18.217 LECTURE 20

Problem Set Presentations

Problem 4.) P-poset, $|P| = n$ (Ivan Motorin)

<u>def</u>: <u>ext(P)</u> = set of order preserving maps $P \to [n]$

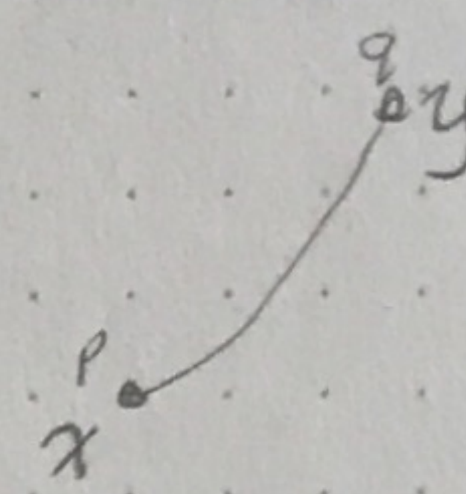
Let $h(x) = |S_x|$ where $S_x = \{y \geq x \mid y \in P\}$

(a) Prove $|ext(P)| \geq \frac{n!}{\prod_{x \in P} h(x)}$

<u>Sltn</u>: $\prod_{x \in P} h(x) \cdot |ext(P)| \geq n!$

$l_{st} \in exp(P)$ $\qquad$ l - <u>any</u> labelling on P

1.) Look at first w.r.t. l_{st} element x and $\underline{S_x}$.
Let us find minimal $y \in S_x$ w.r.t. l. & memorize y

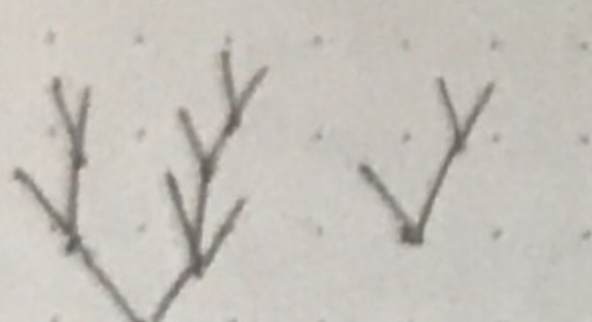


2.) look at 2^{nd} $\qquad$ $S_n \to \prod_{x \in P} S_x \times exp(P)$

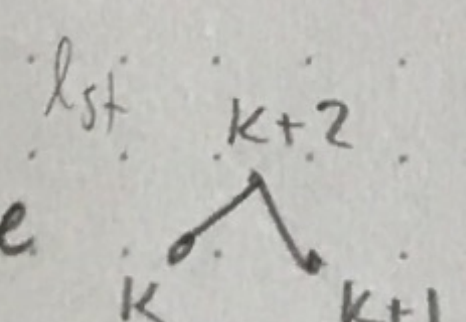
(On i^{th} step, take i^{th} element labelled w.r.t. standard labelling, pick element which has smallest label in l & switch the labels)

(b) Classify all P for which (a) is an equality.

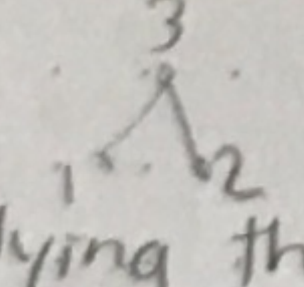
<u>Claim</u>: All rooted forests



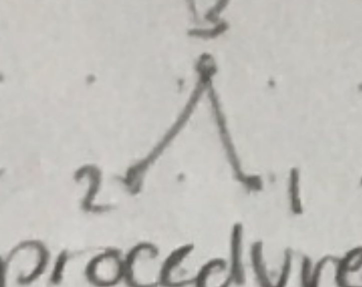
i.e. we don't have



If we have



,



they give the same labelling after applying the procedure.

(And if something contains a piece like this, the rest will be the same after, so the map is not injective)

For rooted forests, both sides reduce to binomial coeff
$\Leftrightarrow$ reduce to 1 tree
Then see algorithm gives bijection.

Problem 7.) (Ilani Axelrod-Freed)

Problem 17.) Kimi (Yihang Sun)

Deduce $\prod_{1\le i<j\le n} \frac{\lambda_j - \lambda_i + j - i}{i-j} = S_\lambda(1,\dots,1)$

$$S_\lambda(x) = \frac{a_{\delta+\lambda}(x)}{a_\delta(x)}$$

$a_\alpha = \det(x_i^{\alpha_j})$

$\delta = (n-1,\dots,0)$

$\delta+\lambda = (\lambda_j + n - j)_j$

Vandermonde: $a_\delta = \det(x_i^{n-j})_{ij}$

$$= \prod_{i<j} x_i - x_j$$

$x_i = t^{i-1}$ then $t \to 1$

$a_{\lambda+\delta} = \det((t^{i-1})^{\lambda_j+n-j})_{ij}$ $\qquad y_j = t^{\lambda_j+n-j}$

$= \det(y_j^{i-1})_{ij}$

$= (-1)^{\binom{n}{2}} \prod_{i<j} y_i - y_j$

$$\frac{a_{\lambda+\delta}}{a_\delta} = \prod_{i<j} \frac{y_i - y_j}{x_i - x_j}$$

$= \dots$

$$= \prod_{i<j} \frac{t^{\lambda_j+n-j} - t^{\lambda_i+n-i}}{t^{i-1} - t^{j-1}}$$

L'opithal $\Rightarrow$ $t \to 1$ = LHS

(Saba Lepsveridze) Same problem, a different way

$$\lim_{x_{k+1}\to 1} \frac{\partial^0_{x_1}\partial^1_{x_2}\dots\partial^{k-1}_{x_k} a^{\lambda+\delta}\big|_{x_1=\dots=x_k=1}}{\partial^0_{x_1}\dots\partial^{k-1}_{x_k} a^{\delta}\big|_{x_1=\dots=x_k=1}}$$

Take limits one by one

$$\det \begin{vmatrix} 1 & n-1 & (n-1)(n-2) & \cdots & x_{k+1}^{n-1} & \\ 1 & n-2 & \vdots & & \vdots & \cdots \\ \vdots & \vdots & & & & \\ 1 & & 1 & & & \\ 1 & 1 & 0 & & x_{k+1} & \\ 1 & 0 & 0 & & 0 & \end{vmatrix} = c_k \prod_{j>k} (x_j - 1)^k \prod_{i>j>k} (x_i - x_j)$$

$$S_\lambda(1, \dots, 1) = \frac{\det \begin{vmatrix} 1 & \lambda_1 + n-1 & \cdots & (\lambda_1 + n-1)(\lambda_1 + n-2) \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n + 0 & & (\lambda_n + 0)(\lambda_n - 1) \cdots \end{vmatrix}}{\det \begin{vmatrix} & * & & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}}$$

$\lambda_{i-i} = \lambda_{j-j}$

18.217 LECTURE 21 10/28

More Problem Set Presentations

Problem 12.) Classify when $f_\lambda^{\text{domino}} > 0$ (Hanna Mularczyk)

↪ When does λ have a domino tiling/numbering.

Give grid checkerboard tiling

Every domino has 1 tile in each color
⇒ must have same # of black & white tiles in λ

Condition λ has same # of W & B boxes

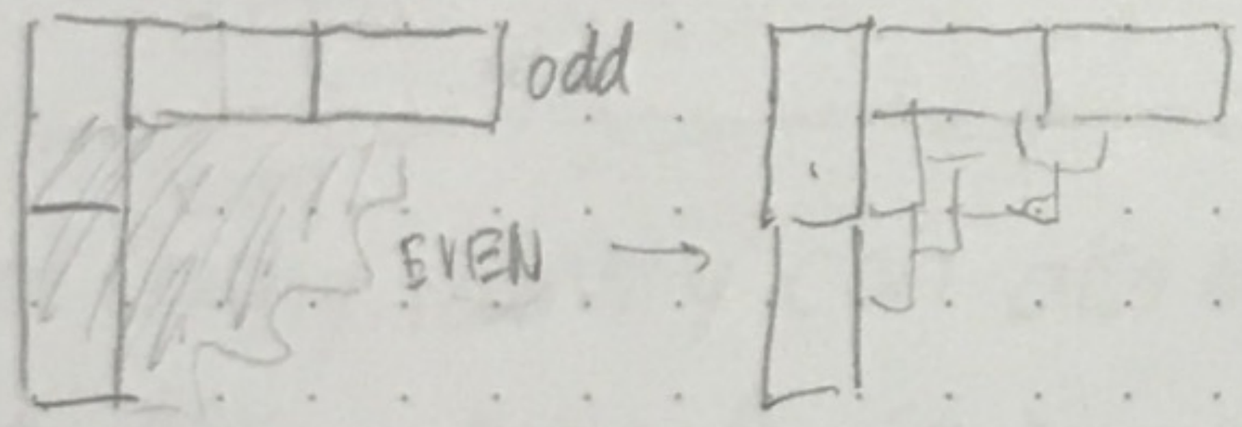
$\Leftrightarrow$ $\lambda = (\lambda_1, \dots, \lambda_n)$. # of odd indexed rows w/ odd # of boxes
= # of even ——— odd # of boxes

This is necessary
Claim: This is sufficient

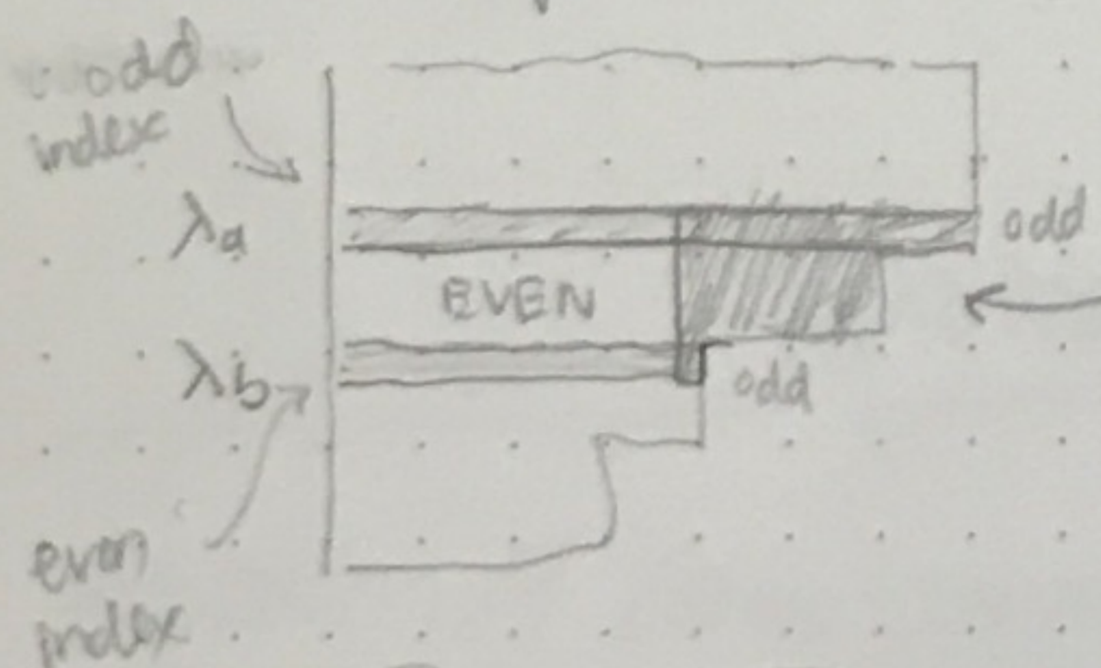
Proof:

Lemma: I can tile $\lambda = (\lambda_1, \dots, 2m-1, 1)$

Proof by picture

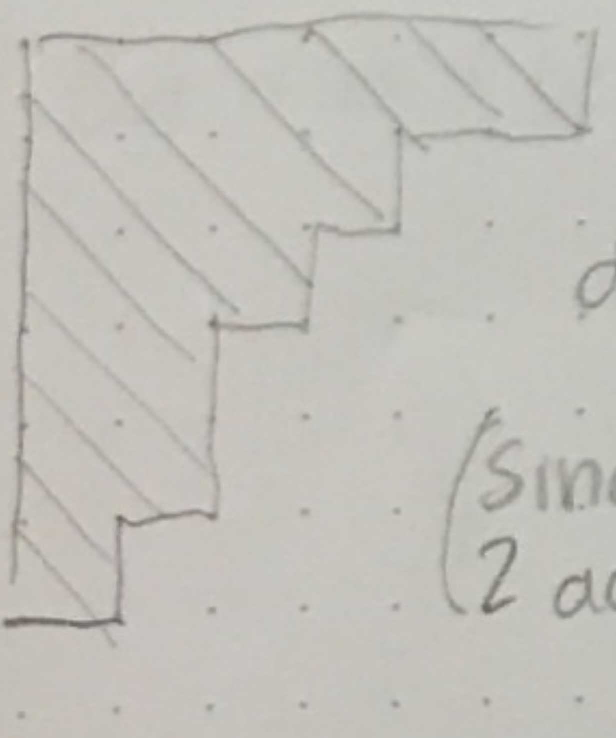


$\exists$ odd part in odd index & odd part in even index w/ only even parts in between



← tile using lemma
Then remove this part
Now we have fewer odd rows

Equivalent condition



$(d_1, \dots, d_r)$ ← diagonal lengths

$d_1 - d_2 + d_3 - \dots = 0$

(Since any domino adds 1 to 2 adjacent diagonals)

Conditions agree b/c odd diagonals are black, Evens are white.

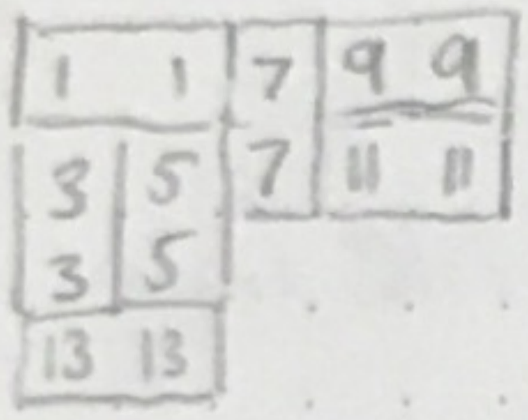
Yannick (Yuan Yao)

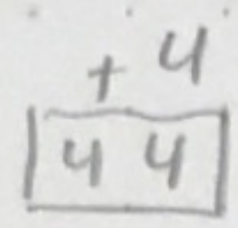
R-S correspondence for domino Tableaux $\rightarrow \left(\sum (f_\lambda^{dom})^2 = 2^n n!\right)$

<u>Goal</u>: Biject signed perm $\leftrightarrow$ pair of dom tableaux

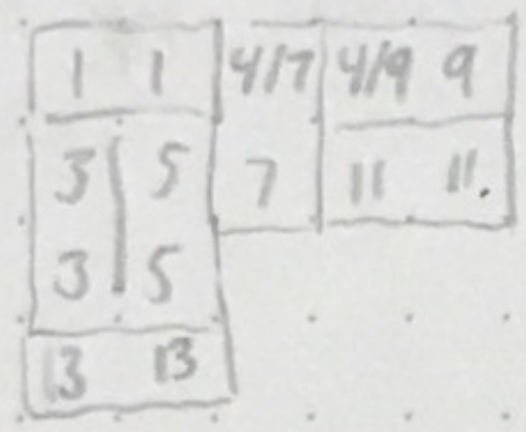
Interpret sign as horizontal (+) or vertical (−)

WLOG start w/ horizontal domino

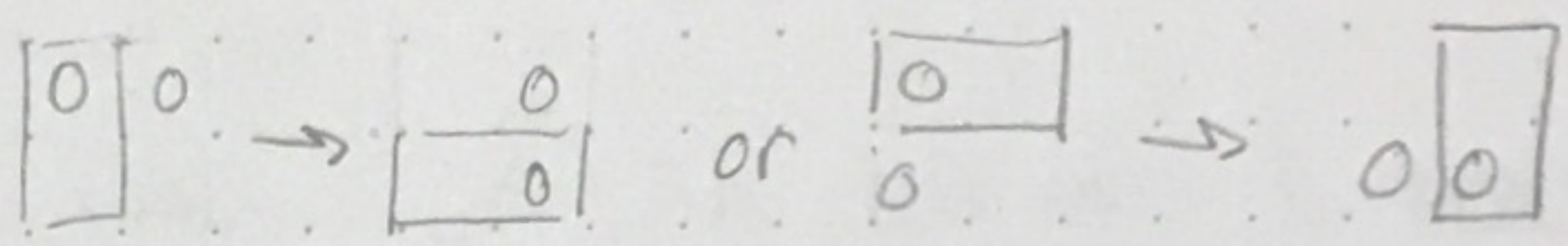
P 

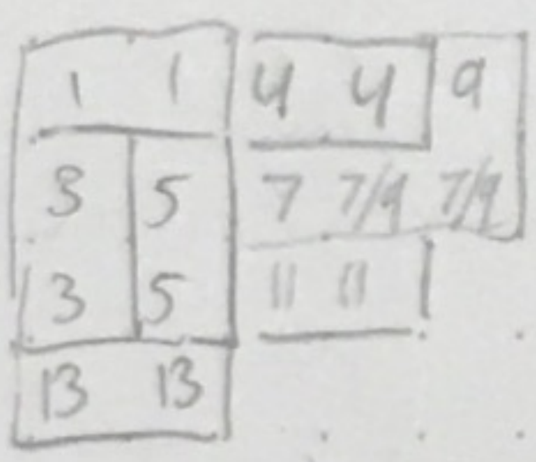
have already. Want to add 

- Insert in first row: (allow temporary overlaps
after last thing < 4

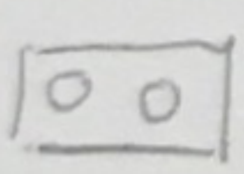
Call overlaps "active" 

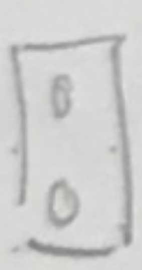
CASE 1: Active boxes in different dominoes

↳ Rotate left/top domino 

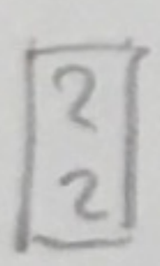
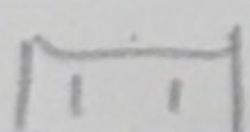


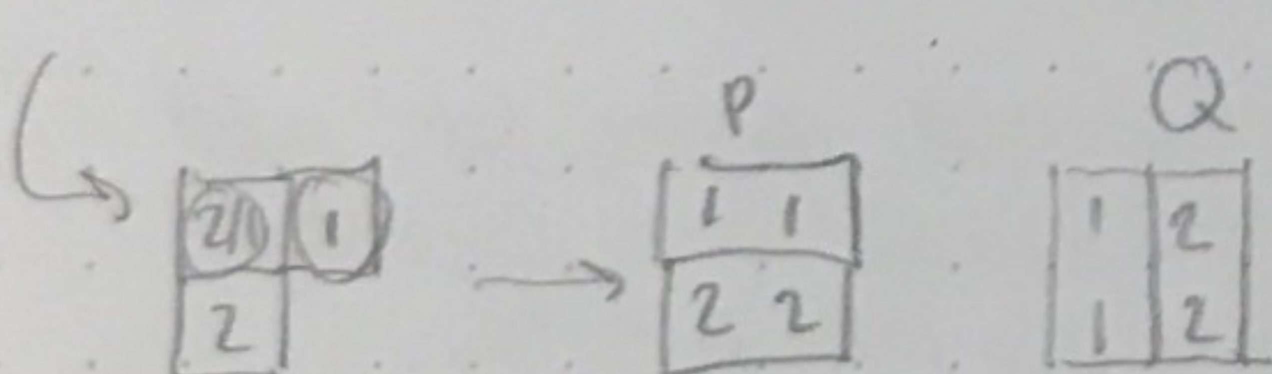
CASE 2: Active boxes in same domino

 → move to next row

 → move to next column

Q Tableaux records how shape of P tableaux changed at each step.

<u>Ex.</u>  insert 



Problem 18.) (Milan Haiman)

show $\sum_{\lambda} s_\lambda(1^m) s_{\lambda'}(1^n) = 2^{mn}$

Will show $\sum_{\lambda} s_\lambda(x_1, \dots, x_m) s_{\lambda'}(y_1, \dots, y_n) = \prod_{\substack{i \in [m] \\ j \in [n]}} (1 + x_i y_j)$ Dual Cauchy identity

Recall Cauchy identity: (Can prove using RSK)

$$\sum_{\lambda} s_\lambda(x_1, \dots, x_m) s_\lambda(y_1, \dots, y_n) = \prod_{\substack{i \in [m] \\ j \in [n]}} \frac{1}{1 - x_i y_j}$$

Turn y part into $s_{\lambda'}$.
What happens to right side?

$$= \prod_{i \in [m]} \sum_{k=0}^{\infty} x_i^k h_k(y_1, \dots, y_n)$$

Now take the dual.

Apply ω_y:

$$\sum_{\lambda} s_\lambda(x_1, \dots, x_m) s_{\lambda'}(y_1, \dots, y_m)$$

$$= \prod_{i \in [m]} \underbrace{\sum_{j \in [n]} x_i^k e_k(y_1, \dots, y_n)}_{\text{generating function for } e_k}$$

$$= \prod_{i \in [m]} \prod_{j \in [n]} (1 + x_i x_j)$$

Dual Cauchy identity can also be proved directly using dual RSK

Dual RSK: bij: $\underset{\substack{m \times n \\ a_{ij} \in \{0,1\}}}{A = (a_{ij})} \longleftrightarrow \left\{ (P, Q) \text{ SSYT's s.t. } \text{shape}(P) = \text{shape}(Q)' \right\}$

Column sums of A = weight of P
row sums of A = weight of Q

Problem 19 (Show combinatorially #GT patterns $= 2^{\binom{n}{2}}$)

is still possibly an open question.

But might be counting Lusage tiling? with some additional conditions.

will talk about next week