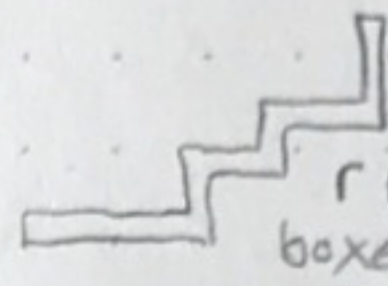
18.217 LECTURE 16

r-Differential Posets

$[D, U] = rI$

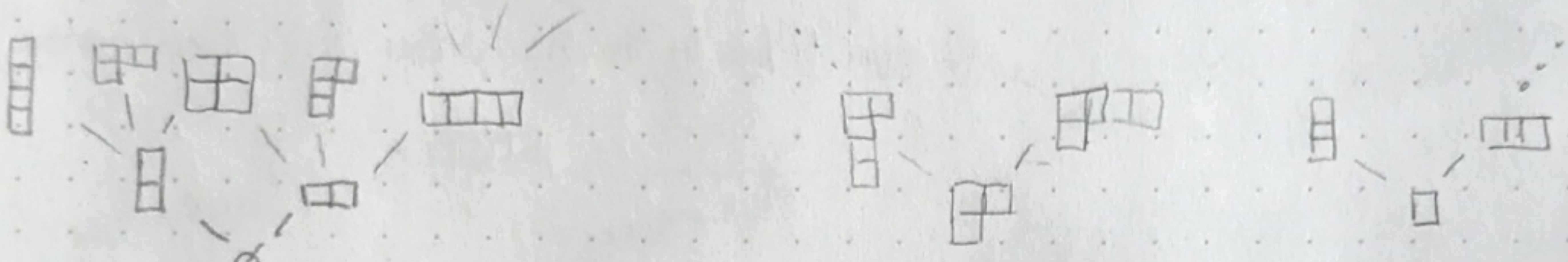
$$\sum_{\lambda \in P,\, rk(\lambda)=n} \#\left\{\begin{array}{c}\text{saturated chains in } P \\ \text{from } \hat{0} \text{ to } \lambda\end{array}\right\} = r^n \cdot n!$$

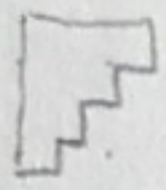
<u>def</u> $\mathbb{Y}^{(r)}$ = all Young diagrams with covering relation $\lambda <_{(r)} \mu$ if μ/λ is an r-ribbon



<u>Lemma</u>: Each connected component of $\mathbb{Y}^{(r)}$ is an r-differential poset

<u>E.x.</u>
<u>r=2</u>:



Any  is minimal elt. b/c impossible to remove domino.

<u>Proposition</u>: Each connected component of $\mathbb{Y}^{(2)}$ has unique minimal elt. of staircase shape. and each conn. comp $\simeq \mathbb{Y} \times \mathbb{Y}$

<u>Exercise</u>: Prove this

Morover, similar is true for any r.

<u>Def</u>: Min. elts of $\mathbb{Y}^{(r)}$ are called <u>r-cores</u>.

<u>Lemma</u>: TFAE

(1) λ is an r-core

(2) λ has no hook length $h(a) = r$

(3) λ has no hook length divisible by r

<u>Prop</u>: $\forall$ r-diff poset P,

\# chains $\underset{\hat{0}}{\lambda^{(0)}}, \lambda^{(1)}, \ldots, \underset{\hat{0}}{\lambda^{(2n)}}$ s.t. $\lambda^{(i+1)} > \lambda^{(i)}$ or $\lambda^{(i)} > \lambda^{(i+1)}$

equals $r^n (2n-1)!!$ Where $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$
$= \#$ of perfect matchings in K_{2n}

Coeff of $\hat{0}$ in $(U+D)^n(\hat{0})$

<u>Prop</u>: $\forall$ r-diff poset P,

\# chains $\underset{\hat{0}}{\lambda^{(0)}}, \lambda^{(1)}, \ldots, \underset{\hat{0}}{\lambda^{(2n)}}$ where $\lambda^{(2i+1)} \gtrdot \lambda^{(2i)}$ or $= \lambda^{(2i)}$ Go up or stay the same

$\lambda^{(2i)} \lessdot \lambda^{(2i-1)}$ or $= \lambda^{(2i-1)}$ Go down or stay the same

Coeff of $\hat{0}$ in $((D+1)(U+1))^n \hat{0}$

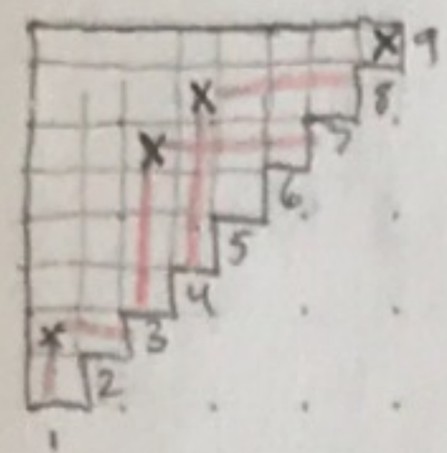
$$= \sum_{\substack{\text{set partition} \\ \pi \text{ of } [n+1]}} r^{n+1 - \#(\text{blocks of } \pi)}$$

$= r^{n+1} B_{n+1}(r^{-1})$ Bell polynomial

For r=1, we get the Bell number $B_{n+1} := \#$ set partitions of $n+1$

Bijection:

$$\left\{\begin{array}{l}\text{rook placement} \\ \text{on } (n, n-1, \dots, 1) \\ \text{with any number of} \\ \text{rooks (can be 0 in a row/column)}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Set partitions} \\ \text{of } [n+1]\end{array}\right\}$$



$$\longmapsto \quad \pi = (137 \mid 489 \mid 5 \mid 6)$$

$\#$ of rooks $= n+1 - \#$ of block in π

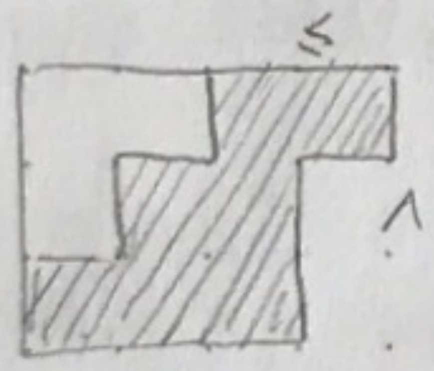
Would be nice if someone presents proving Weyl formula from Hook length formula

Jacobi-Trudi Formulas:

Skew Schur Functions $\lambda \supset \mu$

λ/μ skew Young diagrams

Ex. $(4,3,3)/(2,1,0) =$



$$S_{\lambda/\mu} = \sum_{\substack{T \text{ SSYT of} \\ \text{shape } \lambda/\mu}} x^{\text{weight}(T)}$$

Jacobi-Trudi:

$\lambda = (\lambda_1, \dots, \lambda_n)$

$\mu = (\mu_1, \dots, \mu_n)$

$n \geq \#$ of parts in λ, $\#$ of parts in μ

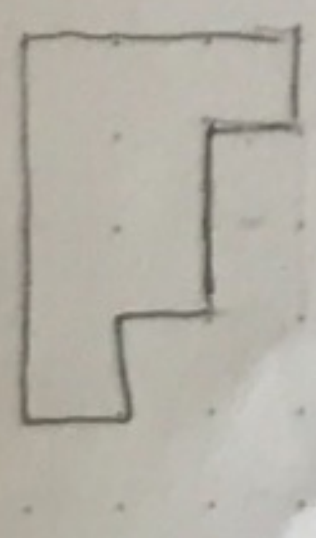
(1) $S_{\lambda/\mu} = \det\left(h_{\lambda_i - i - \mu_j + j}\right)_{i,j \in [n]}$

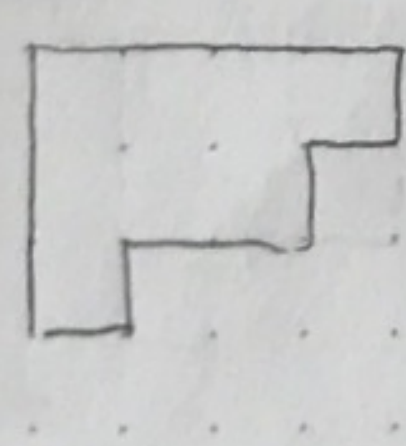
(2) $S_{\lambda/\mu} = \det\left(e_{\lambda'_i - i - \mu'_j + j}\right)_{i,j \in [m]}$ $\quad m \geq \max(\lambda_1, \mu_1)$

$e_0 = h_0 = 1$

$e_k = h_k = 0$ for $k < 0$

Ex. $\lambda = (3,2,2,1)$ $\quad \lambda' = (4,3,1)$





$$S_{(3,2,2,1)} = \begin{vmatrix} h_3 & h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 0 & 1 & h_1 \end{vmatrix}$$

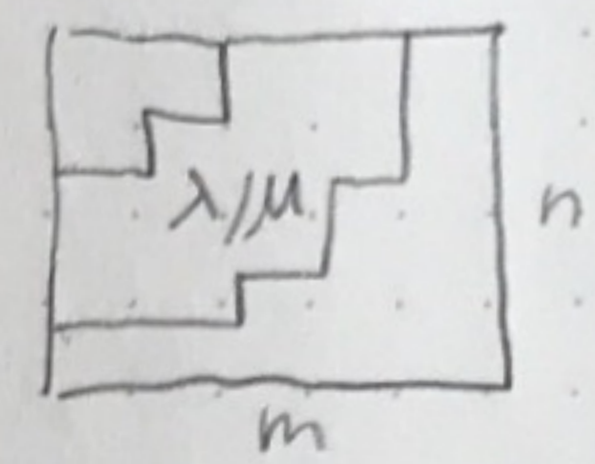
$$= \begin{vmatrix} e_4 & e_5 & e_6 \\ e_2 & e_3 & e_4 \\ 0 & 1 & e_1 \end{vmatrix}$$

Cor: $\omega: S_{\lambda/\mu} \longleftrightarrow S_{\lambda'/\mu'}$

18.217 LECTURE 17

<u>Jacobi-Trudy</u> $\lambda = (\lambda_1, \dots, \lambda_n) \supseteq \mu = (\mu_1, \dots, \mu_n)$

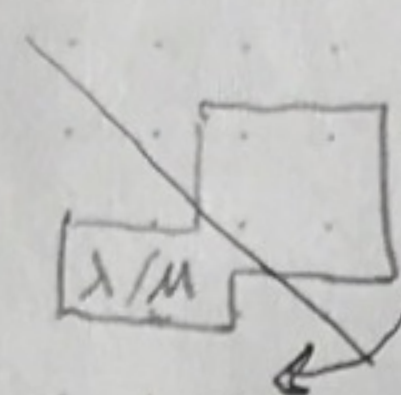
$m \geq \max(\lambda_1, \mu_1)$



1.) $S_{\lambda/\mu} = \det\left((h_{\lambda_i - i - \mu_j + j})_{i,j \in [n]} \right)$

2.) $S_{\lambda/\mu} = \det\left((e_{\lambda'_i - i - \mu'_j + j})_{i,j \in [n]} \right)$

<u>Cor</u>: $\omega : \Lambda \to \Lambda \quad \omega(S_{\lambda/\mu}) \to S_{\lambda'/\mu'}$



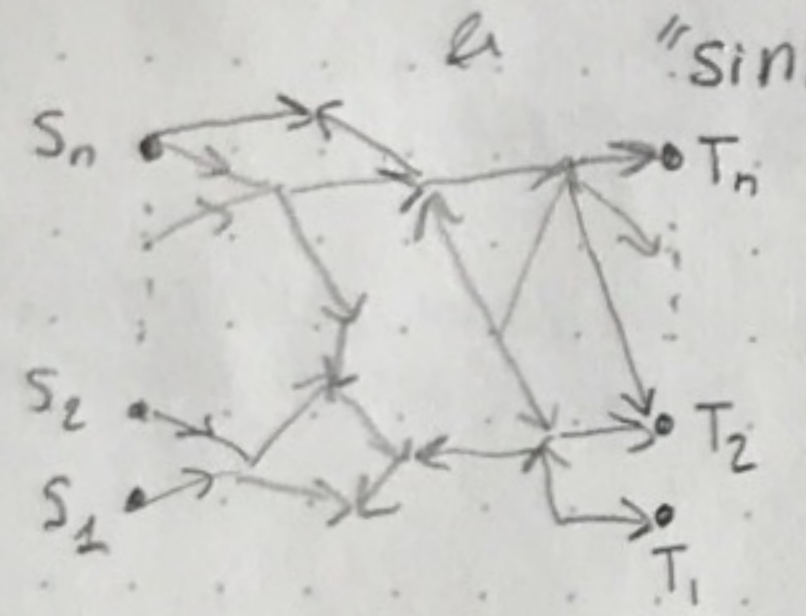
$S_\lambda \cdot S_\mu = \sum c^{\nu}_{\lambda\mu} S_\nu \qquad c^{\nu}_{\lambda\mu} = c^{\nu'}_{\lambda'\mu'}$

Littlewood-Richardson coeffs

<u>Gessel-Viennot Method</u>

<u>Linström Lemma</u>: G weighted *planar* acyclic directed graph

with "source" $S_1, \dots, S_n$ on the left sides of the boundry
& "sinks" $T_1, \dots, T_n$ on the right side of the boundry



with edge weights $x_e > 0$

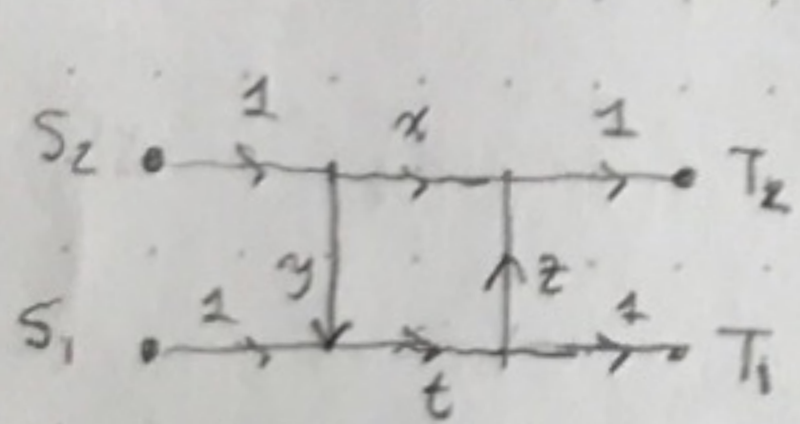
$M = M_G = (M_{ij})$

<u>def</u> $M_{ij} := \sum_{\substack{P : S_i \to T_j \\ \text{directed paths}}} wt(P) \qquad$ where $wt(P) = \prod_{\text{edge } e \in P} x_e$

<u>Lemma</u>: $\det(M) = \sum_{\substack{P_1 : S_1 \to T_1 \\ \vdots \\ P_n : S_n \to T_n}} \prod_{i=1}^{n} wt(P_i)$

n-tuple of non-crossing paths ⟵ P_i & P_j have no common vertices $\forall i, j$

<u>E.x.</u>



$$M_G = \begin{bmatrix} t & tz \\ yt & x + ytz \end{bmatrix} \qquad \det M_G = xt$$

"planarity" ⟷ "positivity"

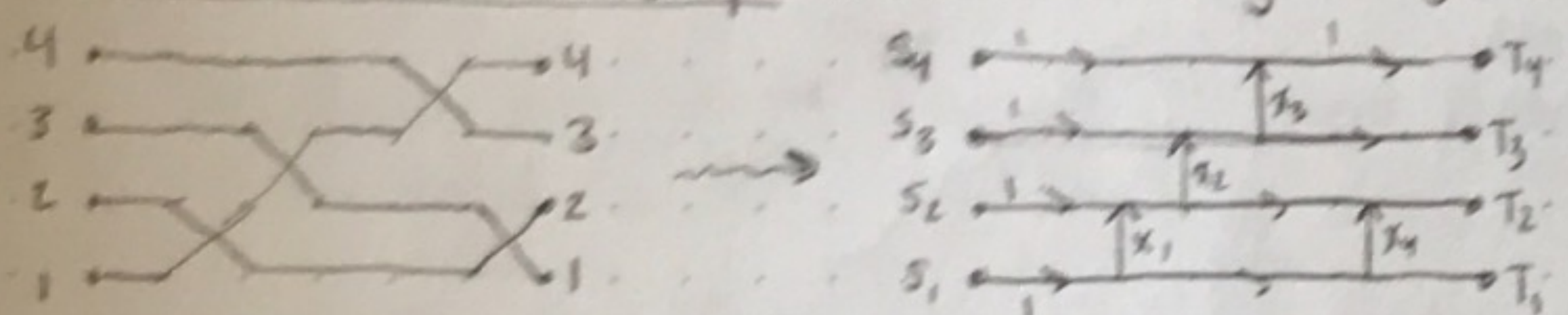
<u>Def</u>: A $n \times n$ matrix M $/\mathbb{R}$ is <u>totally positive</u> (resp totally non-negative) if <u>any minor</u> of M is >0 (≥ 0)

<u>Corollary</u>: M_G is totally non-negative

Since we can just use lemma on set of sources & sinks in that minor of matrix. That will give us that the determinant of that minor is non-negative.

<u>"Inverse-Lindströn Lemma"</u>
$\forall$ totally non-negative matrix M,
$\exists$ weighted planar graph G s.t. $M = M_G$

<u>Fomin-Zelevinsky's double wiring diagrams</u>



$M_G = J_1(x_1) J_2(x_2) J_3(x_3) J_4(x_4)$

where

$$J_i(x) = \begin{array}{r} \\ \\ i \\ i+1 \\ \\ \end{array} \begin{bmatrix} 1 & & & \\ & \ddots & & 0 \\ & & \boxed{\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}} & \\ 0 & & & \ddots \\ & & & & 1 \end{bmatrix}$$
$$\qquad\qquad i \quad i+1$$

<u>Thrm</u> [F,Z]: Any totally non-negative upper triangular matrix M w/ 1's on the diagonal can be written as $M = J_{i_1}(x_1) \cdots J_{i_\ell}(x_\ell)$. $\quad x_i > 0$

<u>Note</u>: By totally positive upper triangular matrix, we mean that any minor that has a chance to be nonzero is positive.

Number of nonzero minors in generic upper triangular matrix:

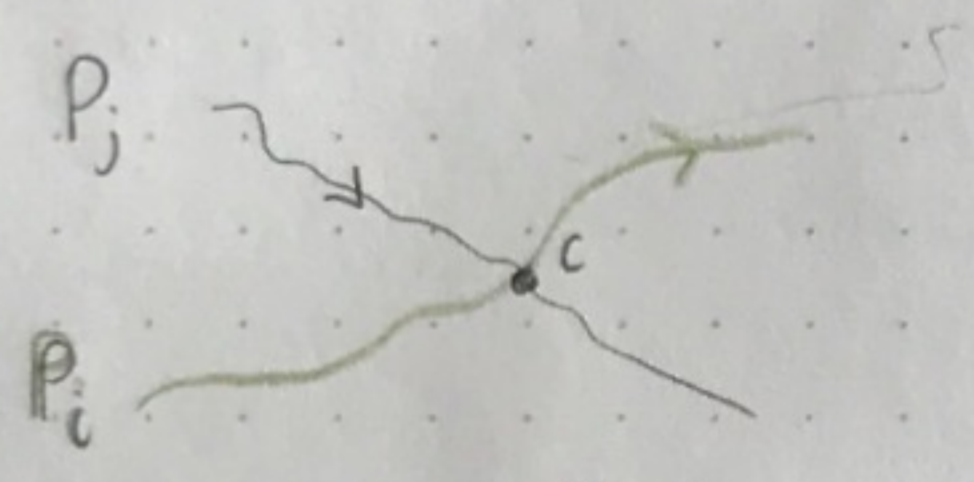
	(1)	$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{bmatrix}$		
	2 (includes empty minor)	$1+3+1 = 5$	$1+6+6+1 = 14$	42	Catalan numbers, C_{n+1}

Proof of Lindström Lemma

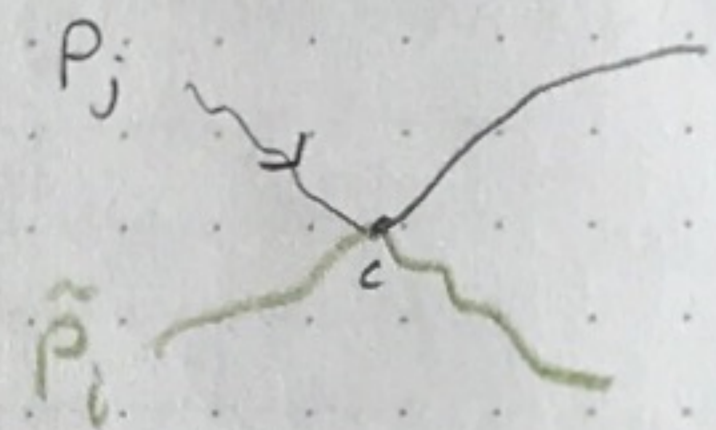
$$\det(M_G) = \sum_{w \in S_n} (-1)^{\ell(w)} \prod_{i=1}^{n} M_{i,w_i}$$

$$= \sum_{w \in S_n} (-1)^{\ell(w)} \sum_{\substack{P_1 : S_1 \to T_{w_1} \\ \vdots \\ P_n : S_n \to T_{w_n}}} \prod_i wt(P_i)$$

$$\stackrel{?}{=} \sum_{\substack{\text{noncrossing} \\ n\text{-tuples} \\ (P_1, \dots, P_n)}}$$



if they cross,
flip the tails



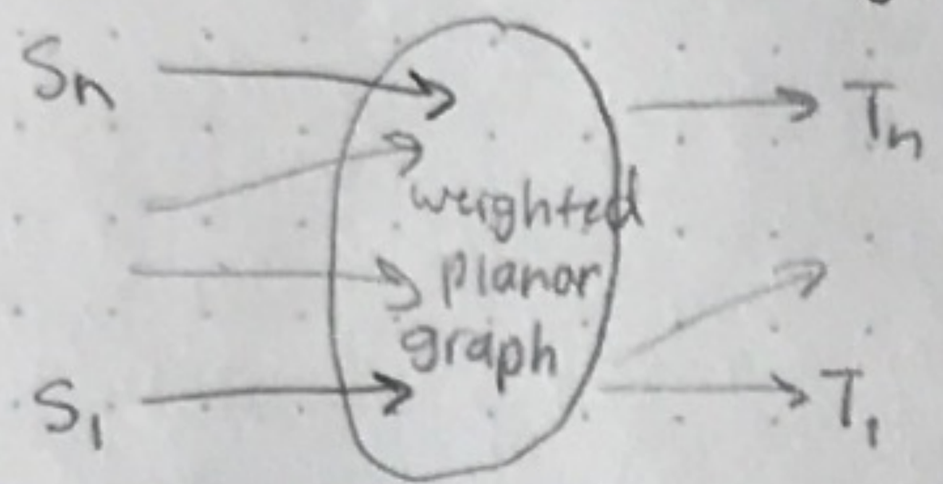
$P_i, P_j \rightsquigarrow \tilde{P}_i \ \tilde{P}_j$

Has 1 transposition of permutation $\rightarrow$ sign of $(-1)^{\ell(w)}$ flips

18.217 LECTURE 18

<u>Last Time</u>: Lindstrom Lemma

Source   Targets



$M = (M_{ij})$ $n \times n$ matrix

$M_{ij} = \sum_{P: S_i \to T_j} wt(P)$

$\det M = \sum_{(P_1, \dots, P_n)} \prod_{i=1}^{n} wt(P_i)$

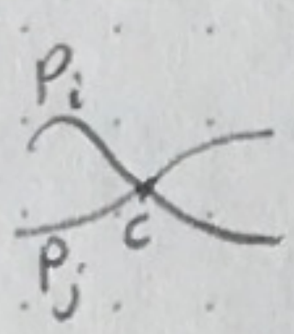
n tuple of noncrossing paths connecting $S_1, \dots, S_n$ with $T_1, \dots, T_n$

$\det(M) = \sum_{(P_1, \dots, P_n)} (-1)^{\ell(w)} \prod wt(P_i) \overset{?}{=} \sum_{(P_1, \dots, P_n)} \prod_{i=1}^{n} wt(P_i)$

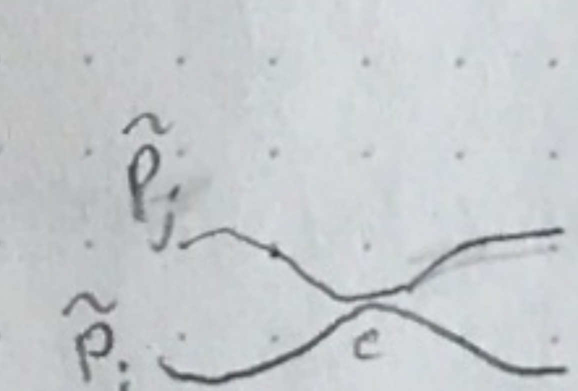
allow crossing → arbitrary n-tuple of paths connecting sources with targets $P: S_i \to T_{w_i}$

non-crossing n-tuples of paths

Find the "first" crossing path



and replace P_i, P_j with $\tilde{P}_i, \tilde{P}_j$



$\tau: (P_1, \dots, P_n) \mapsto (P_1, \dots, \tilde{P}_i, \dots, \tilde{P}_j, \dots \tilde{P}_n)$

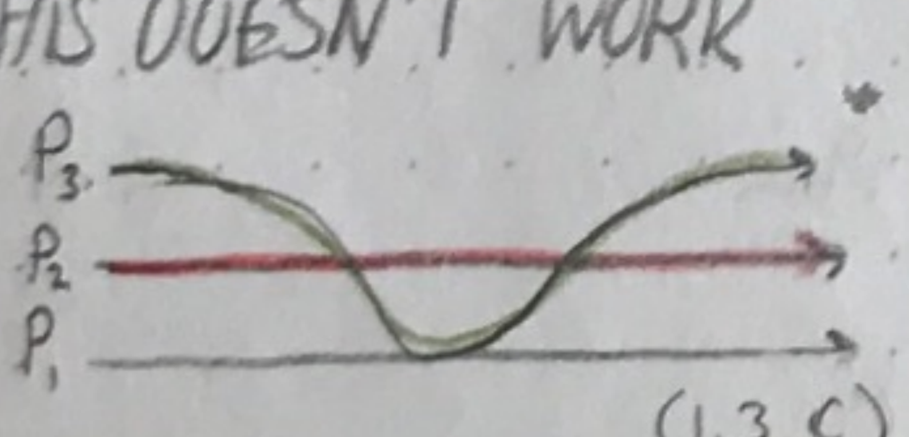
τ is :
- sign reseversing ✓
- weight preserving ✓
- Involution (need to show)

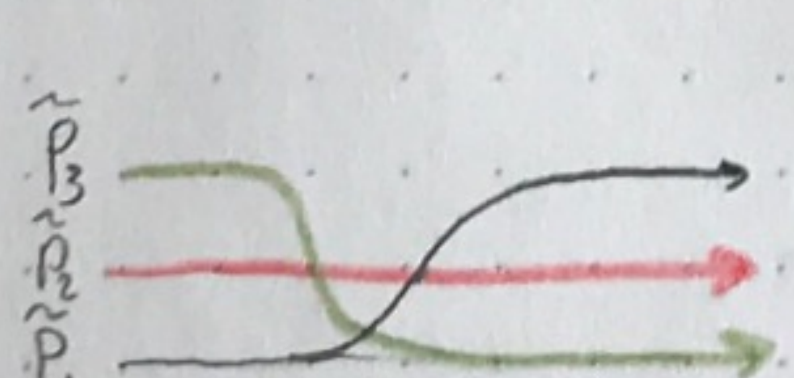
A "crossing" is a triple (i, j, c) s.t. P_i, P_j contain vertex c.

How do we find first crossing?

Guess: Find a crossing with lex min. (i,j) and c the first crossing of P_i, P_j.

↳ THIS DOESN'T WORK



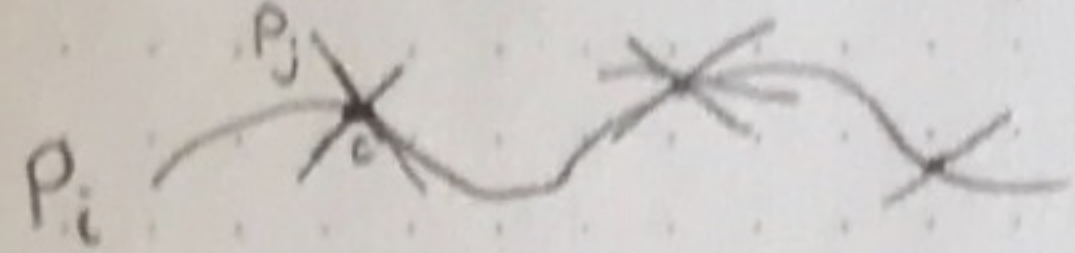


$(1,2,c')$

Correct method: 1.) Find minimal P_i s.t. P_i has a common pt. with another P_j

2.) Find the first vertex $c \in P_i$ that belongs to another path P_j

3.) Find minimal $j \neq i$ s.t. $c \in P_j$



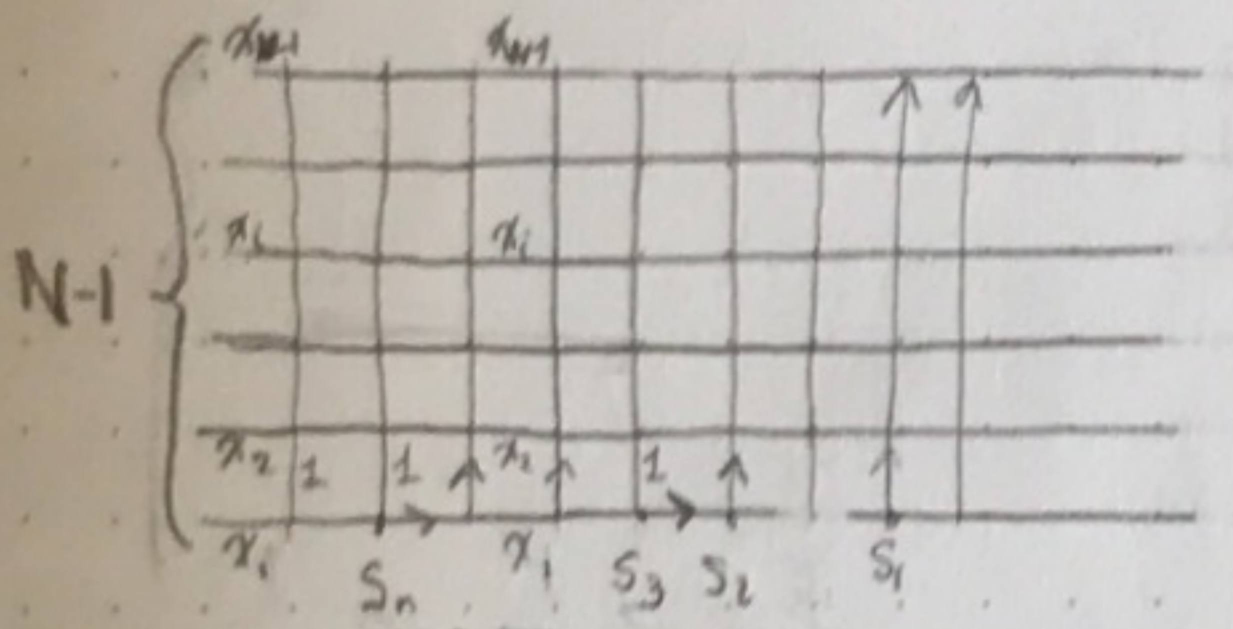
Milan's Method (also correct):
- order all vertices
- Find minimal vertex c involved in a crossing
- Find lex min pair $i \neq j$ s.t. $P_i, P_j \ni c$.

Back to Jacobi-Trudy

$\lambda = (\lambda_1, \ldots, \lambda_n) \supset \mu = (\mu_1, \ldots, \mu_n)$

(1) $S_{\lambda/\mu}(x_1, \ldots, x_N) = \det(h_{\lambda_i - i - \mu_j + j}(x_1, \ldots, x_N))$ $i, j \in [n]$

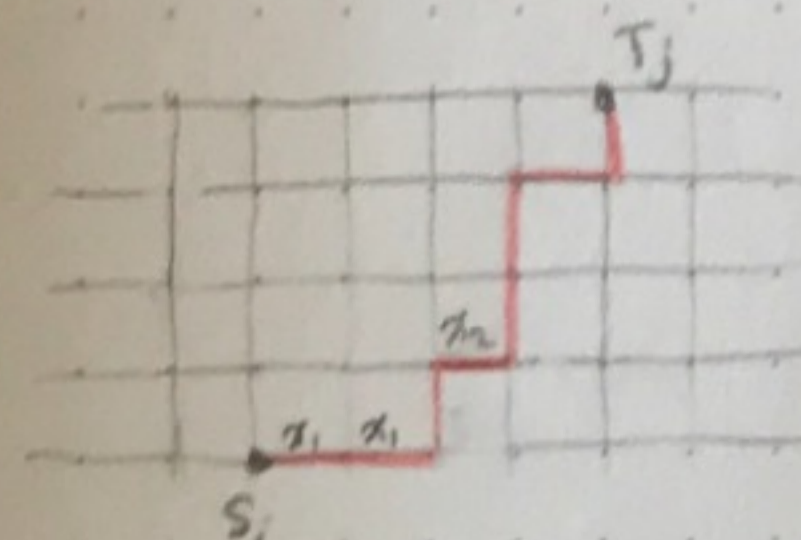
(2) $S_{\lambda/\mu}(x_1, \ldots, x_N) = \det((e_{\lambda'_i - i - \mu'_j + j}(x_1, \ldots, x_N)))$ $i, j \in [m]$



- edges are directed right & up
- vert edges have weight 1
- horizontal edges on lvl i have weight x_i

$S_i = (\mu_i + n - i, 0)$

$T_j = (\lambda_j + n - j, N-1)$



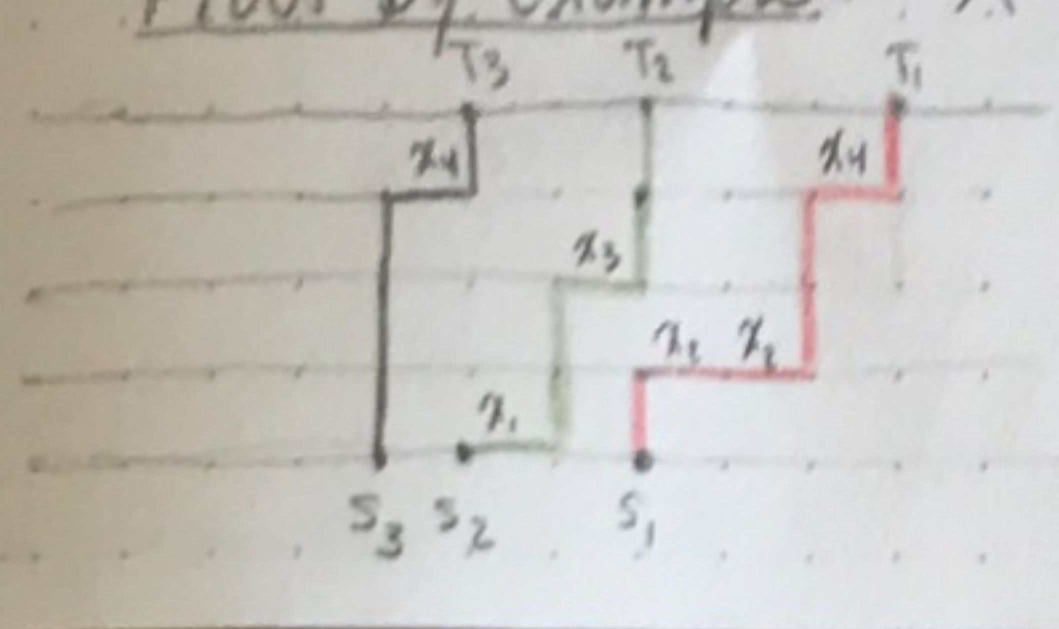
$M_{ij} = h_k(x_1, \ldots, x_N)$

$k = \lambda_j - j - \mu_i + i$

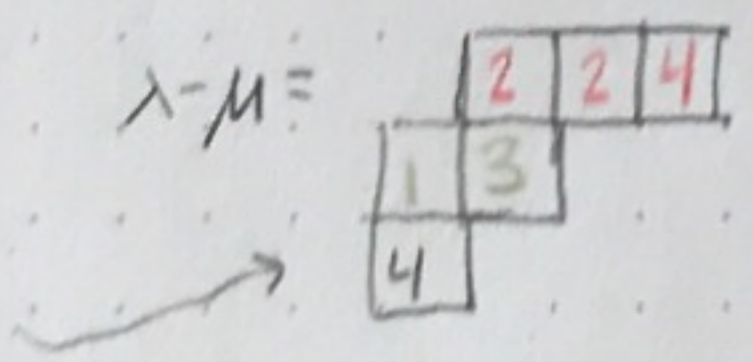
M = transpose of J-T matrix for $S_{\lambda/\mu}(x_1, \ldots, x_N)$

Lemma: $\left\{\begin{array}{c}\text{n-tuples of} \\ \text{noncrossing} \\ \text{paths } (P_1, \ldots, P_n)\end{array}\right\} \xleftrightarrow{\text{bij}} \left\{\begin{array}{c}\text{SSYT's} \\ \text{of shape} \\ \lambda/\mu\end{array}\right\}$

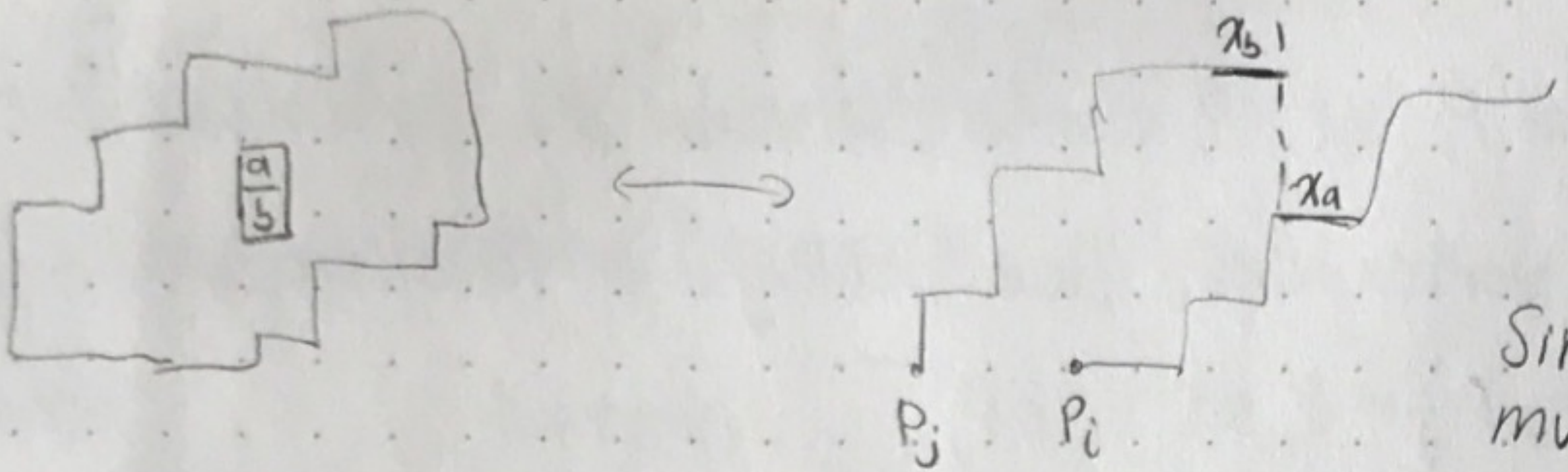
Proof by example: $\lambda = (5,3,2)$, $\mu = (2,1,1)$, $N = 6$



$\lambda - \mu =$



rows will be weekly increasing
columns strictly increasing (otherwise paths would collide)



x_b is first edge to hit start column of x_a.

Since paths are noncrossing, must have $x_b > x_a$

Proof of 2nd part of Jacobi-Trudi

Now look at grid