18.217 LECTURE 22

10/31/22

Introduction to Representation Theory

G finite group

def: A <u>representation</u> of G is a homomorphism

$\rho: G \to GL(V)$ $\qquad$ $V \simeq \mathbb{C}^d$ (finite dim vector space)

Want to consider representations <u>up to isomorphism</u>

def: Two representations

$\rho_1: G \to GL(V_1)$

$\rho_2: G \to GL(V_2)$

if $\exists$ linear isomorphism $A: V_1 \to V_2$
s.t. $\rho_1(g) = A\rho_2(g)A^{-1}$

def: A representation $\rho: G \to GL(V)$ is <u>irreducible</u> if $\nexists$ proper subspace

$W \subsetneq V$, $W \neq \{0\}$ s.t. $\rho(g): W \to W$ $\forall g \in G$

<u>Operations of reps</u> $\rho_1: G \to GL(U)$, $\rho_2: G \to GL(V)$

- <u>Direct sum</u> $\rho_1 \oplus \rho_2 : G \to GL(U \oplus V)$

$$g \longmapsto \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}$$

- <u>Tensor Product</u> $\rho_1 \otimes \rho_2 : G \to GL(U \otimes V)$

<u>Facts</u>

1.) $\exists$ fin. many isom. classes of irreducible reps (irreps) of G
(# of irreps = # of conjugacy classes in G)

2.) Any rep of G decomposes into direct sum of irreps

<u>Notation</u>: $\hat{G} = \{V_1, \dots, V_p\}$ where V_i are all irreps of G (up to isomorphism)
p = # conjugacy classes in g.

$d_i = \dim V_i$

Fundamental isomorphism

$$\underset{\text{group algebra}}{\mathbb{C}[G]} \quad \underset{\text{algebra isomorphism}}{\cong} \quad \bigoplus_{i=1}^{p} \mathrm{End}(V_i)$$

where $\mathrm{End}(V_i)$ is alg. of endomorphisms (lin. operators) $V_i \to V_i$
$= \{\text{all } d_i \times d_i \text{ matrices}\}$

$$\Rightarrow \mathbb{C}[G] \cong \left\{ \begin{pmatrix} \square_{d_1} & & & & 0 \\ & \square_{d_2} & & & \\ & & \square_{d_3} & & \\ & & & \ddots & \\ 0 & & & & \square_{d_p} \end{pmatrix} \right\}$$

Block diagonal matrices

Now assume our group is $\underline{S_n}$

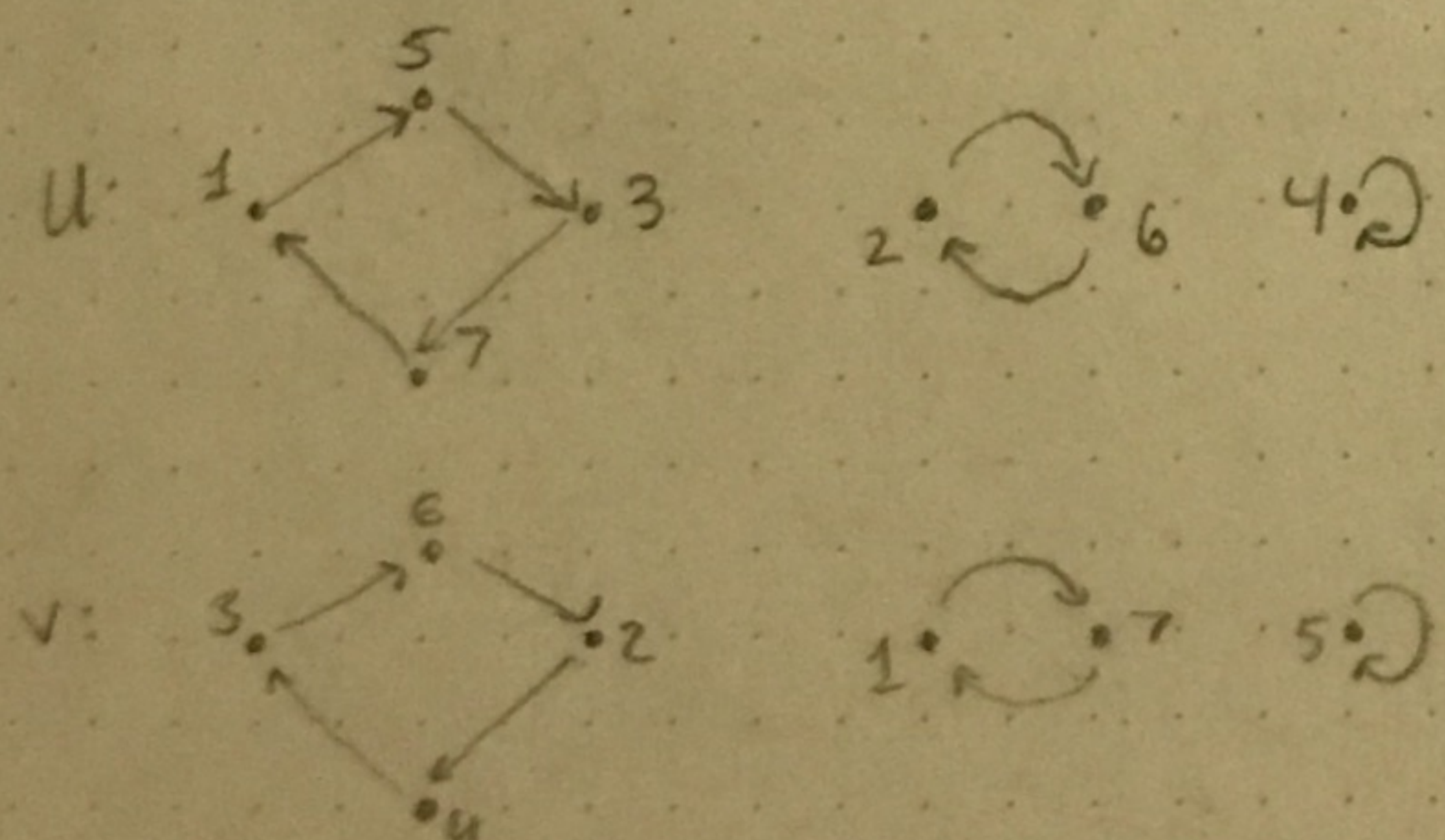
How many conjugacy classes are there?

Conjugate permutations $u, v \in S_n$

$u \sim v$ if $\exists\, w \in S_n$ s.t. $v = wuw^{-1}$

$\Leftrightarrow$ u & v have same cyclic type

Ex.



same cyclic type

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 5 & 6 & 7 & 4 \end{pmatrix}$$

$\Leftrightarrow$ # Conjugacy class in S_n
$=$ # cyclic types
$= p(n)$ (# of partitions of n)
$=$ # of Young Diagrams of n

Classical construction of irreps of S_n

Young Symmetrizer

Pick $\lambda \vdash n$ any tableau T of shape λ

$$T = \begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}$$

$$a_T = \sum_{\substack{u \in S_n \\ \text{preserve rows} \\ \text{in } T}} u \quad , \quad b_T = \sum_{\substack{v \in S_n \\ \text{that preserve} \\ \text{columns of } T}} (-1)^{\ell(v)} v \quad , \quad c_T = a_T b_T \in \mathbb{C}[S_n]$$

$V_T = \mathbb{C}[S_n] \cdot c_T < \mathbb{C}[S_n]$ with left action of S_n

i.e. $\rho : S_n \to GL(V_T)$ given by $\rho(\underset{\in S_n}{w})\sigma = w\sigma$

σ: generator of V_T ($\in S_n$)

Facts:

- $V_T \cong V_{T'}$ if T & T' have same shape.
- $V_\lambda = V_T$ for any T of shape λ.
- $S_n^\wedge = \{ V_\lambda \mid \lambda \vdash n \}$

$$T = \begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\ \hline 1 & 5 \\ \cline{1-2}\end{array} \qquad T' = \begin{array}{|c|c|c|}\hline 5 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

4 perms preserving columns of these rows

E.x. S_3 $\begin{array}{|c|c|c|}\hline 2 & 3 & 1 \\ \hline\end{array}$

$a_T = 1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + \underset{= s_1 s_2 s_1}{w_0}$

$b_T = 1$

$c_T = a_T b_T$

$$\mathbb{C}[S_3]\, c_T = \underbrace{\left\{ \begin{array}{l} \text{subspace spanned} \\ \text{by } w(1 + s_1 + s_2 + s_1 s_2 + w_0 + s_2 s_1) \end{array} \right\}}_{\text{always a multiple of this same vector}} = \underbrace{\{ c(1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + w_0) \mid c \in \mathbb{C} \}}_{\text{1-dim subspace}}$$

Trivial rep of S_3
w acts trivially

Now for column $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline\end{array}$

$a_T = 1$

$b_T = 1 - s_1 - s_2 + s_1 s_2 + s_2 s_1 - w_0$

Sign rep of S_3
w acts as $(-1)^{\ell(w)}$

$V_{\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}} = \mathbb{C}[S_3] \cdot c_T = \{ c(1 - s_1 - s_2 + s_1 s_2 + s_2 s_1 - w_0) \}$

$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

$a_T = 1 + s_2$

$b_T = 1 - s_1$

$c_T = a_T b_T = (1+s_2)(1-s_1)$
$= 1 - s_1 + s_2 - s_1 s_2$

$$V_T = \left\{ \begin{array}{l} \text{subspace of } \mathbb{C}[S_3] \text{ spanned} \\ \text{by } w(1-s_1+s_2-s_1s_2) \text{ for } w \in S_3 \end{array} \right\}$$

$$s_1 \underbrace{(1-s_1+s_2-s_1s_2)}_{(1,-1,1,-1,0,0)} = \underbrace{s_1 - 1 + s_1s_2 - s_2}_{(-1,1,-1,1,0,0)}$$

$$s_2(1-s_1+s_2-s_1s_2) \rightsquigarrow (1,0,1,0,-1,-1)$$

Claim: V_T is 2-dim subspace

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G finite group

$\hat{G} = \{V_1, \ldots, V_n\}$ irreps of G

Each rep of G, $V \simeq \underbrace{V_1 \oplus \ldots \oplus V_1}_{m_1} \oplus \ldots \oplus \underbrace{V_n \oplus \ldots \oplus V_n}_{m_n}$

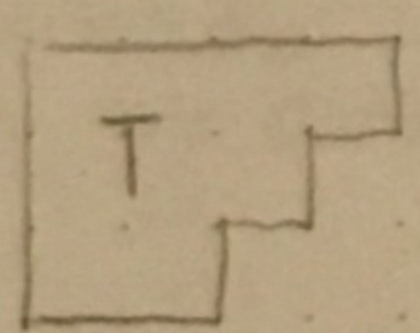
$$= V_1^{\oplus m_1} \oplus \ldots \oplus V_n^{\oplus m_n}$$

<u>Fact</u>: If all $m_i \in \{0,1\}$ (multiplicity free case), then this decomposition is unique

$$\mathbb{C}[G] \simeq \left\{ \begin{bmatrix} \boxed{d_1} & & & 0 \\ & \boxed{d_2} & & \\ & & \ddots & \\ 0 & & & \boxed{d_n} \end{bmatrix} \right\} \qquad d_i = \dim V_i$$

$$|G| = \sum_{i=1}^{n} (d_i)^2$$

<u>E.x.</u> S_n: $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$

λ 

$V_T = \mathbb{C}[S_n]\, a_T b_T$

$\tilde{V}_T = \mathbb{C}[S_n]\, b_T a_T$

<u>Fact</u>: $V_T \cong \tilde{V}_T$

<u>Problem</u>: Find 2 invertible elements $f, g \in \mathbb{C}[S_n]$ s.t.
$a_T b_T = f\, b_T a_T\, g$

This would demonstrate fact combinatorially w/ explicit construction
(currently all known proof of fact are not so direct)

Vershik-Okounkov's "new approach" to reps of S_n

Study all $S_0 < S_1 < S_2 < \ldots$ at once
$S_n < S_{n+1}$ embedded in standard way

$$w = \begin{pmatrix} 1 & \cdots & n \\ w_1 & \cdots & w_n \end{pmatrix} \longmapsto \begin{pmatrix} 1 & \cdots & n & n+1 \\ w_1 & \cdots & w_n & n+1 \end{pmatrix}$$

<u>Exercise</u>: Find example of different embedding $S_n \hookrightarrow S_{n+1}$ which is not isomorphic to this one

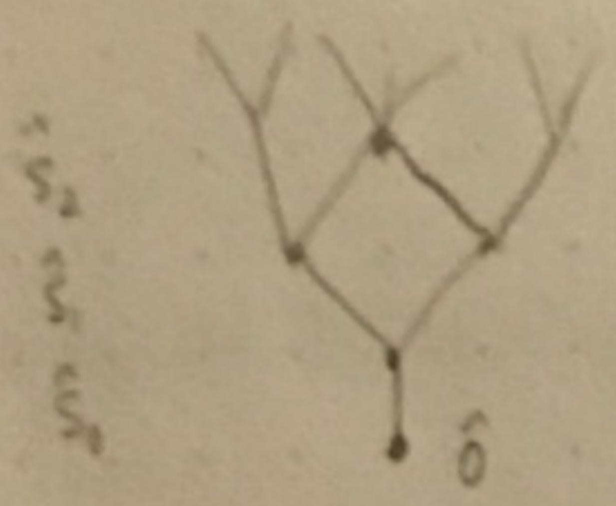
$S_n^\vee = \{V_\lambda \mid \lambda \vdash n\}$ {Assume for now λ are just labelling, but we don't know they are Young Diagram

<u>Crucial Fact</u>: $\mathrm{Res}^{S_n}_{S_{n-1}} V_\lambda \cong \bigoplus_{\mu < \lambda} V_\mu$ is multiplicity free

Rep S_{n-1}

Defined for any series of embedded gps or algebras

$\mu < \lambda$ is covering relation in "Brotteli Diagram"



(Will turn out to be Youngs lattice.)

<u>Def</u>: <u>Gelfand-Tsetlin</u> basis of V_λ unique (up to rescaling basis) defined by the following rule:

$$\mathrm{Res}^{S_n}_{S_{n-1}} V_\lambda = \bigoplus_{\mu < \lambda} V_\lambda$$

$$\mathrm{Res}^{S_{n-1}}_{S_{n-2}} V_\mu = \bigoplus_{\nu < \mu} V_\nu$$

Take vector in each component corresponding to representations of S_0.

<u>Ex.1)</u> S_3

$$V_3 = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}$$

S_3 acts by permutations of x_1, x_2, x_3.

$$\mathrm{Res}^{S_3}_{S_2} V_3 = \langle (1,1,-2) \rangle \oplus \langle (1,-1,0) \rangle$$

GT-basis $\frac{1}{\sqrt{6}}(1,1,-2)$, $\frac{1}{\sqrt{2}}(1,-1,0)$

<u>Ex.2)</u> S_4

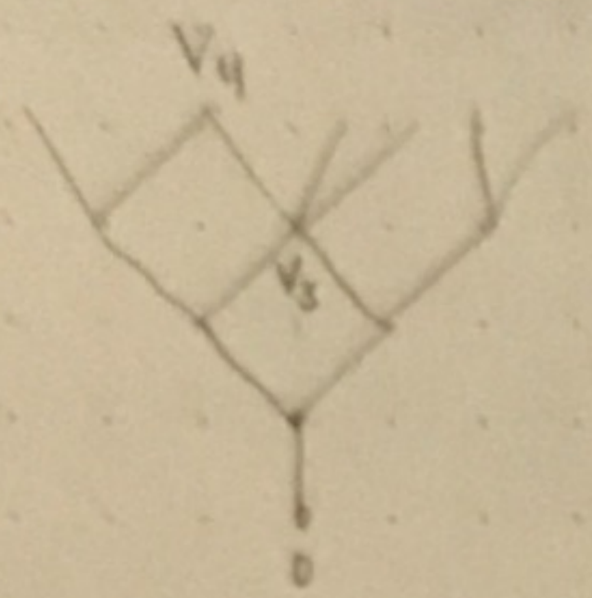
$$V_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \mid x_1 + \dots + x_4 = 0\}$$

$$\mathrm{Res}^{S_4}_{S_3} V_4 = \underbrace{\overset{\dim 2}{\{(x_1, x_2, x_3, 0) \mid x_1 + x_2 + x_3 = 0\}}}_{= V_3} \oplus \overset{\dim 1}{\langle 1,1,1,-3 \rangle}$$

$$\mathrm{Res}^{S_4}_{S_2} = \langle (1,1,-2,0) \rangle \oplus \langle (1,-1,0,0) \rangle + \langle (1,1,1,-3) \rangle$$

GT basis $\frac{1}{\sqrt{6}}(1,1,-2,0)$, $\frac{1}{\sqrt{2}}(1,-1,0,0)$, $\frac{1}{\sqrt{12}}(1,1,1,-3)$

Elements of GT basis for V_λ correspond to saturated chains in the Bratteli diagram from $\hat{0}$ to λ



$\mathbb{C}[S_0] \subset \mathbb{C}[S_1] \subset \mathbb{C}[S_2] \subset \ldots$

<u>Def</u>: $Z_n = Z_{\mathbb{C}[S_n]}$ the <u>center</u> of $\mathbb{C}[S_n]$

$= \{ f \in \mathbb{C}[S_n] \mid f \cdot g = g \cdot f \;\; \forall g \in \mathbb{C}[S_n] \}$

$f = \sum_{w \in S_n} f_w w \qquad f_w \in \mathbb{C}$

when is $f \in Z_n$?

Enough to check $f \cdot u = u \cdot f \;\; \forall u \in S_n$

$f \in Z_n \iff f_w = f_{uwu^{-1}}$

$w \mapsto f_w$ is constant on conjugacy classes of S_n

They are called <u>class functions</u> on S_n

<u>Def</u>: <u>Gelfand-Tsetlin subalgebra</u> of $\mathbb{C}[S_n]$

GT_n is generated by $Z_1, Z_2, \ldots, Z_n$

<u>Lemma</u>: This is a commutative subalgabra of $\mathbb{C}[S_n]$

(True since we picked all the Z_i to commute with all elements of $\mathbb{C}[S_n]$. This includes commuting with all the elements in the other Z_j)

Even more explicit construction: <u>(Young)-Jucys-Murphy elements</u>

$X_1, X_2, \ldots, X_n \in \mathbb{C}[S_n]$

$X_1 = 0$

$X_2 = (1,2)$

$X_3 = (1,3) + (2,3)$

$X_i = (1,i) + (2,i) + \ldots + (i-1,i)$

Properties of these elements

$X_i = \underbrace{\sum \left(\begin{smallmatrix}\text{all transp.}\\ \text{in } S_i\end{smallmatrix}\right)}_{\in Z_i} - \underbrace{\sum \left(\begin{smallmatrix}\text{all transp.}\\ \text{in } S_{i-1}\end{smallmatrix}\right)}_{\in Z_{i-1}} \in GT_n$

<u>Thrm</u>: GT_n is the subalgebra of $\mathbb{C}[S_n]$ generated by $X_1, \ldots, X_n$

<u>To Prove</u>:

$\lambda \vdash n \qquad C_\lambda = \sum_{\substack{\text{all terms}\\ w \in S_n \text{ of}\\ \text{cyclic type } \lambda}} w$

Need to express C_λ in terms of $X_1, \ldots, X_n$

$\to$ And this gives that X_i can be expressed in terms of C_λ