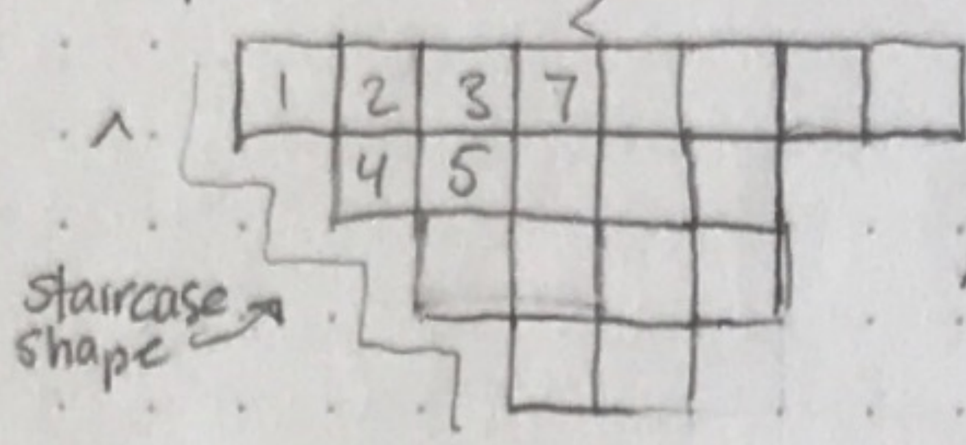
18.217 LECTURE 13 10/7/22

Shifted Shapes and Tableau

$\mu = (\mu_1 > \dots > \mu_\ell)$ strict partition of $n = |\mu|$

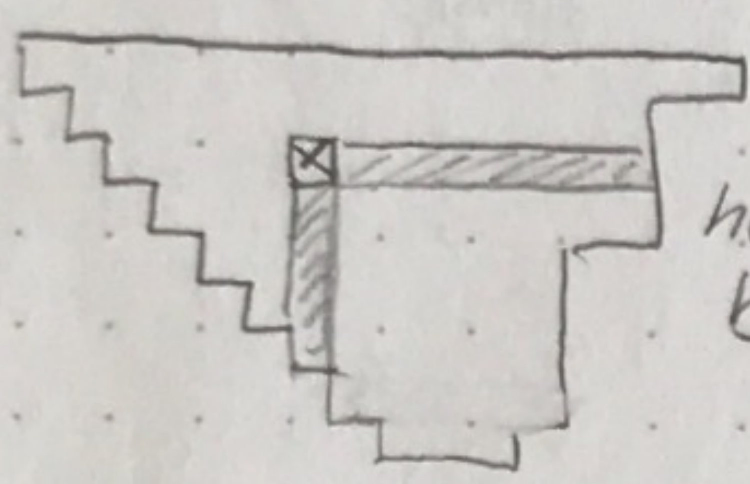
shifted shapes



$\mu = (8,5,4,2)$

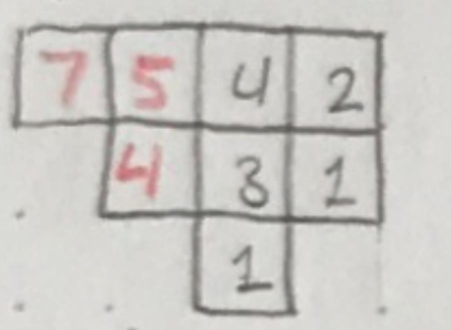
$f^{sh}_\mu := \#\left\{\begin{matrix}\text{SYT of shifted} \\ \text{shape } \mu\end{matrix}\right\}$

<u>Thrm</u>: $f^{sh}_\mu = \dfrac{n!}{\prod_{x\in\mu} h(x)}$ $\qquad h(x)$ hook length of box x.



hook with broken leg.

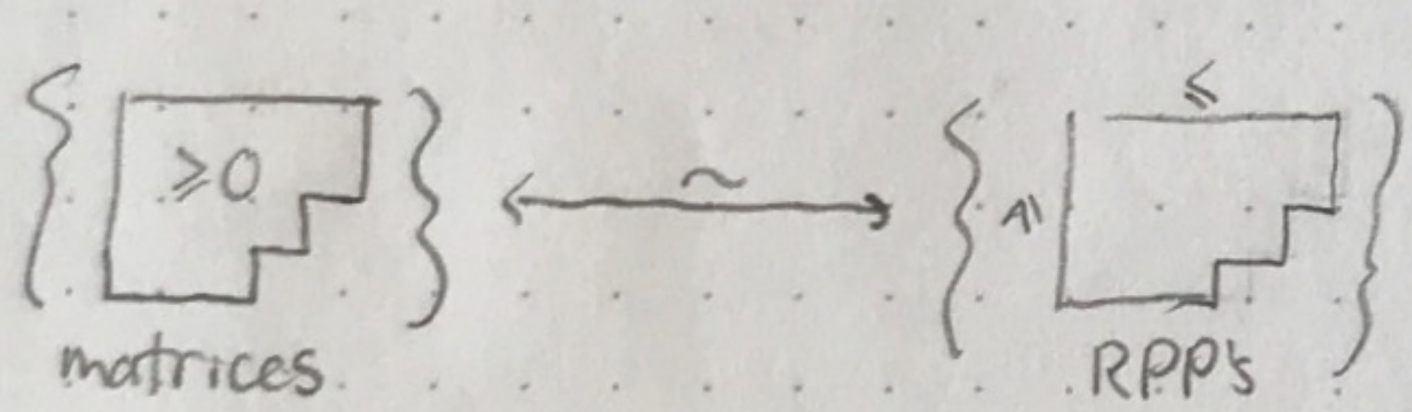
<u>E.x.</u> $\mu = (4,3,1)$



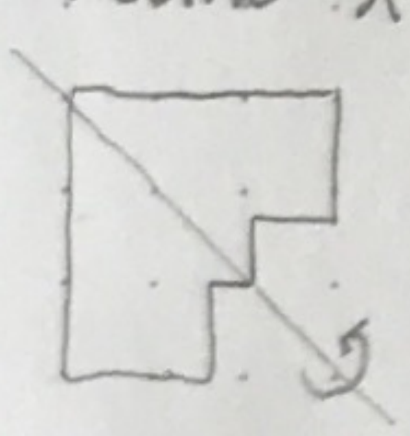
$f^{sh}_\mu = \dfrac{8!}{7\cdot5\cdot4\cdot2\cdot4\cdot3\cdot1\cdot1}$

numbers $h(x)$. - Count broken part of leg in hook length

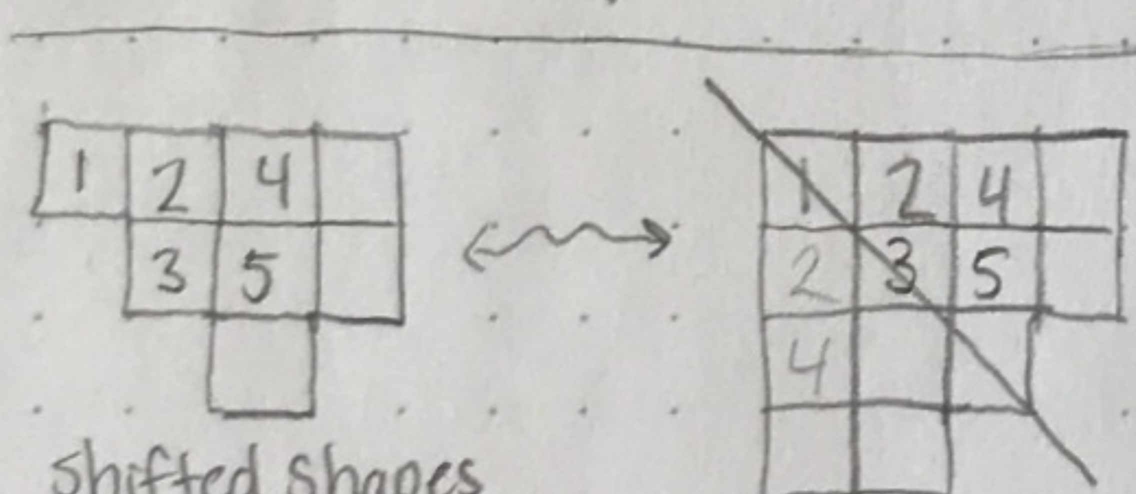
$\varphi_\lambda :=$



Assume $\lambda' = \lambda$



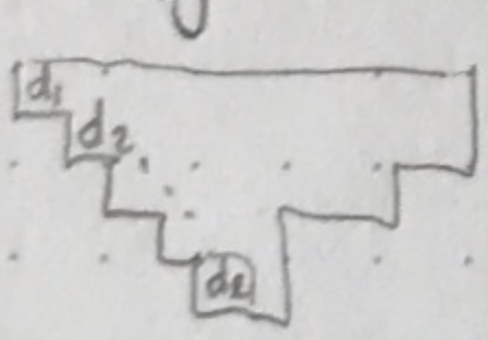
Restrict φ_λ to symmetric matrices



self conj. shapes

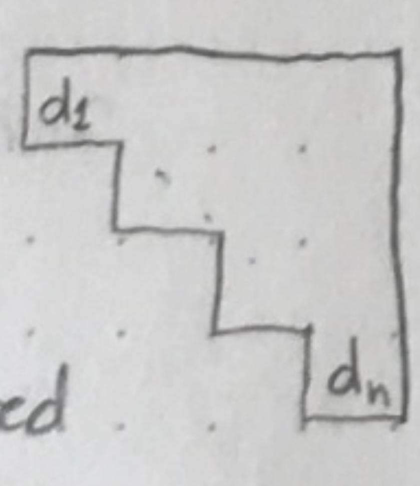
More refined counting:

Fix $d_1, \dots, d_\ell$



\# of SYT of shifted shape with given diagonal entries

Assume $\mu = (n, n-1, \dots, 1)$. "shifted staircase"



$N(d_1, \dots, d_n)$ = # of SYT of shifted shape $(n, n-1, \dots, 1)$ with given diag. entries

What are conditions on $d_1, \dots, d_n$ s.t. $N(d_1, \dots, d_n) \neq 0$?

$d_1 = 1$ $\quad$ $d_1 < d_2 < \dots < d_n$

$d_n = \binom{n+1}{2}$ $\quad$ Moreover, $d_{i+1} - d_i \geq 2$

But there are still more conditions. We will find them all.

Let $a_i = d_{i+1} - d_i - 1$ $\quad$ $i = 1, \dots, n-1$

Make all $d_1, \dots, d_n$ as small as possible lexicographically

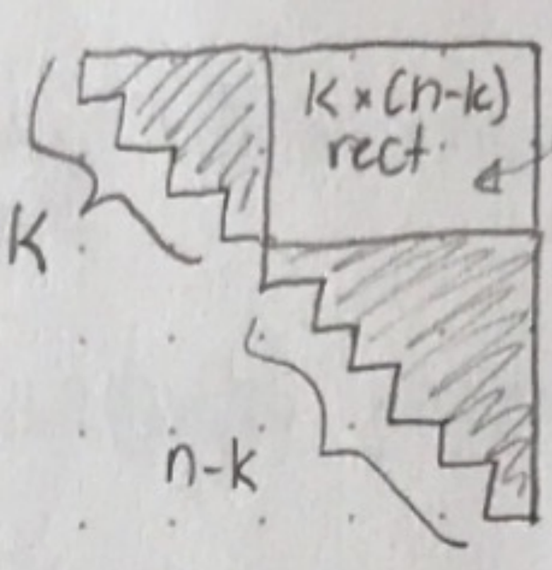
$d_1 = 1$

$d_2 = 3$

$d_3 = 6$

$$\begin{array}{ccc} \boxed{1} & 2 & 4 \\ & 3 & 5 \\ & & \boxed{6} \end{array}$$

Now instead minimize $(d_1, \dots, d_k)$, maximize $(d_n, d_{n-1}, \dots, d_{k+1})$ lexicographically



Any SYT of this rectangular shape

$\underline{a} = (a_1, \dots, a_{n-1})$ $\quad$ $a_i \geq 1$, $\sum a_i = \binom{n}{2}$

<u>Prop</u> $N(1, 2, 3, \dots, k-1, \overset{k(n-k)}{M}, n-k, \dots, 3, 2, 1)$

$= f^{k \times (n-k)}$ (given by usual hook length formula)

<u>E.x.</u> $n = 3$

$$\begin{array}{ccc} 1 & 2 & 4 \\ & 3 & 5 \\ & & 6 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ & 4 & 5 \\ & & 6 \end{array}$$

diag $(1,3,6)$ $\quad$ $(1,4,6)$

a-vectors $(1,2)$ $\quad$ $(2,1)$

<u>Notation</u>: $\Delta_{[i,j]} = \operatorname{conv}(e_i, e_{i+1}, \dots, e_j)$ polytope in $\mathbb{R}^{n-1}$

Assoc_{n-1} = the Minkowski Sum $\sum_{1 \leq i \leq j \leq n-1} \Delta_{[i,j]}$

associahedron

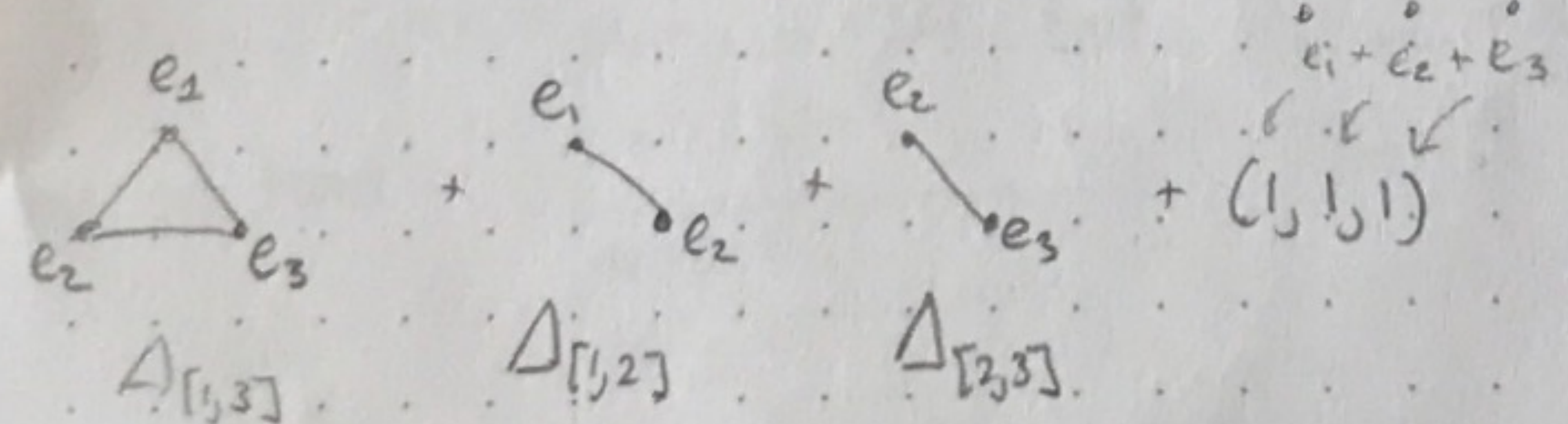
$P, Q < \mathbb{R}^{n-1}$

$P + Q = \left\{ p + q \;\middle|\; \begin{matrix} p \in P \\ q \in Q \end{matrix} \right\}$

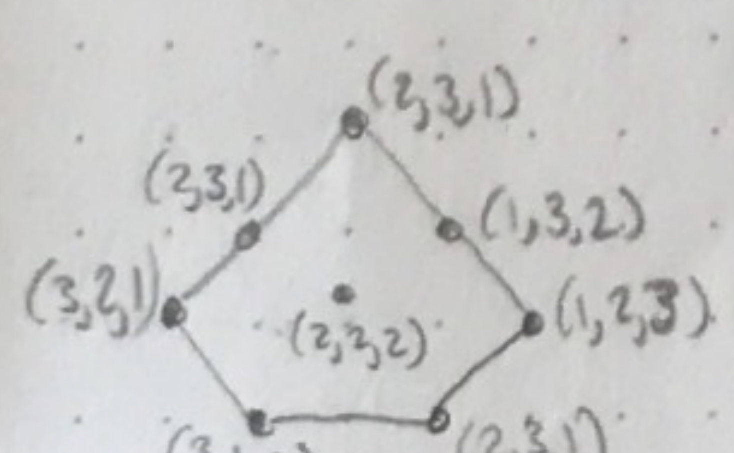
Minkowski sum

<u>E.x.</u> $n = 4$

$\mathrm{Assoc}_3 =$



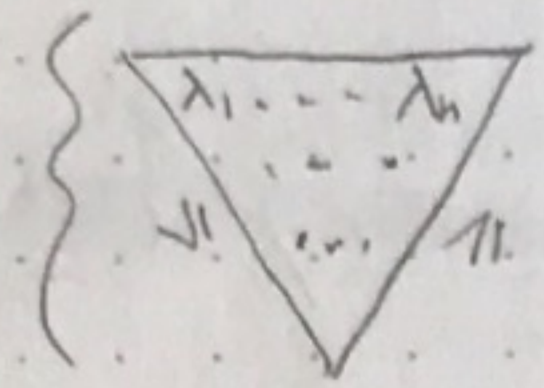
$=$



Thrm: $N(\mathrm{diag}(a_1,\dots,a_{n-1})) \neq 0$ iff $(a_1,\dots,a_{n-1}) \in \underbrace{\mathrm{Assoc}_{n-1} \cap \mathbb{Z}^{n-1}}_{\text{lattice pt of associahedron}}$

$GT(\lambda) \subset \mathbb{R}^{\binom{n}{2}}$

Gelf-Tsetlin polytop



$\mathrm{Vol}\, GT(\lambda) = \boxed{\prod_{1\le i<j\le n} \frac{\lambda_i-\lambda_j}{j-i}}$ True for λ's any numbers, not just integers

WLOG assume that all λ_i's are unique.

Possible linear orders of entries in a GT-pattern

$\updownarrow$ bij

shifted SYT's of shape $(n, n-1, \dots, 1)$.

Ex.

1	2	4
	3	5
		6

$\longleftrightarrow$

$$\begin{array}{ccccc} \overset{\lambda}{\shortparallel} & & \overset{\lambda_2}{\shortparallel} & & \overset{\lambda_3}{\shortparallel} \\ x_0 & & x_3 & & x_1 \\ & x_5 & & x_2 & \\ & & x_4 & & \end{array}$$

$\{\underset{\lambda_3}{\underset{\shortparallel}{x_1}} < x_2 < \underset{\lambda_2}{\underset{\shortparallel}{x_3}} < x_4 < x_5 < \underset{\lambda_3}{\underset{\shortparallel}{x_0}}\}$

$\cong (\lambda_2-\lambda_3)\Delta^1 \times (\lambda_1-\lambda_2)\Delta^2$

triangular prism

$n=3$ $GT(\lambda)$

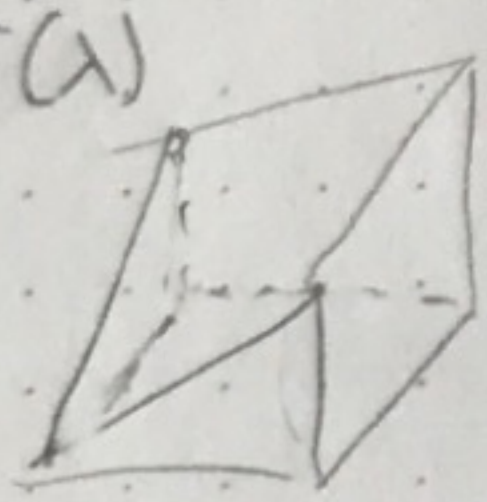


Figure out how to subdivide this polytope into 2 triangular prisms.

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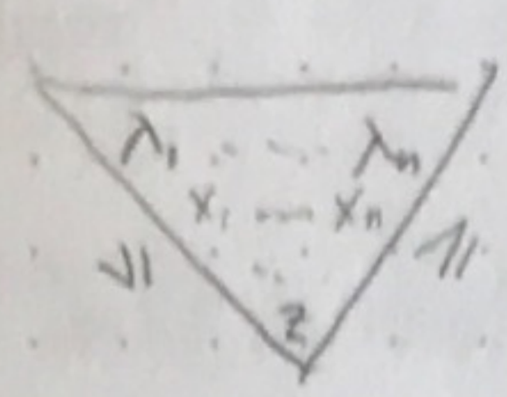
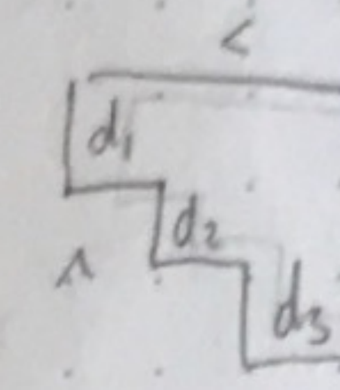
$GT(\lambda) := \Big\{$ 

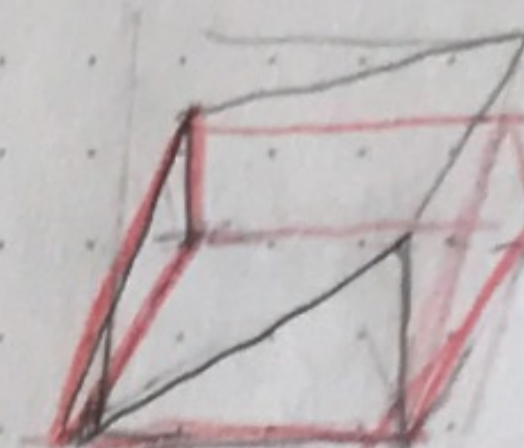
$\Big\} \subset \mathbb{R}^{\binom{n}{2}}$

$$\mathrm{Vol}\, GT(\lambda) = \prod_{1 \le i \le j \le n} \frac{\lambda_i - \lambda_j}{j-i} = \frac{1}{1!2!\cdots(n-1)!} \prod_{i<j} (\lambda_i - \lambda_j)$$

$$= \sum_{d_1,\dots,d_n} \sum_{\substack{T \text{ SYT of} \\ \text{shifted shape} \\ \text{with diag } d_1,\dots,d_n}} \mathrm{Vol}\left((\lambda_{n-1}-\lambda_n)\Delta^{d_2-d_1-1} \times (\lambda_{n-2}-\lambda_{n-1})\Delta^{d_3-d_2-1} \times \cdots \right)$$



$$= \sum_{d_1,\dots,d_n} \sum_{T} \prod_{i=1}^{n-1} \frac{(\lambda_{n-i} - \lambda_{n-i+1})^{d_{i+1}-d_i-1}}{(d_{i+1}-d_i-1)!}$$

$GT(\lambda) =$  $\leftarrow$ 2 triangular prisms

Change variables:

$a_i = d_{i+1} - d_i - 1$

$t_i = \lambda_{n-i} - \lambda_{n-i-1}$ $\qquad i = 1,\dots,n-1$

<u>Thrm</u>: $$\sum_{\substack{a_1,\dots,a_{n-1} \geq 1 \\ a_1+\dots+a_{n-1} = \binom{n}{2}}} N(\mathrm{diag}(a_1,\dots,a_{n-1})) \frac{t_1^{a_1}}{a_1!} \cdots \frac{t_{n-1}^{a_{n-1}}}{a_{n-1}!} = \frac{1}{1!2!\cdots(n-1)!} \prod_{1 \le i \le j \le n} (t_i + t_{i+1} + \dots + t_{j-1})$$

$$\prod_{1 \le i \le j \le n} (t_i + t_{i+1} + \dots + t_{j-1}) = \prod (\lambda_i - \lambda_j)$$

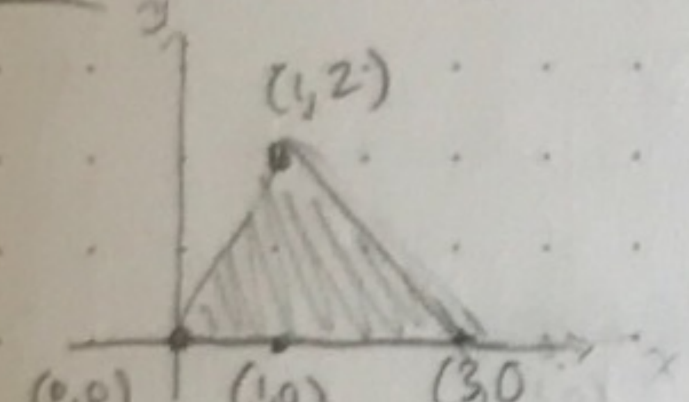
where $\mathrm{diag}(a_1,\dots,a_{n-1}) = (1, a_1+2, a_1+a_2+3, \dots)$

$N(d_1,\dots,d_n) = \#$ shifted SYT w/ diagonal $d_1,\dots,d_n$

$$f(t_1,\dots,t_m) = \sum a_{i_1 \cdots i_m} t_1^{i_1} \cdots t_m^{i_m}$$

<u>def</u>: <u>Newton Polytope</u>: $\mathrm{Newton}(f) = \mathrm{conv}\{(i_1,\dots,i_m) \mid a_{i_1\cdots i_m} \neq 0\}$

<u>Ex.</u> $\mathrm{Newton}(1 + 25x + 70xy^2 + \sqrt{\pi}x^3)$



<u>Def</u>: f has <u>saturated Newton Polytope</u> (SNP) property if $\forall (i_1,\dots,i_m) \in \mathrm{Newton}(f) \cap \mathbb{Z}^m$, we have $a_{i_1\cdots i_m} \neq 0$.

<u>Thrm</u>: $S_\lambda(x_1,\dots,x_m)$ has SNP property. NOTE: This statement equivalent to pset problem 1 (but phrased in very different terms)

<u>Lemma</u>: $\prod_{1 \le i \le j \le n} (t_i + t_{i+1} + \dots + t_{j-1})$ has SNP property.

$$= \sum_{i<j} \Delta_{[i,j-1]} = \mathrm{Assoc}_{n-1}$$

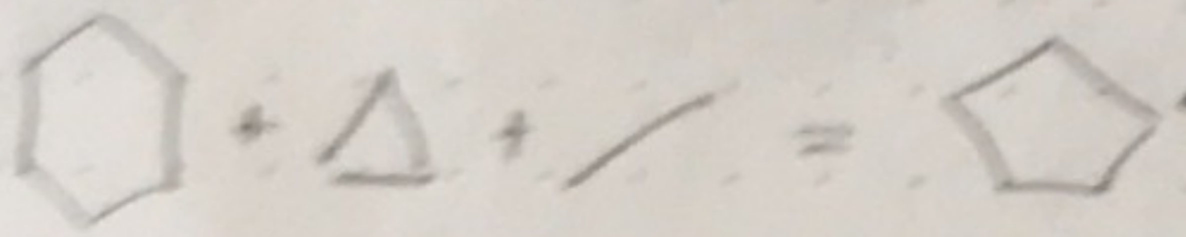
$\text{Newton}(f \cdot g) = \text{Newton}(f) + \text{Newton}(g)$

<u>Corollary</u>: $N(\text{diag}(a_1, \ldots, a_{n-1})) \neq 0$ iff $(a_1, \ldots, a_n) \in \text{Assoc}_{n-1} \cap \mathbb{Z}^{n-1}$

<u>Corollary</u>: $N(\text{diag}(a_1, \ldots, a_{n-1})) = \dfrac{a_1!\, a_2! \cdots a_{n-1}!}{1!\, 2! \cdots (n-1)!} \times \left\{ \begin{matrix} \text{some integer} \\ \text{factor} \end{matrix} \right\}$

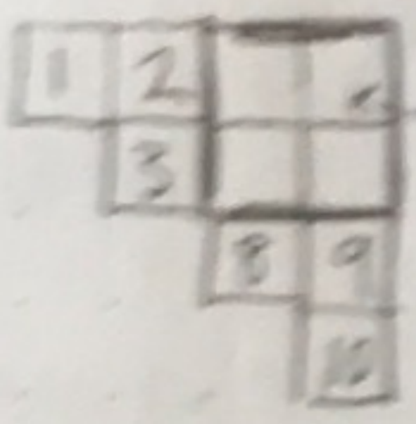
$= 1$ if $(a_1, \ldots, a_{n-1})$ is vertex of Assoc_{n-1}

≥ 2 if $(a_1, \ldots, a_{n-1}) \in \text{Assoc}_{n-1}$ but not a vertex



any vertex has unique way to write in terms of summands

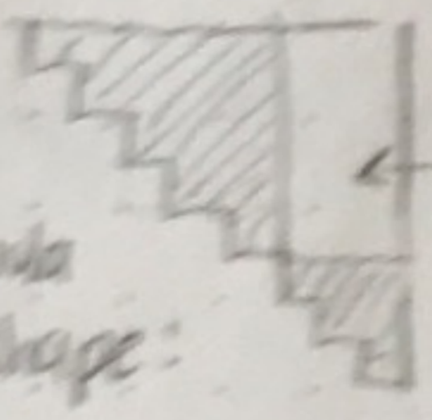
<u>Ex.</u> $n=4$



can fill in with Any SYT of this shape $\Rightarrow N(1,3,8,10) = 2$ check: $\dfrac{1!\,4!\,1!}{1!\,2!\,3!} = 2$ ✓

$(d_1, \ldots, d_4) = (1,3,8,10) \rightarrow (a_1, \ldots, a_{n-1}) = (1,4,1)$

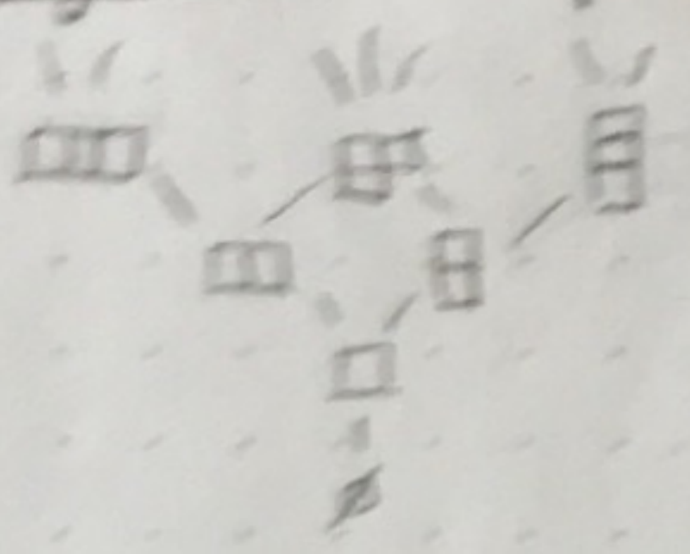
In some special cases, specializes to hook-length formula for rectangular shape:



$k \times (n-k)$

<u>Young's Lattice</u>

<u>def</u>: <u>Young's Lattice</u> = poset of all young diagrams ordered by inclusion

$\mathbb{Y} =$ 

(This is a lattice: Unique min elt above where 2 posets "join"
Unique max elt below — "meet")

Covering relation in $\mathbb{Y}$ is $\lambda \lessdot \mu$ iff $\mu \supset \lambda$ and μ / λ is a single box

Up & down operators that act on the space $\mathbb{C}[\mathbb{Y}]$:

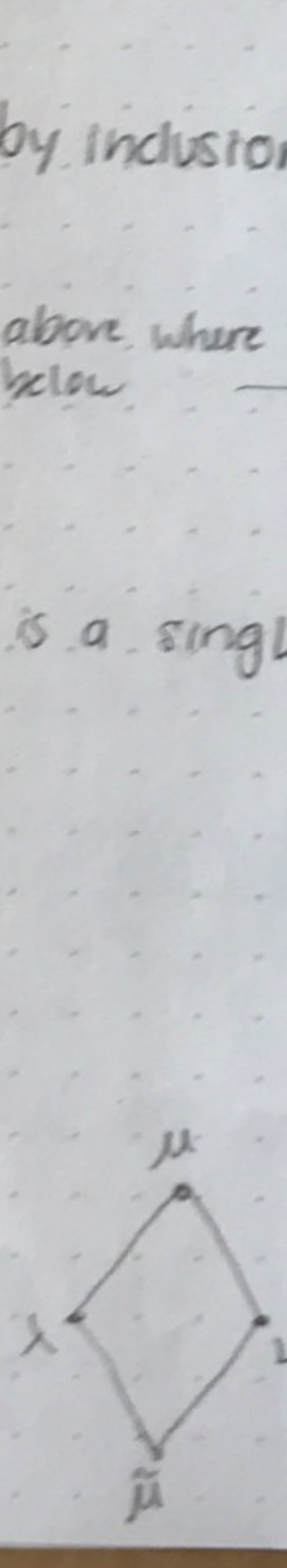
$U : \lambda \longmapsto \sum_{\mu \gtrdot \lambda} \mu$

$D : \lambda \longmapsto \sum_{\mu \lessdot \lambda} \mu$

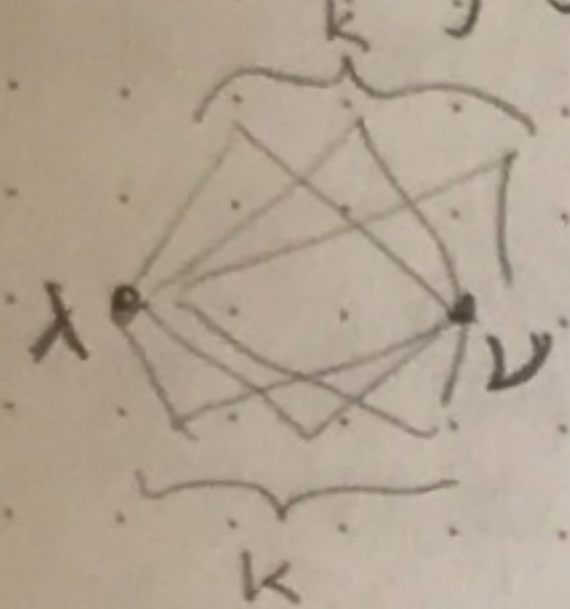
<u>Key lemma</u>: $[D, U] = DU - UD = 1$

<u>Proof</u>: $[D,U]\lambda = DU(\lambda) - UD(\lambda)$ ← all ways to remove then add box

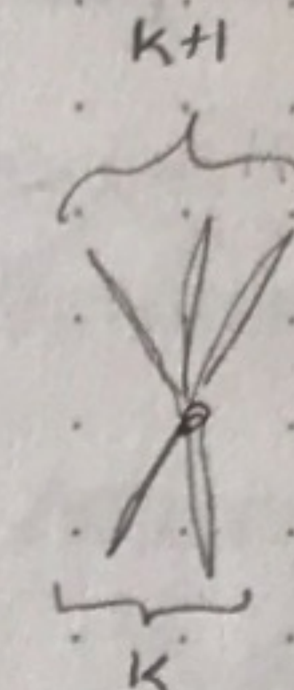
all ways to add then remove box



I. If $\lambda \neq \nu$, should have (# of possible μ) = (# of possible $\tilde{\mu}$)

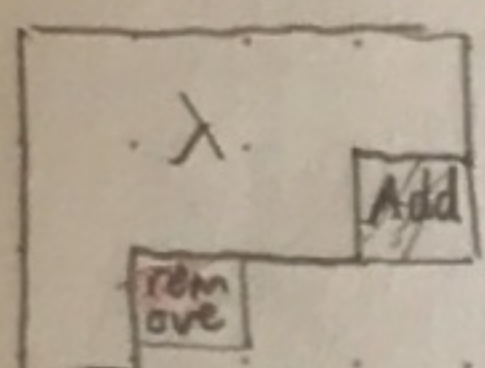


(In $\mathbb{Y}$, $k \in \{0,1\}$)

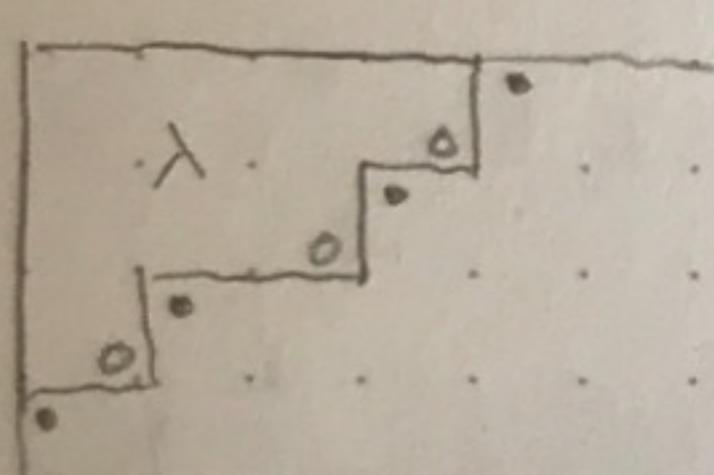


II. If $\lambda = \nu$, (# of possible μ) = (# of possible $\tilde{\mu}$) + 1

<u>Proof</u>

I. 

If $\lambda \neq \mu$, add & remove 2 different boxes.
↳ can reverse the order since they are different boxes

II. 

If $\lambda = \nu$, add & remove same box
Add outer corners, remove inner corners.
↳ there is 1 more outer corner

18.217 LECTURE 15

Def (Stanley): A differential poset P

- P is a locally finite ranked poset, $rk : P \to \mathbb{Z}_{\geq 0}$

  $\lambda \lessdot \mu \quad rk(\mu) = rk(\lambda) + 1$
- P has unique minimal elt $\hat{0}$, $rk(\hat{0}) = 0$
- Up & down operators on $\mathbb{C}[P]$

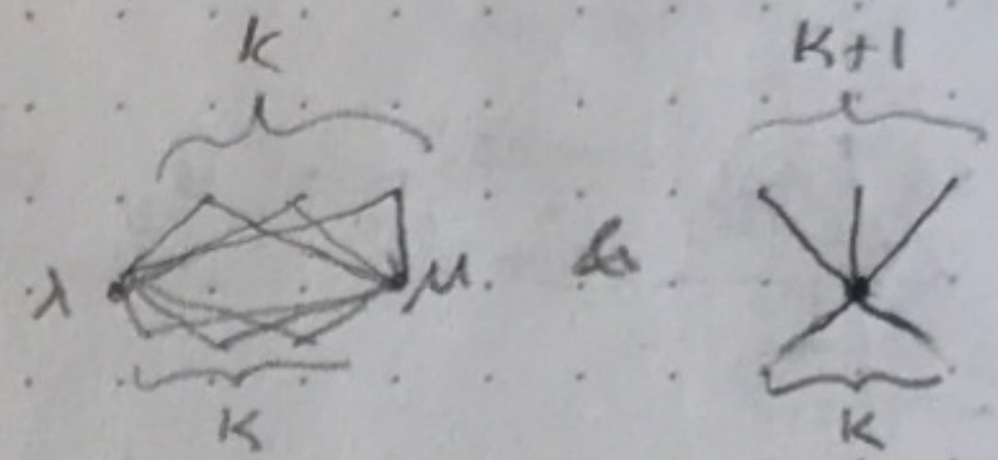
  $U : \lambda \longmapsto \sum_{\mu \gtrdot \lambda} \mu$

  $D : \lambda \longmapsto \sum_{\mu \lessdot \lambda} \mu$

  satisfy $[D, U] = 1$

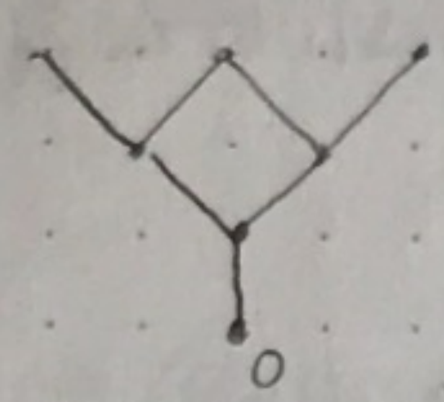
locally finite: P has fin. many elts of rank n

$\left(\Leftrightarrow \forall \lambda \neq \mu \text{ in } P,\right.$



$\left.\right)$

Can start building up: via differential poset rules

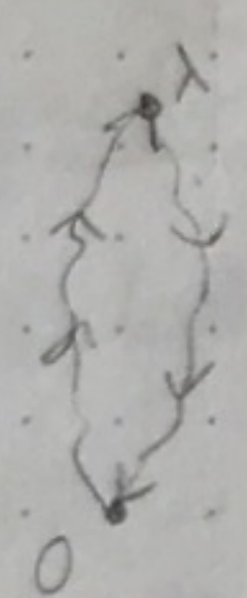


Lemma: Young's Lattice, $\mathbb{Y}$ is a differential poset

$f_\lambda^P = \#\{\text{saturated chains } \hat{0} \lessdot \lambda^{(1)} \lessdot \ldots \lessdot \lambda^{(n)} = \lambda\}$

Thrm: $\sum_{\lambda : rk(\lambda) = n} (f_\lambda^P)^2 = n!$

Proof 1:



LHS = coeff of $\hat{0}$ in $D^n U^n(\hat{0})$

$\boxed{\begin{array}{l} \cdot\, DU = UD + 1 \\ \cdot\, D(\hat{0}) = 0 \end{array}}$

$= \underbrace{DD \cdots D}_{n!}\underbrace{UU \cdots U}_{n!}(\hat{0})$

D can jump over U, or can anihilate with one of the U's (or get to $\hat{0}$ at the end and dissapeare)

$= \#\{\text{matchings between all } U\text{'s } \& \text{ all } D\text{'s}\}\, \hat{0}$

$= n!\, \hat{0}$

Proof 2: Two operators on $\mathbb{C}[x]$

$X : f(x) \longmapsto x f(x)$

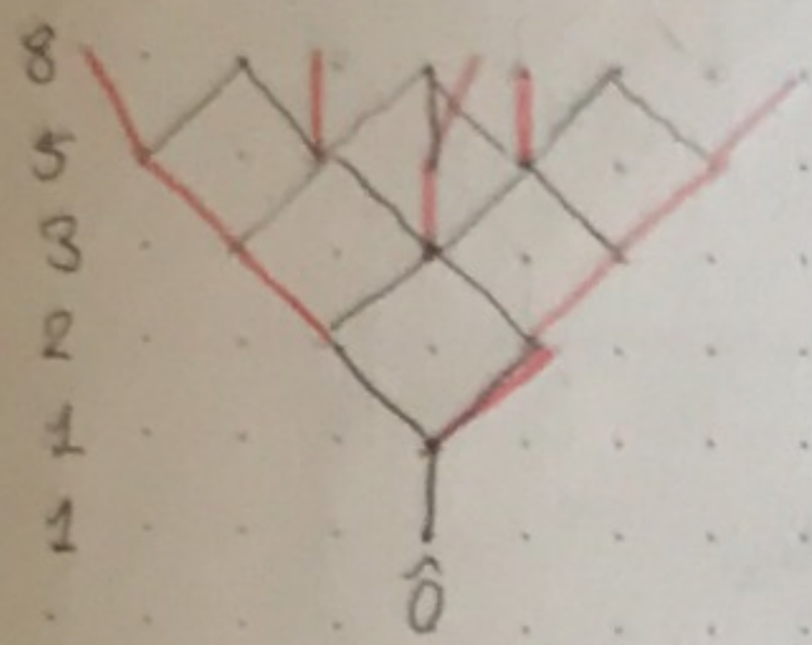
$D : f(x) \longmapsto f'(x)$

Satisfy same relation as D & U, so should produce same coeff

$\boxed{\begin{array}{l} [D, x] = 1 \\ D(1) = 0 \end{array}}$

$x^n \; D^{(n)} = n!$

Construction: For any level, reflect previous level over n^{th} level (the condition 1 automatically holds)
To make condition 2 hold, add 1 extra element above each elt in n^{th} level.



Question: Does this construction give $\mathbb{Y}$? NO
Is this unique?

# of rk n elts of this construction given by Fibonacci numbers

n	0	1	2	3	4	5	6	7
F_n	1	1	2	3	5	8	13	21
$p(n)$	1	1	2	3	5	7	11	15

$p(n)$ = # partitions of n

In $\mathbb{Y}$, 4^{th} level is different (as well as later levels)

<u>(Strd) Oscillating Tableaux</u>

$w = w_{2n} \cdots w_2 w_1$ $\qquad w_i \in \{U, D\}$

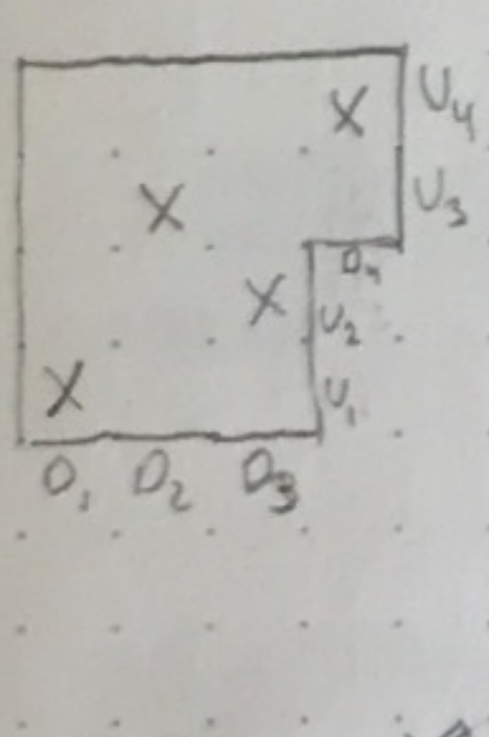
E.x. $w = DDDUUDUU$ $\quad$ n U's, n D's

# of paths in the Hasse diag. of $\mathbb{Y}$ that start & end at $\emptyset$ & have up & down steps given by w?

$= $ coeff of $w(\emptyset)$

= number of matchings between all U's & all D's s.t. each D is matched with some U to the <u>right</u> of it.

$D_1 \; D_2 \; D_3 \; U_1 \; U_2 \; D_4 \; U_3 \; U_4$

$\mathcal{Y}_w =$


→ = # of placements of non-attacking rooks on Young Diagram $\mathcal{Y}_w$.

<u>E.x.</u> # Paths in $\mathbb{Y}$

$\lambda^{(0)} \quad \lambda^{(1)} \quad \ldots \quad \lambda^{(2n)}$
$\lambda^{(0)} = \emptyset$, $\lambda^{(2n)} = \emptyset$

$\forall i, \quad \lambda^{(2i+1)} = \begin{cases} \lambda^{(2i)} \cup \text{a box} \\ \lambda^{(2i)} \end{cases} \qquad \lambda^{(2i)} = \begin{cases} \lambda^{(2i-1)} \setminus \text{box} \\ \lambda^{(2i-1)} \end{cases}$

$$\underbrace{\ldots(D+1)(U+1)(D+1)(U+1)}_{2n \text{ factors}}(\emptyset)$$

$= (\text{some coeff})\,\emptyset + \ldots$