18.217 LECTURE 24

Last time: $\mathbb{C}[S_0] < \mathbb{C}[S_1] < \mathbb{C}[S_2] < \dots$ $\qquad$ $\mathbb{C}[S_n]$ group algebra of S_n

$$\underset{\text{center}}{Z_n = Z(\mathbb{C}[S_n])} = \left\{ \sum_{w \in S_n} f_w w \;\middle|\; f_w \text{ is a class function on } S_n \right\}$$

$$= \langle c_\lambda \mid \lambda \vdash n \rangle$$

(constant on conjugacy classes)

$$c_\lambda := \sum_{\substack{w \in S_n \\ \text{of cycle type } \lambda}} w$$

Jucys-Murphy elts $\quad X_1, \dots, X_n \in \mathbb{C}[S_n]$

$$X_i = \sum_{j<i} (j, i)$$

Gelfand-Tsetlin subalgebra $GT_n < \mathbb{C}[S_n]$

$:=$ subalgebra generated by $Z_1, Z_2, \dots, Z_n$

Thrm: GT_n is generated by $X_1, \dots, X_n$

Lemma Each c_λ can be expressed in $X_1, \dots, X_n$

GT-basis $\{v_T\}$ of irrep V_λ of S_n

"Tableau" T is a path in the Betteli diagram from $\hat{0}$ to V_λ

Saw last time Jucys-Murphy elements generated by $Z_1, \dots, Z_n$.

Exercise Prove $X_1, \dots, X_n$ generate GT_n.

E.x. $c_{2\,1^{n-2}} = X_1 + \dots + X_n$

$$c_{3\,1^{n-3}} = X_1^2 + \dots + X_n^2 - \binom{n}{2}$$

3 cycles
$2\binom{n}{3}$ of them

Why? $\quad X_i^2 = \sum_{\substack{j_1 < i \\ j_2 < i}} (j_1, i)(j_2, i)$

3 cycle if $j_1 \neq j_2$
otherwise is the identity

Recall: Bretelli Diagram

vertices: $\bigcup \hat{S}_n$

edges: $\mathrm{Res}^{S_n}_{S_{n-1}}$ (irreps)

$$\mathbb{C}[S_n] \simeq \left\{ \begin{bmatrix} \boxed{d_1} & & & 0 \\ & \boxed{d_2} & & \\ & & \ddots & \\ 0 & & & \boxed{d_N} \end{bmatrix} \right\}$$

w.r.t GT-basis in each V_λ

$N = |\hat{S}_n|$

$d_1, \dots, d_N$ are dim V_λ

Lemma: $GT_n \cong$ subalgebra of all diagonal matrice

Proof: $Z(Mat(d \times d)) = \left\{ \begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{pmatrix} \right\}$

$$Z_n \cong \left\{ \begin{bmatrix} \boxed{\begin{smallmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & \alpha \end{smallmatrix}} & & 0 \\ & \boxed{\begin{smallmatrix} \beta & 0 \\ 0 & \beta \end{smallmatrix}} & \\ 0 & & \ddots \end{bmatrix} \right\}$$

Cor: GT-basis $\{v_T\}$ is a unique (up to rescaling) basis of V_λ s.t. each basis elt is a common eigenvector of $X_1, \dots, X_n$

$v_T \to (\alpha_1, \dots, \alpha_n)$ $\quad \alpha_1, \dots, \alpha_n \in \mathbb{C}$

s.t. $X_i \vec{v}_T \to \alpha_i \vec{v}_T$. $\quad (\alpha_1, \dots, \alpha_n)$ uniquely describes $\vec{v}_T$

$\underset{\text{"tableaux"}}{T} \xleftrightarrow{\text{bij.}} \{(\alpha_1, \dots, \alpha_n)\}$

Def: Spectrum $Spec(n)$:= the set of sequences of eigenvalues $(\alpha_1, \dots, \alpha_n)$ for all v_T for all irreps V_λ of S_n

Equivalence relation "$\sim$" on $Spec(n)$
$(\alpha_1, \dots, \alpha_n) \sim (\alpha_1', \dots, \alpha_n')$ if the corr to v_T and v_T' in the same irrep V_λ

Thrm: Elts $s_1, \dots, s_{n-1}$ and $X_1, \dots, X_{n-1}$ in $\mathbb{C}[S_n]$ satisfy the relations.

- Coxeter relations for $s_1, \dots, s_{n-1}$
- $X_i X_j = X_j X_i \quad \forall i, j$.
- $s_i X_j = X_j s_i$ if $j \neq i, i+1$.
- $s_i X_i = X_{i+1} s_i - 1$
- $s_{i+1} X_i = X_i s_i - 1$

Degenerated Affine Hecke algebra (DAHA)

Claim: USING THESE RELATIONS, WE CAN DESCRIBE THE WHOLE SPECTRUM

Proof $(\alpha_1, \dots, \alpha_n) \in Spec(n)$.

$\vec{v} = \vec{v}_T \in$ GT basis of V_λ s.t. $X_i \vec{v} = \alpha_i \vec{v} \quad \forall i = 1, \dots, n$

- $\alpha_1 = 0 \qquad (X_1 = 0)$

Suppose $\alpha_i = a$, $\alpha_{i+1} = b$

$X_i \vec{v} = a \vec{v}$, $X_{i+1} \vec{v} = b \vec{v}$

Let $\vec{v}' = s_i(v) \in V_\lambda$

2 cases

I. $\vec{v}$ & $\vec{v}'$ are lin. dependent

$s_i^2 = 1 \quad \Rightarrow \vec{v}' = \pm \vec{v}$

Use $s_i X_i = X_{i+1} s_i - 1$

$\hookrightarrow \pm a\vec{v} + \vec{v} = \pm b\vec{v}$

$\Rightarrow b = a \pm 1$

II. $\vec{v}, \vec{v}'$ are lin. independent

$\Rightarrow$ they span 2-dim subspace $\langle \vec{v}, \vec{v}' \rangle$ in V_λ

On this subspace, X_i, X_{i+1}, s_i acts as

$X_i = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix} \quad X_{i+1} = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix} \quad s_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$X_i \vec{v} = a\vec{v}$

$X_i \vec{v}' = -1\vec{v} + b\vec{v}$

(Will finish next lecture)

<u>Thrm</u>: $(\alpha_1, \dots, \alpha_n) \in \text{Spec}(n)$

(1) $\alpha_1 = 0$

(2) $\alpha_i \neq \alpha_{i+1}$

(3) If $\alpha_i = \alpha_{i+1} \pm 1$, then $s_i(v^T) = \pm v^T$

(4) If $\alpha_i \neq \alpha_{i+1} \pm 1$, then $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_n) \in \text{Spec}(n)$

(5) We cannot have $\alpha = (\dots, a, a+1, a, \dots)$

<u>Claim</u>: These properties uniquely define the spectrum.

$\hookrightarrow$ We can convert this into SYT

18.217 LECTURE 25

<u>Last Week</u>: Vershik-Okounkov's const.

V_λ irreps of S_n $\lambda \in \hat{S_n}$

<u>Jucy's-Murphy elts</u>: $X_i = \sum_{j<i} (j,i) \in \mathbb{C}[S_n]$

V_λ has a unique (up to rescaling) basis $\{v_T\}$ given by common eigenvectors of $X_1, \dots, X_n$

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, there is at most 1 basis elt of V_T s.t. $X_i v_T = a_i v_T$ for $i = 1, \dots, n$ (so we can label v_T by α)

Spec(n) := the set of all $\alpha = (\alpha_1, \dots, \alpha_n)$ for all v_T in all V_λ
$\alpha \sim \beta$ if α & β correspond to vectors in the same V_λ

<u>DAHA relations</u> for $s_1, \dots, s_{n-1}, X_1, \dots, X_n$

- coxeter rel. for $s_1, \dots, s_n$
- $X_i X_j = X_j X_i$
- $s_i X_j = X_j s_i$ if $j \neq i, i+1$
- $s_i X_i = X_{i+1} s_i - 1$
- $s_i X_{i+1} = X_i s_i - 1$

Analysis of Spec(n)

- $\alpha_1 = 0$ (since $X_0 = 0$)
- $\alpha_i = a$, $\alpha_{i+1} = b$ $v = v_T$

$\Rightarrow X_i : v \mapsto av$, $X_{i+1} : \mapsto bv$

$v' := s_i(v)$

<u>2 cases</u> (I) v & v' are lin. dep

$s_i^2 = 1 \Rightarrow v' = \pm v$

(DAHA) $\Rightarrow \underset{\pm av}{\overset{\shortparallel}{av'}} = \pm bv - v$ $\Rightarrow \boxed{b = a \pm 1}$

(II) v, v' are lin. independent
$\langle v, v' \rangle$ 2-dim subspace of V_λ

$X_i = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}$, $X_{i+1} = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}$ on this subspace

$X_i : v \mapsto av$ $X_i : v' \mapsto X_i(s_i v) = (s_i X_{i+1} - 1)v = -v + bv'$

Lemma: $a \neq b$

Proof: We know X_i should be diagonalizable, but if $a=b$, X_i has nontrivial Jordan block $\begin{pmatrix} a & -1 \\ 0 & a \end{pmatrix}$ so can't be diagonalizable

$\tilde{v} = v + (a-b)v'$

$\langle v, v' \rangle = \langle v, \tilde{v} \rangle$ ✓

$$\begin{aligned} X_i : \tilde{v} &\mapsto X_i(v + (a-b)v') \\ &= av + (a-b)(-v + bv') \\ &= b(v + (a-b)v') = b\tilde{v} \end{aligned}$$

$X_{i+1} : \tilde{v} \mapsto a\tilde{v}$

$X_j \tilde{v} = \alpha_j \tilde{v}$ for $j \neq i, i+1$

So $\tilde{v}$ is a common eigenvector of $X_1, \dots, X_n$

$\Rightarrow \tilde{v} \in$ GT-basis with eigenvalues $\tilde{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_n)$

Lemma In case (II), $a-b \neq \pm 1$

Proof: Otherwise $\tilde{v} = v \pm S_i(v)$

and $S_i(\tilde{v}) = S_i(v) \pm v = \pm \tilde{v}$

Then $\tilde{v}$ falls under case (I) $\Rightarrow b = a \pm 1 \Rightarrow a-b = \mp 1$ - contradiction

In general case

Lemma: $\alpha = (\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_n)$

(1) $\alpha_{i+1} \neq \alpha_i$

(2) $\begin{cases} \alpha_i = \alpha_{i+1} \pm 1 & \Longleftrightarrow \text{Case (I)} \\ \alpha_i \neq \alpha_{i+1} \pm 1 & \Longleftrightarrow \text{case (II)} \end{cases}$

and in case(II), $(\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_n) \in Spec(n)$

Lemma: We cannot have $(\alpha_1, \dots, \alpha_n) = (\dots \overset{i}{a}, \overset{i+1}{a\pm 1}, \overset{i+2}{a}, \dots)$

Proof: Assume for contradiction we have $(\dots, a, a+1, a, \dots)$

we are in case(I) so

$S_i S_{i+1} S_i(v) = S_i S_{i+1}(+v) = S_i(-v) = -v$

whereas $S_{i+1} S_i S_{i+1}(v) = v$ $\neq$ - contradiction

(and similarly for $(\dots, a, a-1, a, \dots)$)

Def: An allowed transposition $(\alpha_1, \dots, \alpha_n) \longleftrightarrow (\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_n)$ if $\alpha_{i+1} \neq \alpha_i + 1$

Thrm: $\forall\, \alpha = (\alpha_1, \dots, \alpha_n) \in Spec(n)$ we have

- $\alpha_1 = 0$
- $\alpha_{i+1} \neq \alpha_i \;\forall i$
- we cannot have $\alpha = (\dots a, a\pm 1, a, \dots)$
- For any allowed transposition $\beta = (\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_n)$, $\beta \in Spec(n)$, $\beta \sim \alpha$

Corollaries: (1) $(\alpha_1, \dots, \alpha_n) \in Spec(n)$. all $\alpha_i \in \mathbb{Z}$

Proof: Suppose they are not. Find first noninteger entry.
Allowed transposition w/ everything
$\Leftrightarrow$ can commute until it's in first position, but we know first position has to be 0

(2) If $\alpha_i = \alpha_j = a$, $i<j$ then $a-1$ and $a+1 \in \{\alpha_{i+1}, \dots, \alpha_{j-1}\}$

Proof: Find such pair with smallest distance.
Can commute until we have $(a, a\pm 1, \dots, a\mp 1, a)$
NOTE: if it's the same sign, then
$(a+1, \dots, a+1)$ is at a smaller distance — contradiction

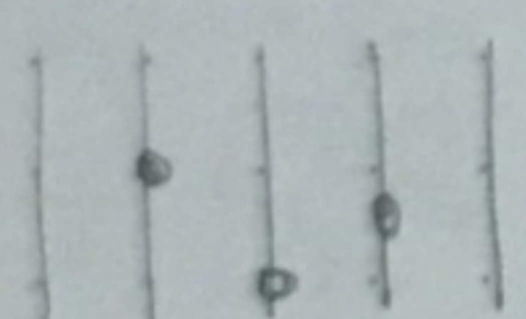
(3) $\forall\, i>1$ α_i+1 or α_{i-1} belongs to $\{\alpha_1, \dots, \alpha_{i-1}\}$

Proof: If $\alpha_i = 0$, then we have $\{0, \dots, 0, \dots\}$
so by property (2), we should have both α_i+1 and α_i-1

If $\alpha_i \neq 0$, and (3) is not true, we can commute until we get a nonzero entry in the first position.

↑ These are all the properties we need to characterize $Spec(n)$

Preview:



beads : α_i's
strings : values of α_i's.

under specific conditions (given above) can commute them

18.217 LECTURE 26

V_λ irreps of S_n, $\{v_T\}$ GT-basis of V_λ

$$v_T \longleftrightarrow \alpha = (\alpha_1, \ldots, \alpha_n) \qquad \text{where} \quad X_i v_T = \alpha_i v_T$$

Spec(n) is the set of all $(\alpha_1, \ldots, \alpha_n)$ for all v_T of all irreps of S_n
$\alpha \sim \beta$ if α, β correspond to basis elts of the same V_λ

Last time

<u>Thrm</u>: $\forall \alpha \in \text{Spec}(n)$
(1) $\alpha_1 = 0$
(2) $\forall i = 1, 2, \ldots, n \qquad \alpha_i - 1$ or $\alpha_i + 1 \in \{\alpha_1, \ldots, \alpha_{i-1}\}$
(3) If $\alpha_i = \alpha_j = a$, $i<j$, both $a-1$ and $a+1 \in \{\alpha_{i+1}, \ldots, \alpha_{j-1}\}$
(4) $\forall \beta$ obtained from α by <u>allowed transposition</u>, $\beta \in \text{Spec}(n)$ and $\alpha \sim \beta$

<u>Def</u>: <u>Allowed transposition</u>: $(\alpha_0, \ldots, \alpha_n) \longleftrightarrow (\alpha_0, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_n)$ if $\alpha_{i+1} - \alpha_i \neq \pm 1$

<u>Def</u>: Cont(n) $\subset \mathbb{Z}^n$ is the set of vectors $\alpha_1, \ldots, \alpha_n$ s.t. α satisfies (1)–(3), and any allowed transp. of α also satisfies these conditions.

Content

Let "$\approx$" be equivalence rel. on Cont(n) generated by allowed transpositions

$$\text{Spec}(n) \subseteq \text{Cont}(n)$$
$$\alpha, \beta, \ \alpha \approx \beta \Rightarrow \alpha \sim \beta$$

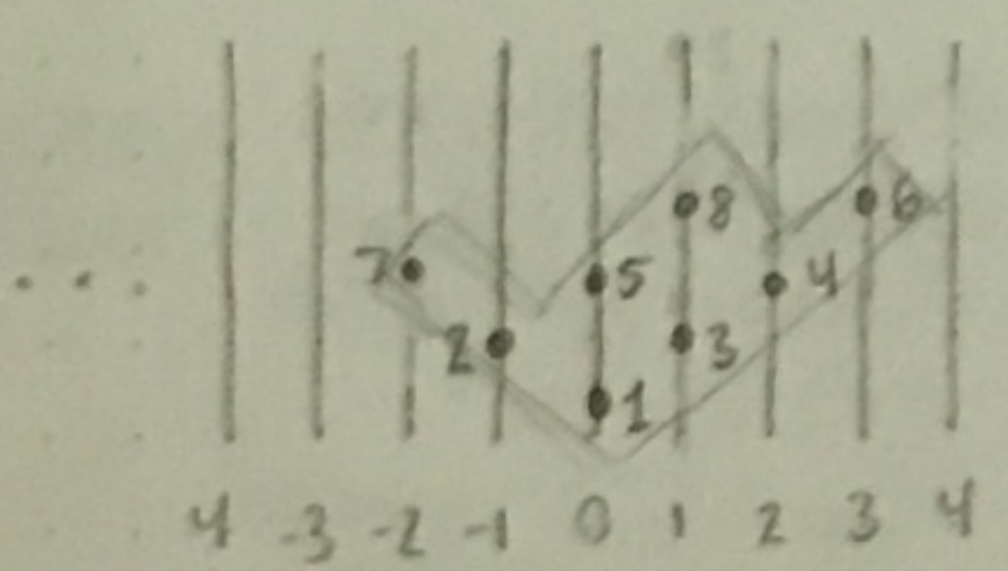
$$\#\,\text{Spec}(n)/\!\sim \ \leq \ \#\,\text{Cont}(n)/\!\approx$$

$p(n)$ ← # of conjugacy class / # of partitions of n

n	Cont(n)
1	(0)
2	(0,1), (0,−1)
3	(0,1,2), (0,1,−1), (0,−1,−2), (0,−1,1)

$\approx$

Lets represent $\alpha_1, \ldots, \alpha_n \in$ Cont(n) by an abacus w/ n beads labelled $1, 2, \ldots, n$



Can't slide beads past that are only 1 apart
→ Every time you drop a bead it should bump into something below

$\alpha = (0, -1, 1, 2, 0, 3, -2, 1)$

<u>Thrm</u>: Cont(n) in bijection with set of all SYT with n boxes, and $\alpha \approx \beta$ iff they correspond to SYT of the same shape

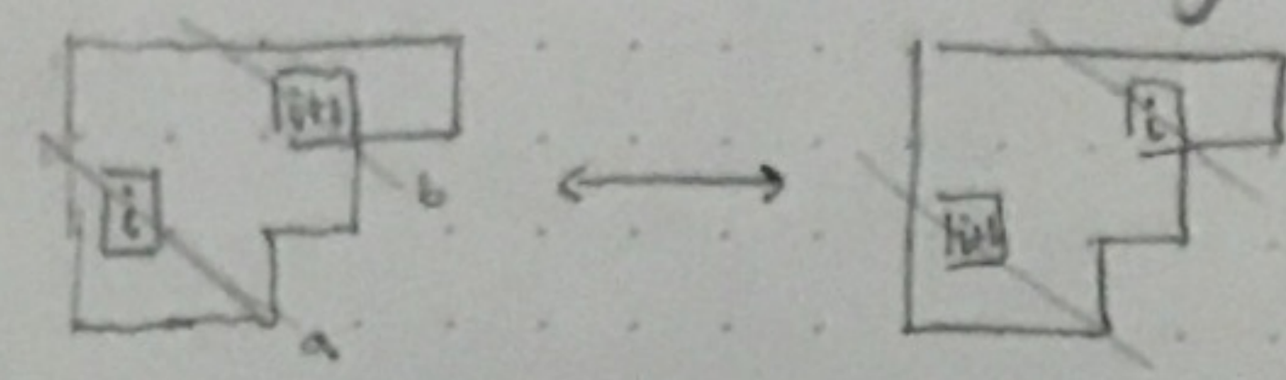
$\alpha = (0, -1, 1, 2, 0, 3, -2, 1) \rightsquigarrow T =$

1	3	4	6
2	5	8	
7			

α_i = Content of the entry i in T

Allowed transposition : Switches beads i & $i+1$ if they are not on adjacent strings

↳ Switches i & $i+1$ in SYT if they are not on adjacent diagonals



$a-b \neq \pm 1$

<u>Lemma</u>: Any two SYT of the same shape can be obtained from each other by a sequence of allowed transpositions.

$\Rightarrow \mathrm{spec}(n) = \mathrm{Cont}(n)$, $\sim \; = \; \approx$

<u>Cor</u>: The Bratteli diagram for $S_0 \subset S_1 \subset S_2 \subset \dots$ is Young's Lattice

$\mathrm{Res}^{S_n}_{S_{n-1}} V_\lambda = \bigoplus_{\substack{\mu \text{ obtained} \\ \text{by removing} \\ \text{a single box}}} V_\mu$ $\qquad (\alpha_1, \dots, \alpha_n) \rightsquigarrow (\alpha_1, \dots, \alpha_{n-1})$

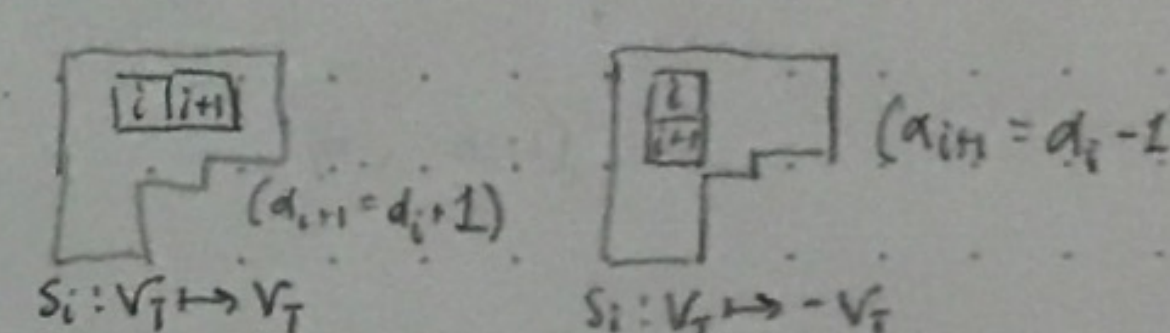
(remove box containing α_n)

<u>Young's Orthogonal Form</u> Murnaghan Nakayama Rule.

Describe the action of S_n on GT-basis. Enough to describe action on $s_1, \dots, s_{n-1} \in S_n$

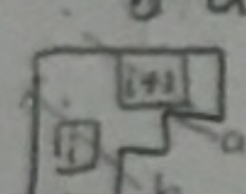
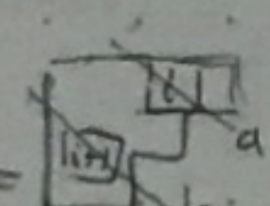
<u>Thrm</u>: (Young's Orth. Form)

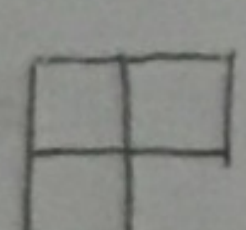
Fix $\lambda \vdash n$. GT-basis $\{v_T \mid T \in SYT(\lambda)\}$



Ⓘ $s_i : v_T \longmapsto \pm v_T$ if boxes i & $i+1$ in adjacent diagonals

Ⓘⓘ $s_i : v_T \longmapsto \frac{1}{b-a} v_T + \sqrt{1 - \frac{1}{(b-a)^2}}\, v_{\tilde{T}}$

if $T =$  $(a-b \neq 1)$ and $\tilde{T} =$  obtained from T by allowed transp.

<u>E.x.</u> $\lambda =$ 

$v_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} \qquad v_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$

$s_1 : v_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} \mapsto v_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$

$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$v_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} \mapsto -v_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$

$s_2 : v_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} \mapsto -\frac{1}{2} v_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} + \frac{\sqrt{3}}{2} v_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$

$s_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

$v_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} \mapsto \frac{1}{2} v_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} + \frac{\sqrt{3}}{2} v_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$