda/\mu)} \prod_{i \in [n] - \{\nu_1, \nu_1+\nu_2, \dots, \nu_1+\dots+\nu_k\}} \frac{1}{c_{i+1}-c_i}$ $\qquad c_i = c_i(T)$ = the content of box $\boxed{i}$ in T

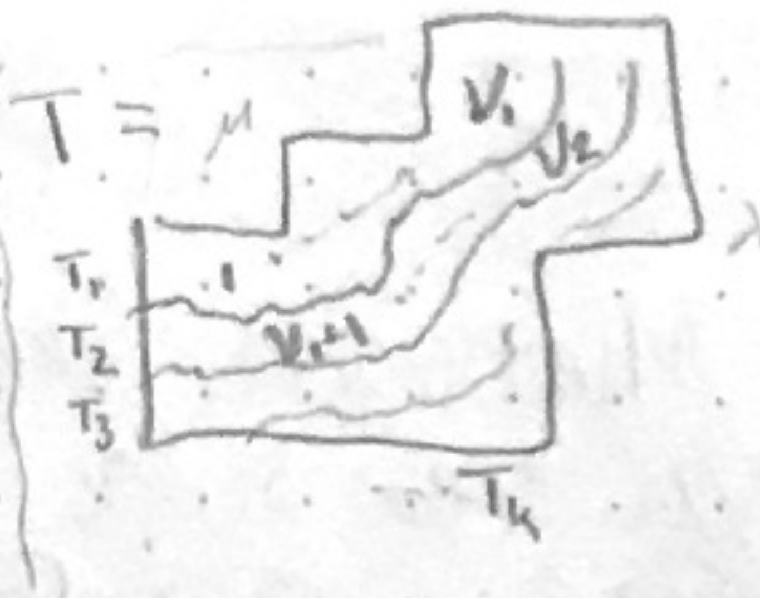
Thrm (The case of MN's rule for $\nu = (n)$)

$$\sum_{T \in SYT(\lambda/\mu)} \frac{1}{c_2-c_1} \frac{1}{c_3-c_2} \cdots \frac{1}{c_n-c_{n-1}} = \begin{cases} (-1)^{ht(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a ribbon} \\ 0 & \text{otherwise} \end{cases} \qquad (\star)$$

Claim: $(\star) \Rightarrow$ general case of MN-rule
Proof: $\chi_{\lambda/\mu}(\nu) \overset{\text{(prop)}}{=} \sum_{T \in SYT(\lambda/\mu)} \prod_{i \in [n] - \{\nu_1, \nu_1+\nu_2, \dots\}} \frac{1}{c_{i+1}-c_i}$

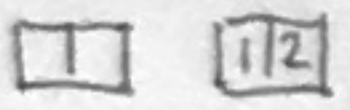
$$= \sum_{\mu \subset \underset{\nu_1}{\lambda^{(1)}} \subset \underset{\nu_2}{\lambda^{(2)}} \subset \cdots \subset \underset{\nu_k}{\lambda^{(k)}} = \lambda} \left(\sum_{T \in SYT(\lambda^{(1)}/\mu)} \frac{1}{c_2 - c_1} \cdots \frac{1}{c_{\nu_1} - c_{\nu_1 - 1}} \right) \cdot \left(\sum_{T \in SYT(\lambda^{(2)}/\mu)} \right) \cdot \ldots$$

break into smaller strips based on what range of numbers they contain

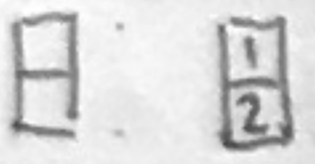


$$\overset{(*)}{=} \sum_{\text{RT ribbon tableau}} (\pm 1)$$

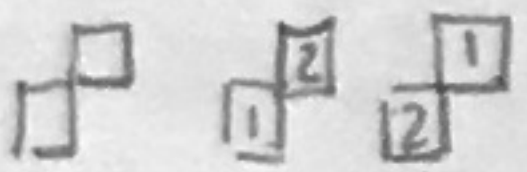
Ex. of $(*)$



$$\frac{1}{c_2 - c_1} = 1$$

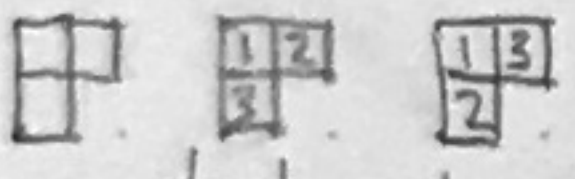


$$\frac{1}{c_2 - c_1} = \frac{1}{-1} = -1$$



$$1/2 - 1/2 = 0$$

disconnected gives 0



$$\frac{1}{1} \cdot \frac{1}{-2} \quad \frac{1}{-1} \cdot \frac{1}{2} = -1$$

ribbon gives ±1

$$\frac{1}{1} \cdot \frac{1}{-2} \cdot \frac{1}{1} \quad \frac{1}{-1} \cdot \frac{1}{2} \cdot \frac{1}{-1} = 0$$

2×2 box gives 0

Multivariate generalization of MN-rule

$\underline{x} = (x_i)_{i \in \mathbb{Z}}$ collection of vars

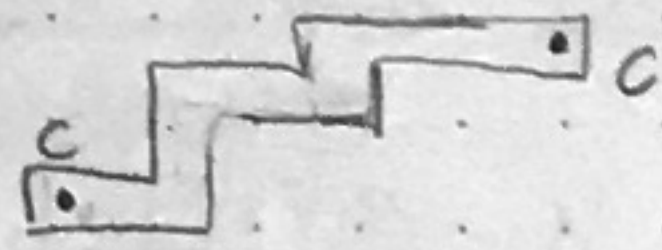
Def: $$\chi^{x}_{\lambda/\mu}(\nu) := \sum_{\substack{T \in SYT(\lambda/\mu) \\ \& \text{ type } \nu}} \prod_{i \in [n] \setminus \{\nu_1, \nu_1 + \nu_2, \ldots\}} \frac{1}{x_{c_i(T)} - x_{c_{i+1}(T)}}$$

Get original case if $x_c = -1 \ \forall c \in \mathbb{Z}$

Thm: $$\chi^{x}_{\lambda/\mu}(\nu) = \sum_{\substack{\text{RT ribbon Tableau} \\ \text{of shape } \lambda/\mu \\ \& \text{ type } \nu}} wt(T)$$

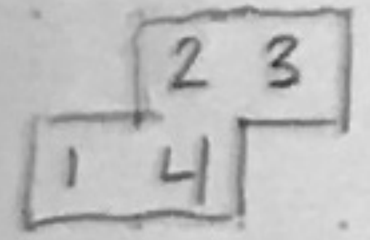
$$wt(\text{ribbon}) = (-1)^{ht} \prod_{i \in [c, c'-1]} \frac{1}{x_i - x_{i+1}}$$

$$wt(RT) = \prod wt(\text{its ribbons})$$



Ex.

$$\frac{1}{x_1 - x_3} \cdot \frac{1}{x_3 - x_2} \cdot \frac{1}{x_2 - x_4} +$$



$$\frac{1}{x_1 - x_3} \cdot \frac{1}{x_3 - x_4} \cdot \frac{1}{x_4 - x_2} + \ldots$$

$\vdots$

$$= -\frac{1}{x_1 - x_2} \frac{1}{x_2 - x_3} \frac{1}{x_3 - x_4}$$

LECTURE 29

Last time: "Multivariable Murnaghan-Nakayama Rule"

$(x_i)_{i\in\mathbb{Z}}$ $\quad\lambda/\mu$ skew shape with n boxes

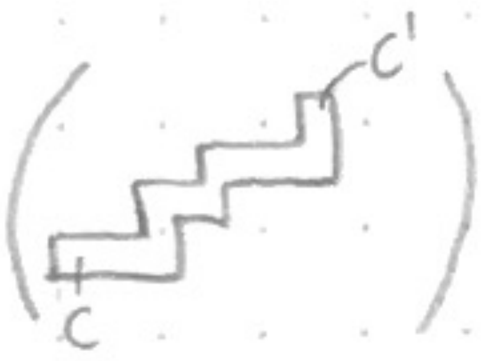
$\nu=(\nu_1,\dots,\nu_k)$ composition of n

Def: $\chi^x_{\lambda/\mu}(\nu) := \sum_{T\in SYT(\lambda/\mu)} \prod_{i\in[n]\setminus\{\nu_1,\nu_1+\nu_2,\dots,\nu_1+\dots+\nu_k\}} \frac{1}{x_{c(i)}-x_{c(i+1)}}$

$c(i)$ = the content (diagonal #) of box $\boxed{i}$ in T.

Thrm: $\chi^x_{\lambda/\mu}(\nu) = \sum_{\substack{RT \text{ ribbon tableau} \\ \text{of shape } \lambda/\mu \text{ and} \\ \text{type } (\nu)}} wt(T)$

$wt(T) = \prod wt\binom{\text{its}}{\text{ribbons}}$

wt  $= (-1)^{ht} \prod_{i\in[c,c'-1]} \frac{1}{x_i-x_{i+1}}$

Ex. RT 

Classical case: $x_i = -i \;\; \forall i$

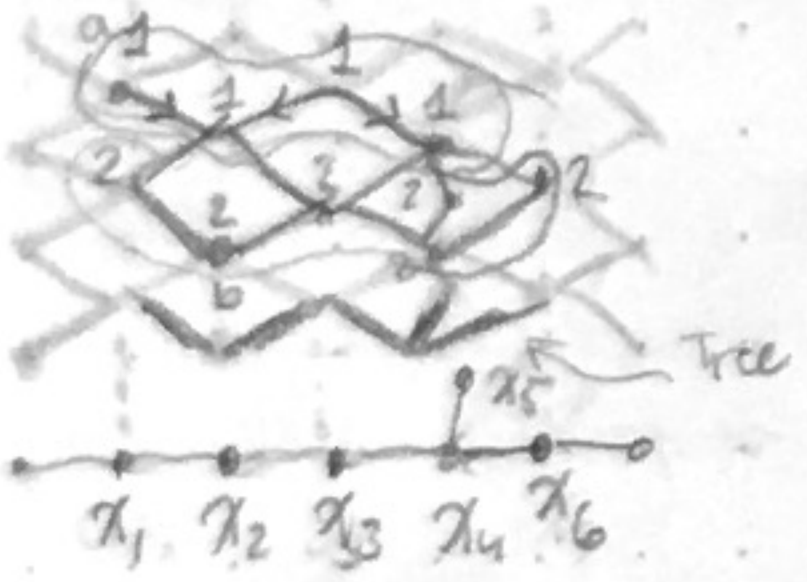
$\chi^x_{\lambda/\mu}(\nu)$

usual char. of reps of S_n

Generalize even more!

Abacus drawing of λ/μ. 

More generally, fix a tree $\mathbb{T}$ (possibly infinite) $\rightsquigarrow P_{\mathbb{T}}$ Abacus poset of T

Ex. $P_{\mathbb{T}}$ 

A shape K is a finite subset of $P_{\mathbb{T}}$ with n elts s.t. $a,b\in K \Rightarrow [a,b]\in K$

SYT(K) = the set of lin. extensions of K

"ribbons" $\rightsquigarrow$ a ______ s.t. their Hasse diagram are directed trees

Def: A tree tableaux TT of shape K and type ν is a labelling $TT: K \to \{1,\dots,k\}$ s.t.

- the labels weakly increasing
- $\forall i\in[k]$, $\{a\in K \mid TT(a)=i\}$ is a directed tree with ν_i elts.

$wt(TT) = \prod wt\binom{\text{its}}{\text{trees}}$

$wt(\text{tree}) = \prod_{\substack{i\to j \\ \text{edge in } T}} \frac{1}{x_{c(i)}-x_{c(j)}}$

Ex. $wt(\;) = \left(\frac{1}{x_1-x_2}\frac{1}{x_3-x_2}\frac{1}{x_3-x_4}\right)\left(\frac{1}{x_1-x_2}\frac{1}{x_3-x_2},\dots\right)$

For $\nu=(n)$

Thrm: $\chi_K([n]) = \begin{cases} wt(K) & \text{if } K \text{ is a tree shape} \\ 0 & \text{otherwise} \end{cases}$

! follow from following 3 Lemmas

$\langle i_1, i_2, \dots, i_\ell\rangle = \frac{1}{x_{i_1}-x_{i_2}} \cdot \frac{1}{x_{i_2}-x_{i_3}} \cdots \frac{1}{x_{i_{\ell-1}}-x_\ell}$

Lemma 1: A, B nonempty disjoint subset in $\mathbb{Z}$

$$\sum_{c \in \text{shuffle}(A,B)} \langle c \rangle = 0$$

Exercise 1,2,3:
Prove Lemmas 1,2,3

E.x. $A = \{1\}$, $B = \{2,3\}$
$\langle 1,2,3\rangle + \langle 2,1,3\rangle + \langle 2,3,1\rangle = 0$

Lemma 2: $\{1\}$, A, B disjoint & nonempty

$$\sum_{\text{shuffle}(A,B)} \langle 1, c, 1\rangle = 0$$

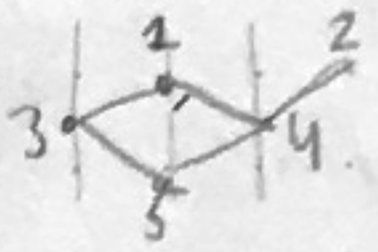
E.x. $A = \{2\}$ $B = \{3,4\}$

$$\langle 1,2,3,4,1\rangle + \langle 1,3,3,4,1\rangle + \langle 1,3,4,2,1\rangle = 0$$

This lemma explains why we get 0 when Hasse diagram contains a cycle

If not a tree, will have 2 elts on same string

E.x.

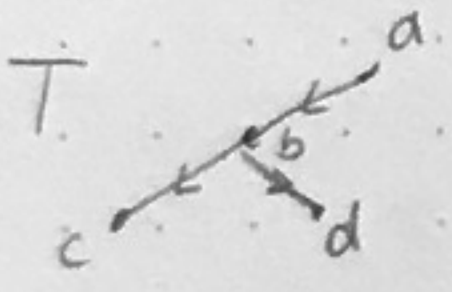


$\rightsquigarrow \langle 2,4,1,3,2\rangle$
+ (sum over shuffles of labels
↳ automatically 0

Lemma 3: T directed tree

$$\sum_{\substack{w_1,\dots,w_n \\ \text{lin. ext of } T}} \langle w_1, \dots, w_n\rangle = \prod_{\substack{i \to j \\ \text{edge of } T}} \frac{1}{x_i - x_j}$$

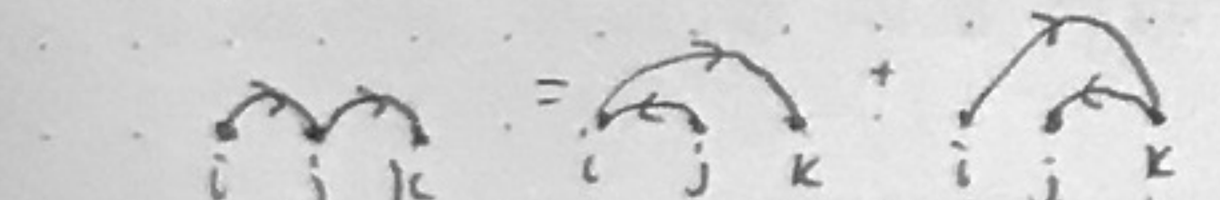
E.x. T



$$\langle a,b,c,d\rangle + \langle a,b,d,c\rangle = \frac{1}{x_a - x_b} \cdot \frac{1}{x_b - x_c} \cdot \frac{1}{x_b - x_d}$$

Key Identity:

$$\frac{1}{x_i - x_j} \cdot \frac{1}{x_j - x_k} = \frac{1}{x_i - x_k} \cdot \frac{1}{x_i - x_j} + \frac{1}{x_i - x_k} \cdot \frac{1}{x_j - x_k}$$



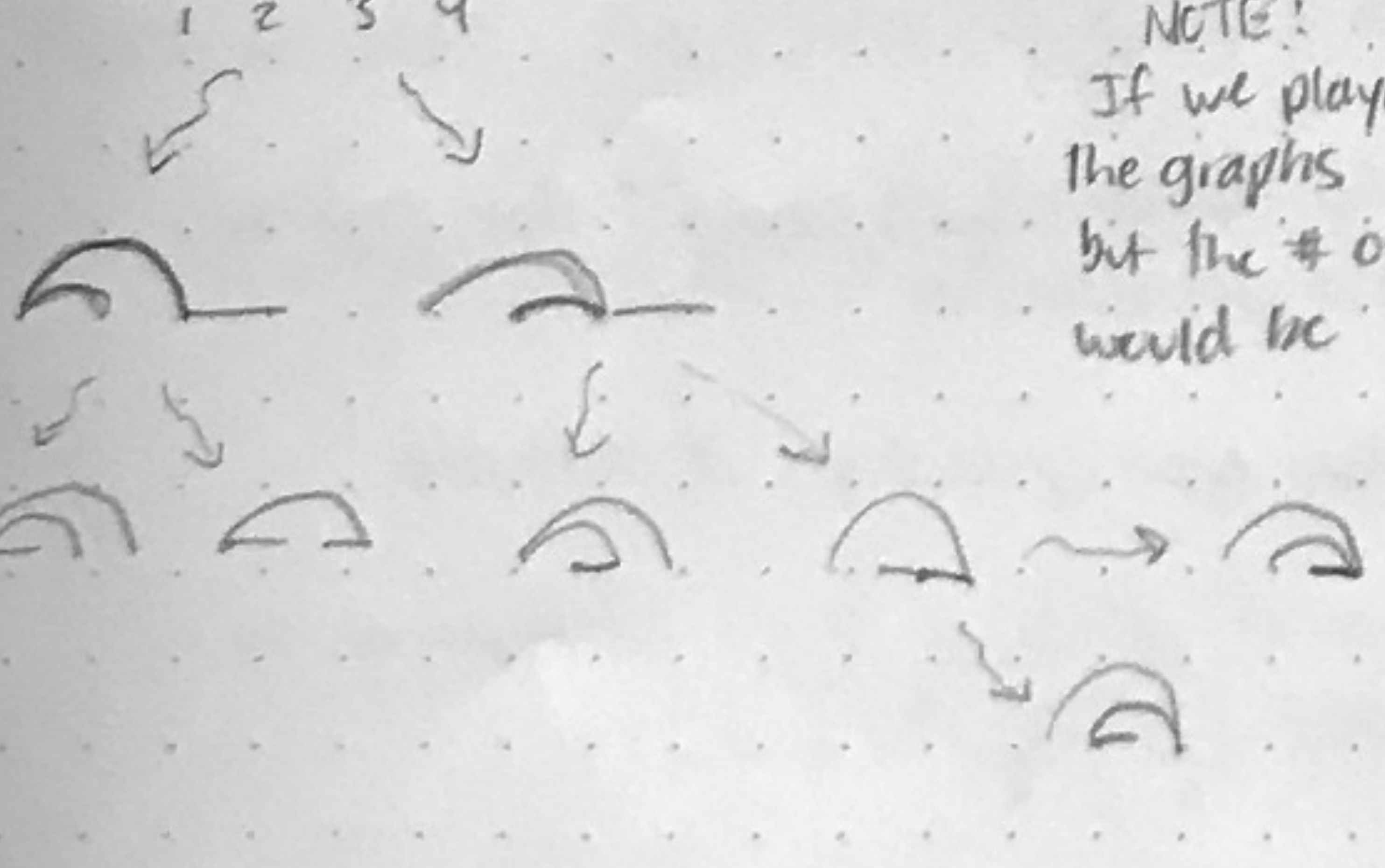
All Thms today are certain identities in
Oulik-Terao Alg (for type A braid arr.)

OT_n : commutation alg. w/ generators a_{ij} $i,j \in [n]$

- $a_{ij} = -a_{ij}$
- $a_{ij} a_{jk} = a_{ik} \cdot a_{ij} + a_{ik} a_{jk}$

Some more identities:

$a_{12} a_{23} a_{34}$



NOTE!
If we played differently, the graphs would be different but the # of endpts would be the same

n chain $\rightsquigarrow$ C_{n-1} end pts

$K_n \to C_{n-1} \cdot C_{n-2} \cdots C_1$ end pts

This fact is known, but not a combinatorial proof

18.217 LECTURE 30

(Arnold) Orlik-Soloman & Orlik-Terao Algebras (for type A braid arrangement)

OS_n anti-commutative alg (over $\mathbb{C}$)

generators: a_{ij} $i \neq j \in [n]$

$a_{ij} = a_{ji}$

Relations: $a_{ij} a_{kl} = -a_{kl} a_{ij}$

$(a_{ij}^2 = 0)$

OT_n commutative alg

generators: c_{ij} $i \neq j \in [n]$

$c_{ij} = -c_{ij}$

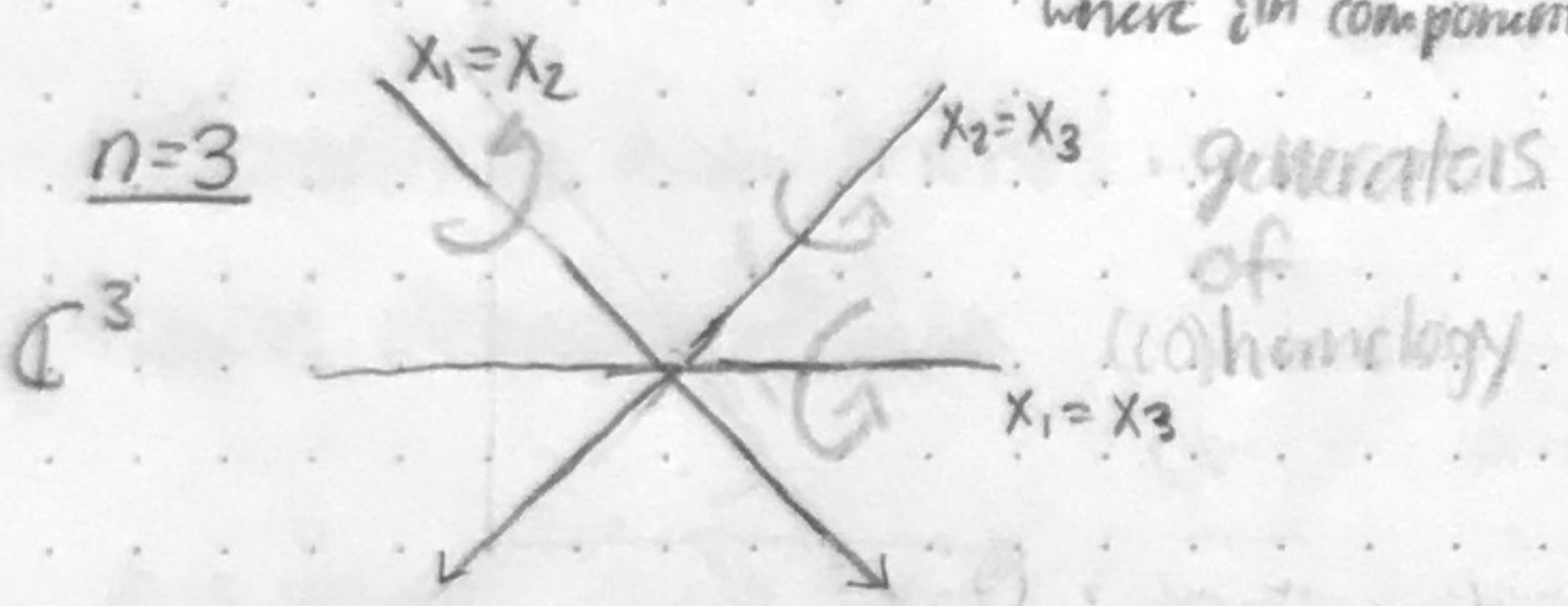
Relations: $c_{ij} c_{kl} = c_{kl} c_{ij}$

$a_{ij} a_{jk} = a_{ik} a_{ij} + a_{jk} a_{ik}$ ← order of terms matters for a_{ij}'s

$c_{ij} c_{jk} = c_{ik} c_{ij} + c_{jk} c_{ik}$

Thrm [Arnold]: $H^*(\mathbb{C}^n \setminus \{x_i = x_j, i<j\}, \mathbb{C}) \simeq OS_n$

↖ diagonal hyperplanes where i^{th} component = j^{th} component



$a_{ij} \rightsquigarrow \frac{d(x_i - x_j)}{x_i - x_j} = d\log(x_i - x_j)$

$c_{ij} \rightsquigarrow \frac{1}{x_i - x_j}$

Hilbert Series

$\mathrm{Hilb}_{OS_n}(t) = \sum_{k \geq 0} \dim OS_n^k \, t^k$ ← k^{th} graded component

fin. dim'l

$\mathrm{Hilb}_{OT_n}(t) = \ldots$

infinite dim'l

n=3 K	0	1	2	3	4
$\dim OS_n^k$	1	3	2	0	0 ...
$\dim OT_n^k$	1	3	5	≠ 0	≠ 0 ...
generators		a_{11}, a_{23}, a_{13}	$a_{12} a_{23}$ $a_{12} a_{23}$ $a_{23} a_{13}$	$a_{12} a_{13} a_{23}$ $= 0$	

$\mathrm{Hilb}_{OS_3}(t) = 1 + 3t + 2t^2$
$= (1+t)(1+2t)$

Thrm: $\mathrm{Hilb}_{OS_n}(t) = (1+t)(1+2t) \cdots (1+(n-1)t)$

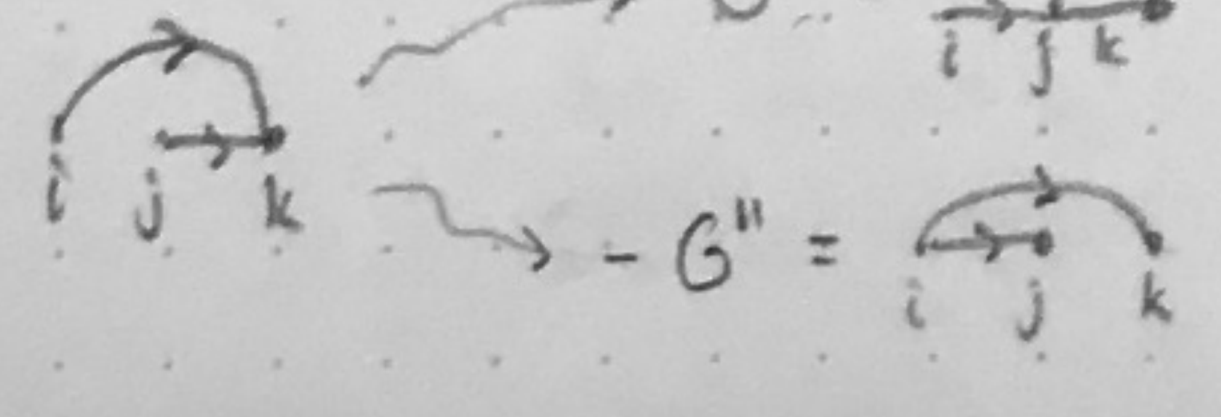
$\mathrm{Hilb}_{OT_n}(t) = \mathrm{Hilb}_{OS_n}\left(\frac{t}{1-t}\right) = \frac{1}{(1-t)^n}(1+t)(1+2t) \cdots (1+(n-2)t)$

Exercise: Prove this

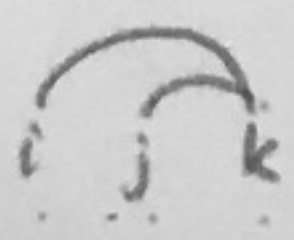
A linear basis of OS_n. Draw as graphs. Cycles are 0. Edges always directed from smaller to larger vertex
⇒ need lin. ind forests

Game on graphs:

NOTE: we are constructing basis up to sign of basis el'ts



keep playing game → Alternating sum of poly'ls in a_{ij}'s corresponed to graphs w/out



Claim: No matter how we play the game we get the same result. This is a basis.

Def: $F \subset K_n$ is an <u>increasing forest</u> if each conn. comp. has root = its min elt and the labels are increasing as we go away from the root

Ex.

<u>Claim</u>. Monomials corr. to increasing forests in K_n form a linear basis of OS_n
(And allowing mult. edges gives lin. basis of OT_n)

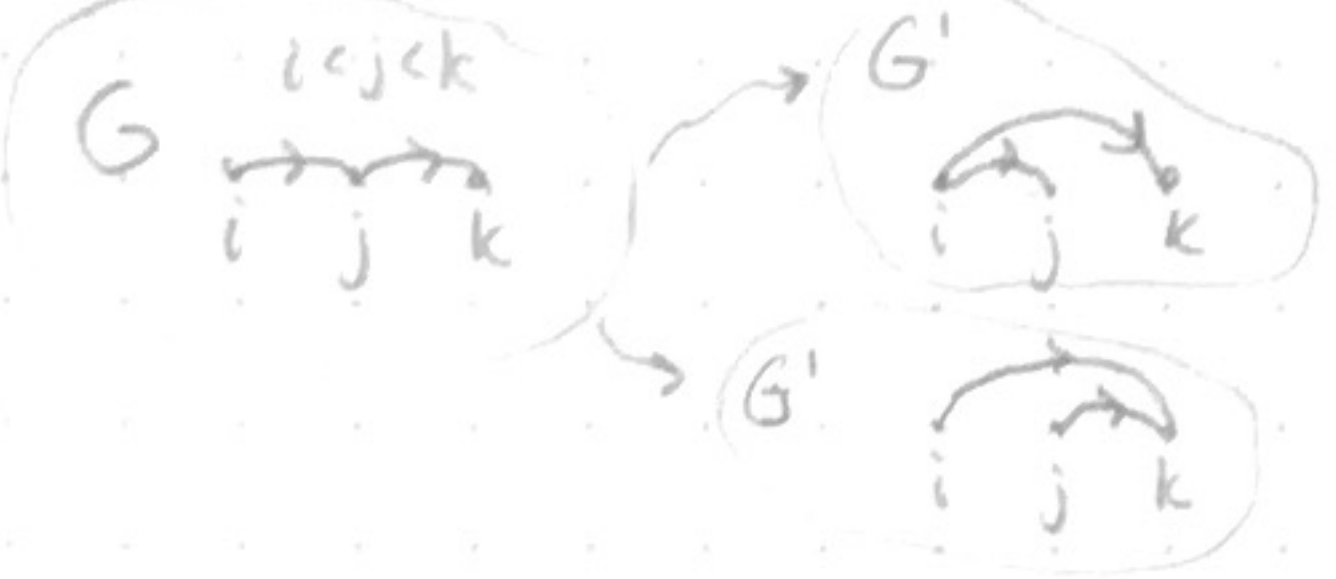
Another lin. basis of OS_n is given by forests s.t. each connected component is a chain starting at the minimal vertex of this comp.

Ex.

$$\sum_{\substack{F\subset K_n\\ \text{increasing}\\ \text{forest}}} f^{\#\text{ edges in } F} = (1+t)(1+2t)\cdots(1+(n-1)t) = \sum_{k=0}^{n-1} c(n,n-k)\,t^k$$

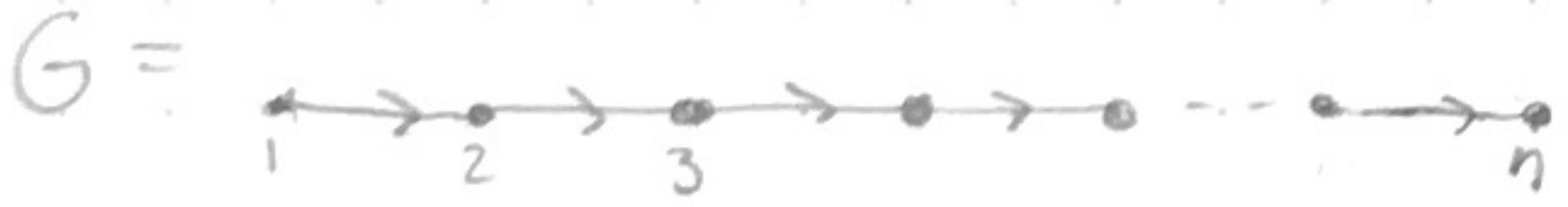
signless Stirling #'s of the first kind $\overset{\text{def}}{:=}$ # perms in S_n with $n-k$ cycles

<u>Another game on graphs</u> (representing monomials in OT_n)

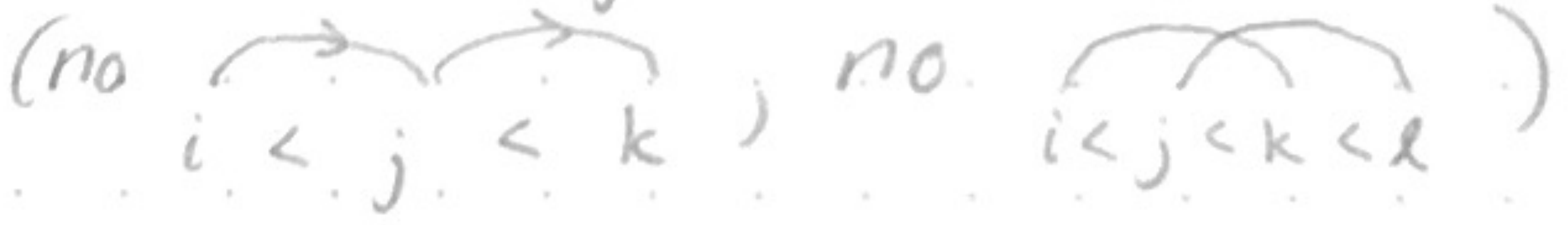


⇝ lin comb some graphs w/out 2 chains
alternating graphs (every vertex either source or sink)
get different ending graphs depending on how you play

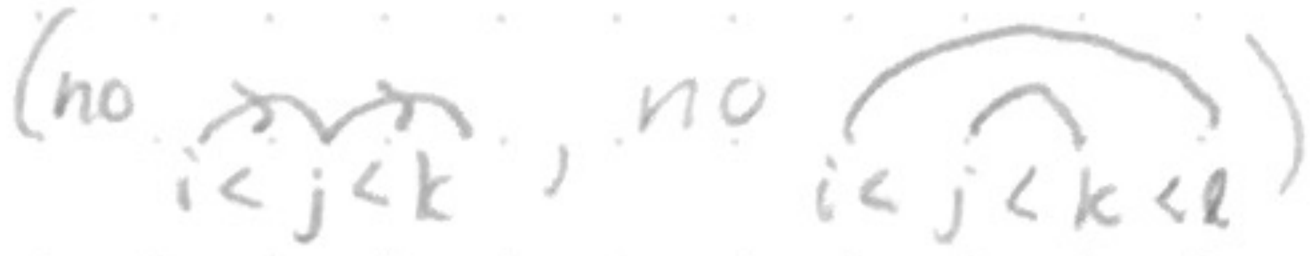
Assume that G is a tree



one way to play ⟷ all alternating non-crossing trees in K_n



alternate way to play ⇝ All alternating non-nesting trees



<u>Ex.</u> <u>n=4</u>

<u>Root polytope</u> $R_G = \operatorname{conv}(0, e_i - e_j\ \forall \text{ edges } (i,j)\in G\ i<j)$

Can break into pieces
# of simplices should be same however you do it.

(LECTURE 31 missed for Thanksgiving)

18.217 LECTURE 32

Previously: V_λ irreps of S_n, $V_{\lambda/\mu}$ more general set of reps

$\chi_\lambda = \chi_{V_\lambda}$ form a basis in the space of class functions on S_n (functions const on conjugacy classes)

"characters" = $\mathbb{Z}_{\geq 0}$-linear combinations of χ_λ's.

"virtual character" = arbitrary class functions = arbitrary lin. comb. of χ_λ's (not necessarily over $\mathbb{Z}_{\geq 0}$, could be over $\mathbb{C}$ or something)

Frobenius Character map

$$ch : \left\{ \begin{matrix} \text{class functions} \\ \text{on } S_n \end{matrix} \right\} \longrightarrow \Lambda^n$$ (the space of homogeneous symmetric functions (in infinitely many vars) of degree n.)

$\chi : S_n \to \mathbb{C}$ (any virtual character)

$$ch(\chi) := \frac{1}{n!} \sum_{w \in S_n} \chi(w) P_{\text{type}(w)}$$

type(w) = the cyclic type of w

$\nu = (\nu_1, \nu_2, \ldots)$ partition of n

$P_\nu = P_{\nu_1} P_{\nu_2} P_{\nu_3} \cdots$

and $P_k := x_1^k + x_2^k + \ldots$ (power s.f)

Equivalently,
$$ch(\chi) = \sum_{\nu \vdash n} \chi(\nu) \frac{P_\nu}{z_\nu}, \text{ where}$$

$$\frac{1}{z_\nu} = \frac{1}{n!} \# \left\{ \begin{matrix} \text{permutations in } S_n \\ \text{of cyclic type } \nu \end{matrix} \right\}$$

Lemma: $z_\nu = \prod_{i \geq 1} i^{m_i} \cdot m_i!$ where m_i = # parts i in ν.

E.x. $\nu = (4,2,2,1,1,1,1)$ $m_1 = 4$, $m_2 = 2$, $m_3 = 0$, $m_4 = 1$

$z_\nu = 4!\, 2^2 2!\, 4^1 1!$

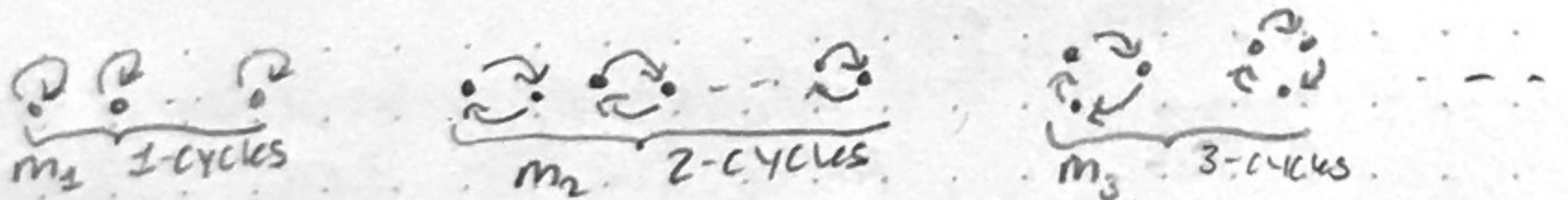
Proof: Fix a perm $w \in S_n$ of cyclic type ν → The orbit of S_n-action on S_n by conjugation $u : w \mapsto uwu^{-1}$

$\frac{n!}{z_\nu} := \#\{\text{permutations of type } \nu\} = |\text{Orbit}(w)|$

$|\text{Orbit}(w)| = \frac{|S_n|}{|\text{Fix}(w)|}$ where $\text{Fix}(w) = \{u \in S_n \text{ s.t. } uwu^{-1} = w\}$

so $z_n = |\text{Fix}(w)|$

$w =$



↳ Can permute labels in cycles & permute cycles themselves

$\Rightarrow |\text{Fix}(w)| = m_1! \cdot 2^{m_2} m_2! \cdot 3^{m_3} m_3! \cdots$

Thm (Frobenius): $ch(\chi_\lambda) = S_\lambda$

More generally, $ch(\chi_{\lambda/\mu}) = S_{\lambda/\mu}$ ← skew Schur functions

Original Formulation: $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, $\rho = (k-1, k-2, \ldots, 1, 0)$

$\chi_\lambda(\nu)$ = the coeff of $x^{\lambda+\rho}$ in $\left(\prod_{1 \leq i < j \leq k} (x_i - x_j)\right) P_\nu(x_1, \ldots, x_k)$.

Exercise: Confirm these two formulations are equivalent using classical definition of Schur functions

Logic of Proofs

Claims: A.) MN-rule for $\chi_{\lambda/\mu}$ ✓ (we already proved)

B.) Frobenius Char. formula

C.) MN rule for $S_{\lambda/\mu}$

Knowing any 2 of these claims implies the 3rd

We will prove C.), and then this will automatically imply B.)

Murnaghan-Nakayama (version for symmetric functions)

Thrm: $$S_{\lambda/\mu} = \sum_{\substack{\nu \\ |\nu| = |\lambda/\mu|}} \sum_{\substack{\text{RT ribbon tableaux} \\ \text{of shape } \lambda/\mu \text{ and} \\ \text{type } \nu}} \frac{\rho_\nu}{z_\nu}$$

Thrm: $$\rho_k \cdot S_\mu = \sum_{\substack{\lambda \supseteq \mu \\ \lambda/\mu \text{ is a } k\text{-ribbon}}} (-1)^{ht(\lambda/\mu)} S_\lambda$$

Cor: $\rho_k = S_k - S_{(k-1,1)} + S_{(k-2,2)} - \dots$

Ex. $\rho_2 = S_{\square\square} - S_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = h_2 - e_2 = x_1^2 + x_2^2 + \dots$ $\qquad \rho_3 = S_{\square\square\square} - S_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} + S_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$

Proof: $$H(t) = \sum_{k \geq 0} h_k t^k = \prod_{i \geq 1} \frac{1}{1 - x_i t}$$

$$E(t) = \sum_{k \geq 0} e_k t^k = \prod_{i \geq 1} (1 + x_i t)$$

$$P(t) = \sum_{k \geq 1} \rho_k t^{k-1} = \sum_{i \geq 1} (x_i + x_i^2 t + x_i^3 t^2 + \dots) = \sum_{i \geq 1} \frac{x_i}{1 - x_i t}$$

(1) $H(t)E(-t) = 1$

(2) $H'(t) = P(t)H(t)$ $\Leftrightarrow$ Newton's formulas

(3) $E'(t) = P(-t)E(t)$

$$P(t) \overset{(2)}{=} \frac{H'(t)}{H(t)} = H'(t)E(-t)$$

$\Updownarrow$

Lemma: $$\rho_k = \sum_{r=1}^{k} r \cdot h_r (-1)^{k-r} e_{k-r}$$

$$\rho_k \cdot S_\mu = \sum_{\substack{r \geq 1 \\ \lambda \geq 0 \\ r + \lambda = k}} r (-1)^{\ell} e_\ell \cdot (h_r \cdot S_\lambda)$$

Sign = $(-1)^{\#\text{ green boxes}}$