# 18.217 LECTURE 10

9/30/22

Operators on $\mathbb{C}[x_1, \dots, x_n]$

div. diff $\partial_i : f \longmapsto \frac{1}{x_i - x_{i+1}}(1 - s_i) f$

Demozure $D_i : f \longmapsto \frac{1}{1 - x_{i+1}/x_i}\left(1 - \frac{x_{i+1}}{x_i} s_i\right) f$

$X_i : f \longmapsto x_i f$

Then $D_i = \partial_i X$

For $w = s_{i_1} s_{i_2} \dots s_{i_l}$ (reduced word)

$\partial_{w_i} := \partial_{i_1} \dots \partial_{i_l}$

$D_{w_i} := \partial_{i_1} \dots \partial_{i_l}$

For $\beta = (\beta_1, \dots, \beta_n)$, $X^\beta = X_1^{\beta_1} \cdots X_n^{\beta_n}$

<u>Exercise</u>: For longest word, $\partial_{w_i} : f \longmapsto \frac{1}{\prod_{1 \le i < j \le n} x_i - x_j} \left( \sum_{w \in S_n} (-1)^{\ell(w)} w(f) \right)$

<u>Thrm</u>: $D_{w_0} = \partial_{w_0} X^\delta$  $\delta = (n-1, n-2, \dots, 1, 0)$

<u>Lemma</u>: (1) $\partial_i$ commutes with $X_j$ if $j \ne i, i+1$

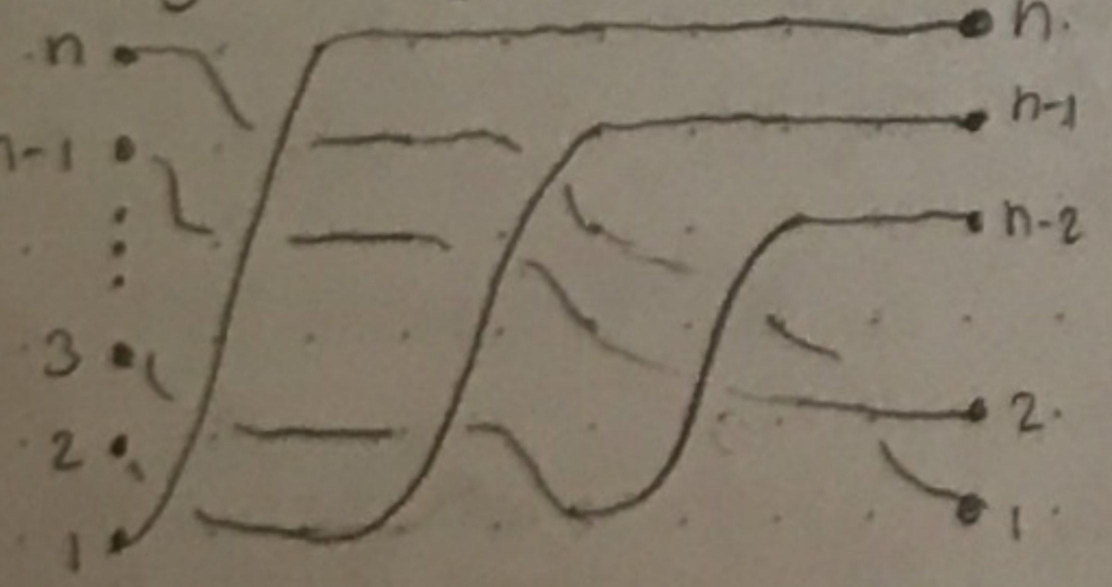
(2) $\partial_i$ commutes with any $f(x_1, \dots, x_n)$ s.t. $s_i(f) = f$.

<u>E.x.</u> $\partial_1$ commutes with $x_1 + x_2$ & $x_1 x_2$.

$S_n \ni w = (s_1 s_2 s_3 \dots s_{n-1})(s_1 s_2 \dots s_{n-2}) \cdots (s_1 s_2)(s_1)$

cycles $(1, 2, \dots, n)$  $(1, 2, \dots, n-1)$ $\cdots$ $(1,2)$ $(1)$

Wiring diagram:




(The maximal word)

$$D_{w_0} = \left.\begin{array}{l} (\partial_1 X_1 \partial_2 X_2 \dots \partial_{n-1} X_{n-1}) \\ (\partial_1 X_1 \dots \partial_{n-2} X_{n-2}) \\ \vdots \\ (\partial_1 X_1 \partial_2 X_2) \\ (\partial_1 X_1) \end{array}\right\} = \begin{array}{l} (\partial_1 \partial_2 \dots \partial_{n-1} X_1 X_2 \dots X_{n-1}) \\ (\partial_1 \partial_2 \dots \partial_{n-2} X_1 \dots X_{n-2}) \\ \vdots \\ (\partial_1 X_1) \end{array}$$

Now can commute all $X_i$'s to the end together $\Rightarrow$ we get $\partial_{w_0} X^\delta$

$D_1 = \partial_1 x_1$ , $D_2 = \partial_2 x_2$

$D_{w_0} = \partial_{w_0} x^{\delta}$

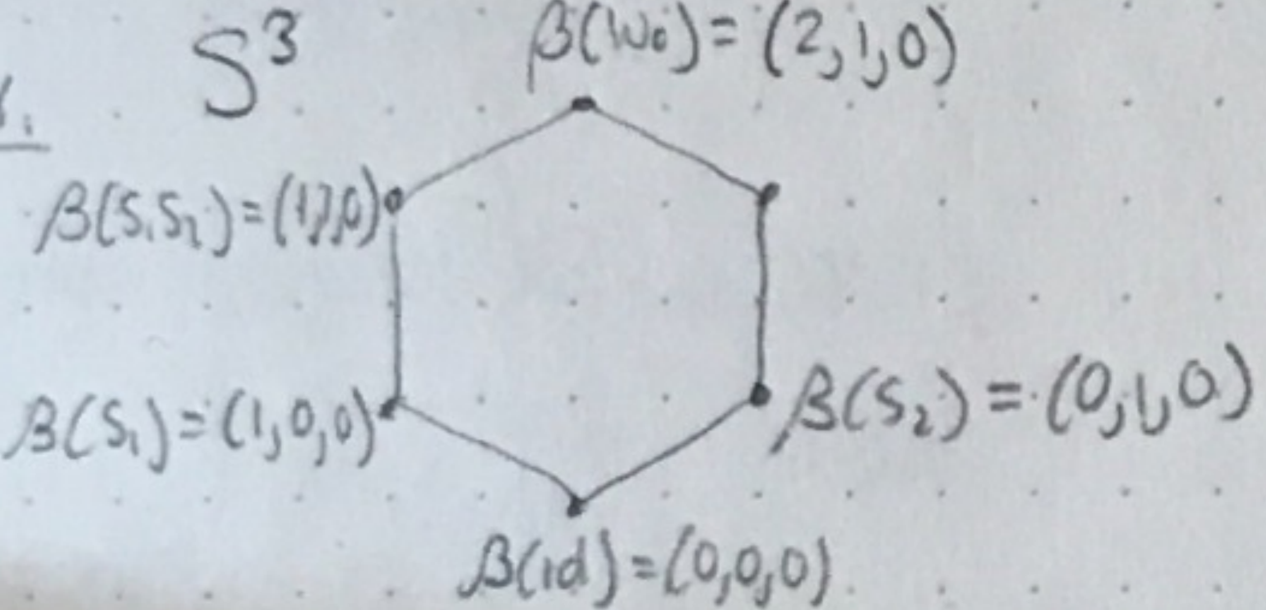
$D_{s_1 s_2} = D_1 D_2$

$= \partial_1 x_1 \partial_2 x_2 = \partial_{s_1 s_2} x_1 x_2$

but $D_{s_2 s_1} = \partial_2 x_2 \partial_1 x_1 \neq \partial_2 \partial_1 x^{\beta}$

<u>Problem</u>: Characterize permutations $w \in S_n$ s.t.
$D_w = \partial_w x^{\beta}$ for some $\beta = \beta(w) = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$ (will be On PSET)

<u>E.x.</u> $S^3$




<u>Hint</u>: Use
<u>Lehmer Code</u>: $\mathrm{code}(w) = (c_1, \dots, c_n)$ of $w \in S_n$ where $c_i = \#\{j > i \mid w(i) < w(j)\}$

$\beta$ will be Lehmer code of permutation

$\mathrm{code}(w) \leq \delta$

In general,

$D_w = \partial_w x^{\mathrm{code}(w)} + (\text{something else})$ $\leftarrow$ Question: Find what the something else is

<u>Thrm</u>: $S_\lambda^{comb} \overset{?}{=} S_\lambda^{class} = S_\lambda^{schub} \overset{\checkmark}{=} S_\lambda^{perm}$

follows from Lemma in This lecture

proved before

<u>Symmetric Functions</u>

$\Lambda$ ring of symmetric functions in $x_1, x_2, x_3, \dots$

<u>Elementary symm functions</u>: $e_k := \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ — sum of all square free monomials of deg $k$

<u>Complete homogeneous fcntns</u>: $h_k := \sum_{j_1 \leq \dots \leq j_k} x_{j_1} \cdots x_{j_k}$ — Sum of <u>all</u> monomials of deg $k$

$e_k = S_{1^k}$ $\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_k \\ \hline \end{array}$ $\wedge$ $\qquad h_k = S_{(k)}$ $\overset{\leq}{\begin{array}{|c|c|c|c|}\hline j_1 & j_2 & \cdots & j_k \\ \hline \end{array}}$

<u>Fundamental Thrm of Symm Functions</u>

(1) $\Lambda = \mathbb{C}[e_1, e_2, \dots]$
(2) $\Lambda = \mathbb{C}[h_1, h_2, \dots]$

How to express $e_k$ in terms of $h_\ell$'s and vice versa $\rightarrow$ (1) $\Leftrightarrow$ (2) for Thrm.

<u>Lemma</u>: $e_0 = h_0 = 1$

$e_1 h_0 - e_0 h_1 = 0$

$e_2 h_0 - e_1 h_1 + e_0 h_2 = 0$

$$\boxed{\sum_{k=0}^{n} (-1)^k e_k h_{n-k} = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n>0 \end{cases}}$$

<u>Proof</u>: $\sum_{k=0}^{n} (-1)^k e_k h_{n-k} = \sum_{k=0}^{n} (-1)^k \left[\sum_{i_1<\dots<i_k} x_{i_1} \cdots x_{i_k}\right]\left[\sum_{j_1 \geq \dots \geq j_{n-k}} x_{j_1} x_{j_2} \cdots x_{j_{n-k}}\right]$

WTS all terms cancell.

Use sign reversing involution on set of pairs $\left\{ (i_1, \dots, i_k), (j_1, \dots, j_{n-k}) \,\middle|\, \begin{smallmatrix} i_1 < \dots < i_k \\ j_1 \geq \dots \geq j_{n-k} \end{smallmatrix} \right\}$

$$\varphi : ((i_1, \dots, i_k), (j_1, \dots, j_{n-k})) \longmapsto \begin{cases} ((i_1, \dots i_k, j_1), (j_2, \dots, j_{n-k})) & \text{if } i_k < j_1 \\ ((i_1, \dots, i_{k-1}), (i_k, j_1, \dots, j_{n-k})) & \text{if } i_k \geq j_1 \end{cases}$$

<u>Monomial Symmetric Functions</u>

$\lambda = (\lambda_1, \dots, \lambda_\ell)$ partition

$m_\lambda = x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}$ + (all other monomials obtained by permutation)

$\hookrightarrow$ $\{m_\lambda\}$ is a linear basis of $\Lambda$

Also define $e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_\ell}$

$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_\ell}$

FTSF $\Leftrightarrow$ (1) $\{e_\lambda\}$ is a lin. basis of $\Lambda$

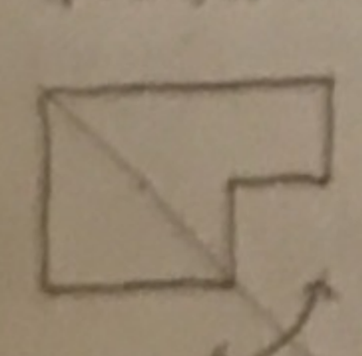
(2) $\{h_\lambda\}$ is a lin. basis of $\Lambda$

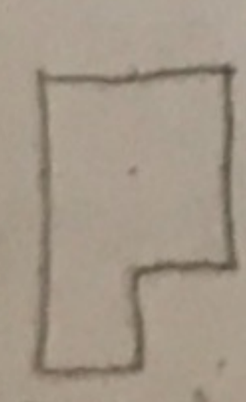
Would actually work for any extension of dominance ordering

Lex order on all partitions of $n$

$\lambda \geq_{lex} \mu$ if $\lambda_i = \mu_i$ for $i = 1, \dots, k-1$ $\lambda_k > \mu_k$ for some $k \leq \ell$

Conj. Partition $\lambda'$ of $\lambda$

$\lambda =$ 

$\lambda' =$ 

$(3,2) = \lambda$ $\quad$ $(2,2,1) = \lambda'$

<u>Lemma</u>: $e_\lambda = m_{\lambda'} + \sum_{\mu <_{lex} \lambda'} a_{\lambda\mu} m_\mu$

some coeff

# 18.217 LECTURE 11

$m_\lambda$ monomial symm. funcns

$s_\lambda = s_\lambda^{comb}$ Schur symmetric functions

$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$ elementary s.f.

$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ complete homogeneous s.f.

<u>Thm</u> $\{m_\lambda\}, \{s_\lambda\}, \{e_\lambda\}, \{h_\lambda\}$ are lin. bases of $\Lambda$ ← ring of s.f.

FTSF

$\Leftrightarrow \Lambda = \mathbb{C}[e_1, \ldots, e_n] = \mathbb{C}[h_1, \ldots, h_n]$

<u>Proof</u> $m_\lambda$ ✓

$\boxed{s_\lambda:}$ $\quad s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu \qquad K_{\lambda\mu}$ = Kotska number = # SSYT of shape $\lambda$, weight $\mu$

$(K_{\lambda\mu})$ is upper triangular matrix: (w/ $\lambda, \mu$ indexed by Lex order)

- $K_{\lambda\lambda} = 1$
- $K_{\lambda\mu} = 0$ unless $\lambda \geq_{Lex} \mu$

→ Form a basis

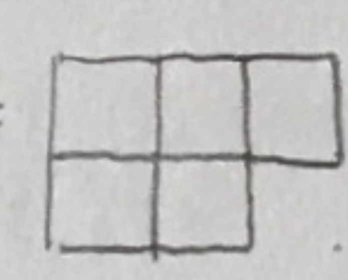
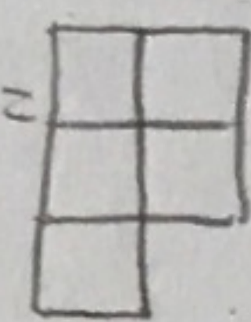
$\boxed{e_\lambda:}$ $\quad e_\lambda = \sum_\mu c_{\lambda\mu} m_\lambda$

$\lambda'$ the conjugate partition

- $c_{\lambda\lambda'} = 1$
- $c_{\lambda\mu'} = 0$ unless $\mu \leq_{Lex} \lambda'$

⇒ again get an upper triangular matrix

E.x. $e_{(3,2)} = e_3 \cdot e_2 = (x_1x_2x_3 + x_1x_2x_4 + \ldots) \cdot (x_1x_2 + x_1x_3 + \ldots)$

$\lambda =$  $\quad \lambda' =$ 

$$= \underbrace{(x_1^2x_2^2x_3 + \ldots)}_{m_{221}} + \underbrace{(x_1^2x_2x_3x_4 + \ldots)}_{m_{2111}} + \underbrace{(x_1x_2x_3x_4x_5 + \ldots)}_{m_{11111}}$$

$$= m_{221} + a m_{2111} + b m_{11111}$$

$\boxed{h_\lambda:}$ $\quad h_\lambda = \sum_\mu d_{\lambda\mu} m_\mu$

some coeff

$d_{\lambda\mu} \neq 0$ for any $\lambda, \mu$ s.t. $|\lambda| = |\mu|$

<u>Lemma</u>: $\sum_{k=0}^{n} (-1)^k e_k h_{n-k} = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n>0 \end{cases}$

Let $E(t) := e_0 + e_1 t + e_2 t^2 + \ldots$

$H(t) := h_0 + h_1 t + h_2 t^2 + \ldots$

Thm Lemma $\Leftrightarrow E(t)\cdot H(-t) = 1$

$$E(t) = \prod_{i=1}^{\infty} (1 + x_i t)$$

$$H(t) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} = \prod_{i=1} (1 + x_i t + x_i^2 t_i^2 + \dots)$$

$$\Downarrow$$

$$E(t) H(t) = 1$$

$$H = \begin{bmatrix} 1 & & & & \\ h_1 & 1 & & 0 & \\ h_2 & h_1 & 1 & & \\ \vdots & h_2 & \ddots & \ddots & \\ h_2 & \dots & h_2 & h_1 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & & & & \\ -e_1 & 1 & & 0 & \\ e_2 & -e_1 & 1 & & \\ -e_3 & e_2 & \ddots & \ddots & \\ & -e_3 & & & \\ \pm e_n & -e_3 & e_2 & -e_1 & 1 \end{bmatrix}$$

$(n+1)\times(n+1)$ matrix

Lemma $\Leftrightarrow \boxed{E = H^{-1}}$

$\tilde{H}, \tilde{E}$ switch which matrix has alternating $(-)$ signs

Lemma $e_n = \begin{vmatrix} h_1 & 1 & & 0 \\ h_2 & h_1 & 1 & \\ & h_2 & \ddots & \\ \vdots & & & 1 \\ h_n & & h_2 & h_1 \end{vmatrix}$

determinants of $n \times n$ matrix

$$h_n = \begin{vmatrix} e_1 & 1 & & \\ e_2 & e_1 & 1 & \\ \vdots & e_2 & \ddots & 1 \\ e_n & \dots & e_2 & e_1 \end{vmatrix}$$

Involution $\omega$

Lemma: $\exists!$ algebra homomorphism

$\omega : \Lambda \to \Lambda$ s.t.

$\omega : e_k \mapsto h_k$
$\quad h_k \mapsto e_k \qquad \forall k$

FTSF $\Rightarrow$

Lemma: $\{f_\lambda\}, \{g_\lambda\}$ collection of symm. funcs. s.t.

(1) $\deg(f_\lambda) = \deg(g_\lambda) = |\lambda|$

(2) $f_\emptyset = g_\emptyset = 1$

Lemma cont

(3) $e_n \cdot f_\lambda = \sum_\mu d_{\lambda\mu n} f_\mu$

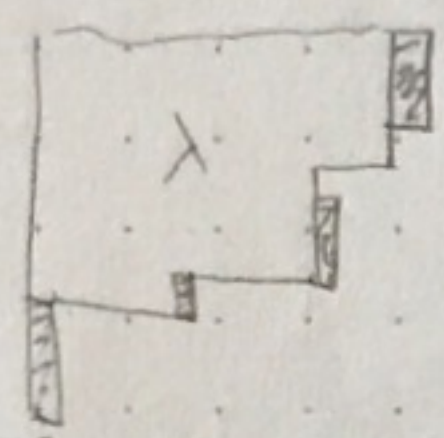
$e_n g_\lambda = \sum_\mu d_{\lambda\mu n} g_\mu$ $\qquad \forall \lambda, n.$

Then $\{f_\lambda\}$ and $\{g_\lambda\}$ are linear basis of $\Lambda$ and $f_\lambda = g_\lambda \ \forall \lambda$.

## Pieri Rules

The following rules hold for both $S_\lambda^{comb}$ and $S_\lambda^{class}$

(1) $e_n \cdot S_\lambda = \sum_{\substack{\mu \text{ st } \mu/\lambda \\ \text{is a vertical} \\ n\text{-strip}}} S_\mu$




$\mu - \lambda$ has $n$ boxes, any row contains $\leq 1$ box

(2) $h_n \cdot S_\lambda = \sum_{\substack{\mu \text{ st. } \mu/\lambda \\ \text{is horizontal} \\ n\text{-strip}}} S_\mu$

For $S_\lambda^{comb}$ (1) & (2) can be directly checked using <u>RSK</u>

For $S_\lambda^{class} = a_{\lambda+\delta}/a_\delta$, we can check (1) directly
(and (2) follows using involution on $\omega$)

<u>Thm</u>: $\omega : S_\lambda \longmapsto S_{\lambda'}$

<u>Proof #1</u>: Conjugate $\lambda$ ~ shifting between (1) & (2) above.
vertical strip ⇝ horizontal strip, while $\omega: e_n \rightsquigarrow h_n$

<u>Proof #2</u>: <u>Couchy identities</u>

<u>Proof #3</u>: Check it's a ring homo (sends products to products).

$$S_\lambda \cdot S_\mu = \sum C^\nu_{\lambda\mu} S_\nu$$

$$C^\nu_{\lambda\mu} = C^{\nu'}_{\lambda'\mu'}$$